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Metric unconditionality and Fourier analysis

by

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Abstract. We investigate several aspects of almost 1-unconditionality. We characterize the metric unconditional approximation property (umap) in terms of "block unconditionality". Then we focus on translation invariant subspaces $L^p_E(\mathbb{T})$ and $C_E(\mathbb{T})$ of functions on the circle and express block unconditionality as arithmetical conditions on E. Our work shows that the spaces $L^p_E(\mathbb{T})$, p an even integer, have a singular behaviour from the almost isometric point of view: property (umap) does not interpolate between $L^p_E(\mathbb{T})$ and $L^{p+2}_E(\mathbb{T})$. These arithmetical conditions are used to construct counterexamples for several natural questions and to investigate the maximal density of such sets E. We also prove that if $E = \{n_k\}_{k\ge1}$ with $|n_{k+1}/n_k| \to \infty$, then $C_E(\mathbb{T})$ has (umap) and we get a sharp estimate of the Sidon constant of Hadamard sets. Finally, we touch on the relationship of metric unconditionality and probability theory.

1. Introduction. We study isometric and almost isometric counterparts to the following two properties of a separable Banach space Y:

(ubs) Y is the closed span of an unconditional basic sequence;

(uap) Y admits an unconditional finite-dimensional expansion of the identity.

We focus on the case of translation invariant spaces of functions on the torus group $\mathbb{T}$, which will provide us with a bunch of natural examples. Namely, let E be a subset of $\mathbb{Z}$ and X be one of the spaces $L^p(\mathbb{T})$ $(1 \le p < \infty)$ or $C(\mathbb{T})$. If $\{e^{int}\}_{n\in E}$ is an unconditional basic sequence ((ubs) for short) in X, then E is known to satisfy strong conditions of lacunarity: E must be in Rudin's class $\Lambda(q)$, $q = p \vee 2$, and a Sidon set respectively. We raise the following question: what kind of lacunarity is needed to get the following stronger property:

(umbs) E is a metric unconditional basic sequence in X: for any $\varepsilon > 0$, one may lower its unconditionality constant to $1+\varepsilon$ by removing a finite set from it.

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In the case of $\mathcal{C}(\mathbb{T})$, E is a (umbs) exactly when E is a Sidon set with constant asymptotically 1.

In the same way, call $\{T_k\}$ an approximating sequence (a. s. for short) for Y if the T_k's are finite rank operators that tend strongly to the identity on Y; if such a sequence exists, then Y has the bounded approximation property. Denote by $\Delta T_k = T_k - T_{k-1}$ the difference sequence of T_k. Following Rosenthal (see [14, §1]), we then say that Y has the unconditional approximation property ((uap) for short) if it admits an a. s. $\{T_k\}$ such that for some C,

$$(1) \qquad \Big\|\sum_{k=1}^{n} \lambda_k \Delta T_k\Big\|_{\mathcal{L}(Y)} \le C \quad \text{for all } n \text{ and scalar } \lambda_k \text{ with } |\lambda_k| = 1.$$

By the uniform boundedness principle, (1) just means that $\sum \Delta T_k y$ converges unconditionally for all $y \in Y$. We now ask the following question: which conditions on E do yield the corresponding almost isometric property, first introduced by Casazza and Kalton [7, §3]:

(umap) The span $Y = X_E$ of E in X has metric (uap): for any $\varepsilon > 0$, one may lower the constant C in (1) to $1 + \varepsilon$ by choosing an appropriate a. s. $\{T_k\}$.

This has been studied by Li [31] for $X = \mathcal{C}(\mathbb{T})$; he obtains remarkably large examples of such sets E, in particular Hilbert sets. Thus, the second property seems to be much weaker than the first (although we do not know whether $\mathcal{C}_E(\mathbb{T})$ has (umap) for all (umbs) E in $\mathcal{C}(\mathbb{T})$).

In fact, both problems lead to strong arithmetical conditions on E that are somewhat complementary to the property of quasi-independence (see [45, §3]). In order to obtain them, we apply Forelli's [15, Prop. 2] and Plotkin's [47, Th. 1.4] techniques in the study of isometric operators on L^p: see Theorem 2.9 and Lemma 7.3. This may be done at once for (umbs). In the case of (umap), however, we need a more thorough knowledge of its connection with the structure of E: this is the duty of Theorem 6.2.3. As in Forelli's and Plotkin's results, we find that the spaces $X = L^p(\mathbb{T})$ with p an even integer play a special rôle. For instance, they are the only spaces which admit 1-unconditional basic sequences $E \subseteq \mathbb{Z}$ with more than two elements: see Proposition 2.5.

There is another fruitful point of view: we may consider elements of E as random variables on the probability space $(\mathbb{T}, dm)$. They have uniform distribution and if they were independent, then our questions would have trivial answers. In fact, they are strongly dependent: for any $k, l \in \mathbb{Z}$, Rosenblatt's [50] strong mixing coefficient

$$\sup\{|m[A \cap B] - m[A]m[B]| : A \in \sigma(\mathrm{e}^{\mathrm{i}kt}) \text{ and } B \in \sigma(\mathrm{e}^{\mathrm{i}lt})\}$$

has its maximum value, 1/4. But lacunarity of E enhances their independence in several weaker senses (see [2]). Properties (umap) and (umbs) can be seen as an expression of almost independence of elements of E in the "additive sense", i.e. when appearing in sums. We show their relationship to the notions of pseudo-independence (see [42, §4.2]) and almost i.i.d. sequences (see [1]).

The gist of our results is the following: almost isometric properties for the spaces X_E in "little" Fourier analysis may be read as a smallness property of E. They rely in an essential way on the arithmetical structure of E and distinguish between real and complex properties. In the case of $L^{2n}(\mathbb{T})$, n integer, these arithmetical conditions are finite in number and turn out to be sufficient, because these spaces have a polynomial norm. Furthermore, the number of conditions increases with n in that case. In the remaining cases of $L^p(\mathbb{T})$, p not an even integer, and $\mathcal{C}(\mathbb{T})$, these arithmetical conditions are infinitely many and become much more coercive. In particular, if our properties are satisfied in $\mathcal{C}(\mathbb{T})$, then they are satisfied in all $L^p(\mathbb{T})$ spaces, $1 \le p < \infty$.

We now turn to a detailed discussion of our results: in Section 2, we first characterize the sets E and spaces X such that E is a 1-(ubs) in X (Prop. 2.5). Then we show how to treat similarly the almost isometric case and obtain a range of arithmetical conditions $(\mathcal{I}_n)$ on E (Th. 2.9). These conditions turn out to be identical whether one considers real or complex unconditionality; this is surprising and in sharp contrast to what happens when $\mathbb{T}$ is replaced by the Cantor group. They also do not distinguish amongst $L^p(\mathbb{T})$ spaces with p not an even integer and $\mathcal{C}(\mathbb{T})$, but single out $L^p(\mathbb{T})$ with p an even integer: this property does not "interpolate". This is similar to the phenomenons of equimeasurability (see [29, introduction]) and $\mathcal{C}^\infty$-smoothness of norms (see [8, Chapter V]). These facts may also be appreciated from the point of view of natural renormings of the Hilbert space $L^2_E(\mathbb{T})$.

In Section 3, of purely arithmetical nature, we give many examples of (umbs) thanks to an investigation of property $(\mathcal{I}_n)$. As expected with lacunary series, number theoretic conditions show up (see especially Prop. 3.2).

In Section 4, we first return to the general case of a separable Banach space Y and show how to connect (umap) with a simple property of "block unconditionality". Then a skipped blocking technique (see [5]) gives a canonical way to construct an a. s. that realizes (umap) (Th. 4.2.4).

In Section 5, we introduce the p-power approximation property ℓ_p-(ap) and its metric counterpart, ℓ_p-(map). It may be described as simply as (umap). Then we connect ℓ_p-(map) with the work of Godefroy, Kalton, Li and Werner [28], [18] on subspaces of L^p which are almost isometric to ℓ_p.

Section 6 focuses on (uap) and (umap) in the case of translation invariant subspaces X_E. The property of block unconditionality may then be expressed in terms of "beginning" and "tail" of E: see Theorem 6.2.3.

In Section 7, we proceed as in Section 2 to obtain a range of arithmetical conditions $(\mathcal{J}_n)$ for (umap) and metric unconditional (fdd) (Th. 7.5 and Prop. 7.7). These conditions are similar to $(\mathcal{I}_n)$, but are decidedly weaker: see Proposition 8.2(i). This time, real and complex unconditionality differ; again $L^p(\mathbb{T})$ spaces with even p are singled out.

In Section 8, we continue the arithmetical investigation begun in Section 3 with property $(\mathcal{J}_n)$ and obtain many examples for (umap).

However, the main result of Section 9, Theorem 9.3, shows how a rapid (and optimal) growth condition on E allows avoiding number theory in any case considered. We therefore get a new class of examples for (umbs) and (umap). A sharp estimate of the Sidon constant of Hadamard sets is obtained as a byproduct (Cor. 9.4).

Section 10 uses combinatorial tools to give some rough information about the size of sets E that satisfy our arithmetical conditions. In particular, we answer a question of Li [31]: for $X \neq L^2(\mathbb{T}), L^4(\mathbb{T})$, the maximal density of E is zero if X_E has (umap) (Prop. 10.2).

Section 11 is an attempt to describe the relationship between these notions and probabilistic independence. Specifically, the Rademacher and Steinhaus sequences show the way to a connection between metric unconditionality and the almost i.i.d. sequences of [1]. We note further that the arithmetical property $(\mathcal{I}_\infty)$ of Section 2 is equivalent to Murai's [42, §4.2] property of pseudo-independence.

In Section 12, we collect our results on (umbs) and (umap) and conclude with open questions.

Notation and definitions. $(\mathbb{T}, dm)$ denotes the compact abelian group $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ endowed with its Haar measure dm; $m[A]$ is the measure of a subset $A \subseteq \mathbb{T}$. Let $\mathbb{D} = \{-1, 1\}$. $\mathbb{U}$ will denote either the complex ($\mathbb{U} = \mathbb{T}$) or real ($\mathbb{U} = \mathbb{D}$) choice of signs. For a real function f on $\mathbb{U}$, the *oscillation* of f is

$$\operatorname*{osc}_{\lambda\in\mathbb{U}} f(\lambda) = \sup_{\lambda\in\mathbb{U}} f(\lambda) - \inf_{\lambda\in\mathbb{U}} f(\lambda).$$

The dual group $\{e_n : \lambda \mapsto \lambda^n : n \in \mathbb{Z}\}$ of $\mathbb{T}$ is identified with $\mathbb{Z}$. We write $\#[B]$ for the cardinality of a set B.

For a not necessarily increasing sequence $E = \{n_k\}_{k\geq 1} \subseteq \mathbb{Z}$, let $\mathcal{P}_E(\mathbb{T})$ be the space of trigonometric polynomials spanned by E. Let X_E be the complex Banach space of those elements of $X \in \{\mathcal{C}(\mathbb{T}), \mathcal{M}(\mathbb{T}), L^p(\mathbb{T})\,(1 \leq p < \infty)\}$ whose Fourier transform vanishes off E.

Denote by $\pi_k : X_E \to X_E$ the orthogonal projection onto $X_{\{n_1,\ldots,n_k\}}$. It is given by

$$\pi_k(f) = \widehat{f}(n_1)e_{n_1} + \ldots + \widehat{f}(n_k)e_{n_k}.$$

Then the π_k commute. They form an a. s. for X_E if and only if E is a basic sequence. For a finite or cofinite $F \subseteq E$, π_F is similarly the orthogonal projection of X_E onto X_F.

For a given Banach space X, B_X is the unit ball of X and I denotes the identity operator on X. For a given sequence $\{U_k\}$ let $\Delta U_k = U_k - U_{k-1}$ (where $U_0 = 0$).

The functional notions of (ubs), (umbs) and the unconditionality constants $C_p(E)$ are defined in 2.1. The functional notions of a. s., (uap) and (umap) are defined in 4.1.1. Properties ℓ_p-(ap) and ℓ_p-(map) are defined in 5.1.1. The functional property $(\mathcal{U})$ of block unconditionality is defined in 6.2.1. The sets Ξ^m and Ξ^m_n of arithmetical relations are defined before 2.5. The arithmetical properties $(\mathcal{I}_n)$ of almost independence and $(\mathcal{J}_n)$ of block independence are defined in 2.8 and 7.2 respectively. The pairing $\langle \xi, E\rangle$ is defined before 3.1.

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2. Metric unconditional basic sequences (umbs). We start with the definition of metric unconditional basic sequences ((umbs) for short).

Definition 2.1. Let $E \subseteq \mathbb{Z}$ and $X = L^p(\mathbb{T})$ $(1 \leq p < \infty)$ or $X = \mathcal{C}(\mathbb{T})$ $(p = \infty)$.

(i) E is an *unconditional basic sequence* (*ubs*) in X if there is a constant C such that

$$\Big\| \sum_{q\in G} \lambda_q a_q e_q \Big\|_p \leq C \Big\| \sum_{q\in G} a_q e_q \Big\|_p \tag{2}$$

for all finite $G \subseteq E$, $a_q \in \mathbb{C}$ and scalar λ_q with $|\lambda_q| = 1$. We write C_p^{c} (resp. $C_p^{\mathrm{r}}(E)$) for the infimum of such C with complex (resp. real) λ_q.

(ii) E is a *complex* (resp. *real*) *metric unconditional basic sequence* (*umbs*) in X if for each $\varepsilon > 0$ there is a finite set F such that $C_p^{\mathrm{c}}(E \setminus F)$ (resp. $C_p^{\mathrm{r}}(E \setminus F)$) is less than $1 + \varepsilon$.

In fact, if (2) holds, then E is a basis of its span in $X \in \{\mathcal{C}(\mathbb{T}), L^p(\mathbb{T})\,(1 \leq p < \infty)\}$, which is X_E by Weierstraß' theorem. We have the following relationship between the unconditionality constants of E for the spaces $X \in \{\mathcal{C}(\mathbb{T}), L^p(\mathbb{T})\,(1 \leq p < \infty)\}$:

PROPOSITION 2.2. *Let $E \subseteq \mathbb{Z}$.*

(i) $C^{\mathrm{r}}_\infty(E) = \sup_{1\le p<\infty} C^{\mathrm{r}}_p(E)$ *and* $C^{\mathrm{c}}_\infty(E) = \sup_{1\le p<\infty} C^{\mathrm{c}}_p(E)$.

(ii) *If E is a (umbs) in $\mathcal{C}(\mathbb{T})$, it is a (umbs) in $L^p(\mathbb{T})$ for all $1 \le p < \infty$.*

This follows from the well-known (see e.g. [22])

LEMMA 2.3. *Let $E \subseteq \mathbb{Z}$ and T be a multiplier on $\mathcal{C}_E(\mathbb{T})$. Then $\|T\|_{\mathcal{L}(\mathcal{C}_E)} = \sup_{1\le p<\infty} \|T\|_{\mathcal{L}(L^p_E)}$.*

Proof. The linear functional $f \mapsto Tf(0)$ on $\mathcal{C}_E(\mathbb{T})$ extends to a measure $\mu \in \mathcal{M}(\mathbb{T})$ such that $\|\mu\|_{\mathcal{M}} = \|T\|_{\mathcal{L}(\mathcal{C}_E)}$. Let $\check\mu(t) = \mu(-t)$. Then $Tf = \check\mu * f$ for $f \in \mathcal{C}_E(\mathbb{T})$ and

$$\|T\|_{\mathcal{L}(L^p_E)} \le \|\check\mu\|_{\mathcal{M}} = \|T\|_{\mathcal{L}(\mathcal{C}_E)}.$$

Furthermore, if $\|Tf\|_p \le C\|f\|_p$ for all $1 \le p < \infty$, then $\|Tf\|_\infty \le C\|f\|_\infty$ by passing to the limit. ∎

REMARK. There is no interpolation theorem for such relative multipliers. The forthcoming Theorem 2.9 shows that there can be no metric interpolation. Furthermore, [16] furnishes an example of an $E \subseteq \mathbb{Z}$ such that the π_k are uniformly bounded on $L^1_E(\mathbb{T})$ but not on $\mathcal{C}_E(\mathbb{T})$.

It is known that E is an (ubs) in $\mathcal{C}(\mathbb{T})$ (resp. in $L^p(\mathbb{T})$) if and only if it is a Sidon (resp. $\Lambda(2\vee p)$) set. To see this, let us recall the relevant definitions.

DEFINITION 2.4. Let $E \subseteq \mathbb{Z}$.

(i) (See [26].) E is a *Sidon set* if there is a constant C such that

$$\sum_{q\in G} |a_q| \le C\Big\|\sum_{q\in G} a_q \mathrm{e}_q\Big\|_\infty \quad \text{for all finite } G \subseteq E \text{ and } a_q \in \mathbb{C}.$$

The infimum of such C is E's *Sidon constant.*

(ii) (See [52, Def. 1.5].) Let $p > 1$. E is a $\Lambda(p)$ *set* if there is a constant C such that $\|f\|_p \le C\|f\|_1$ for $f \in \mathcal{P}_E(\mathbb{T})$.

In fact, the Sidon constant of E is equal to $C^{\mathrm{c}}_\infty(E)$. Thus E is a complex (umbs) in $\mathcal{C}(\mathbb{T})$ if and only if tails of E have Sidon constants arbitrarily close to 1. We may also say: its Sidon constant is asymptotically 1.

Furthermore, E is a $\Lambda(2\vee p)$ set if and only if $L^p_E(\mathbb{T}) = L^2_E(\mathbb{T})$. Thus $\Lambda(2\vee p)$ sets are (ubs) in $L^p(\mathbb{T})$. Conversely, if E is an (ubs) in $L^p(\mathbb{T})$, then by Khinchin's inequality

$$\Big\|\sum_{q\in G} a_q\mathrm{e}_q\Big\|_p \approx \text{average}\Big\|\sum_{q\in G} \pm a_q\mathrm{e}_q\Big\|_p \approx \Big(\sum_{q\in G}|a_q|^2\Big)^{1/2} = \Big\|\sum_{q\in G} a_q\mathrm{e}_q\Big\|_2$$

for all finite $G \subseteq E$ (see [52, proof of Th. 3.1]).

The corresponding isometric question: when do we have $C^{\mathrm{c}}_X(E) = 1$? admits a rather easy answer. To this end, introduce the following notation: let $A_n = \{\alpha = \{\alpha_p\}_{p\ge1} : \alpha_p \in \mathbb{N} \ \&\ \alpha_1 + \alpha_2 + \ldots = n\}$; if $\alpha \in A_n$, all but a finite number of the α_p vanish and the multinomial coefficient $\binom{n}{\alpha} = n!/(\alpha_1!\alpha_2!\ldots)$ is well defined. Let $A^m_n = \{\alpha \in A_n : \alpha_p = 0 \text{ for } p > m\}$. Note that A^m_n is finite. We call E *n-independent* if

$$\sum \alpha_i p_i = \sum \beta_i p_i \Rightarrow \alpha = \beta \quad \text{for all } \alpha,\beta \in A^m_n \text{ and distinct } p_1,\ldots,p_m \in E.$$

In Rudin's [52, §1.6(b)] notations, if E is n-independent then the number $r_n(E;k)$ of representations of $k \in \mathbb{Z}$ as a sum of n elements of E is at most $n!$ for all k. This may also be expressed in the framework of arithmetical relations $\Xi^m = \{\xi \in \mathbb{Z}^{*m} : \xi_1 + \ldots + \xi_m = 0\}$ and $\Xi^m_n = \{\xi \in \Xi^m : |\xi_1| + \ldots + |\xi_m| \le 2n\}$. Note that Ξ^m_n is finite, and void if $m > 2n$. Then E is n-independent if and only if $\sum \xi_i p_i \ne 0$ for all $\xi \in \Xi^m_n$ and distinct $p_1,\ldots,p_m \in E$. We shall prefer to treat arithmetical relations in terms of Ξ^m_n rather than A^m_n.

PROPOSITION 2.5. *Let $E \subseteq \mathbb{Z}$.*

(i) *If $p \in [1,\infty]$ is not an even integer, then $C^{\mathrm{c}}_p(E) = 1$ if and only if E has at most two elements.*

(ii) *If p is an even integer, then $C^{\mathrm{c}}_p(E) = 1$ if and only if E is $p/2$-independent. There is a constant $C_p > 1$, depending only on p, such that either $C^{\mathrm{c}}_p(E) = 1$ or $C^{\mathrm{c}}_p(E) > C_p$.*

Proof. (i) Let p be not an even integer. We may suppose $0 \in E$; let $\{0,k,l\} \in E$. If we had $\|1 + \mu a\mathrm{e}_k + \nu b\mathrm{e}_l\|_p = \|1 + a\mathrm{e}_k + b\mathrm{e}_l\|_p$ for all $\lambda,\mu \in \mathbb{T}$, then

$$\begin{aligned}\int |1 + a\mathrm{e}_k + b\mathrm{e}_l|^p\,dm &= \int |1 + \mu a\mathrm{e}_k + \nu b\mathrm{e}_l|^p\,dm(\mu)\,dm(\nu)\,dm\\ &= \int |1 + \mu a + \nu b|^p\,dm(\mu)\,dm(\nu).\end{aligned}$$

Denoting by $\theta_i : (\lambda_1,\lambda_2) \mapsto \lambda_i$ the projections of $\mathbb{T}^2$ onto $\mathbb{T}$, this would mean that $\|1 + a\mathrm{e}_k + b\mathrm{e}_l\|_p = \|1 + a\theta_1 + b\theta_2\|_{L^p(\mathbb{T}^2)}$ for all $a,b \in \mathbb{C}$. By [53, Th. I], $(\mathrm{e}_k,\mathrm{e}_l)$ and (θ_1,θ_2) would have the same distribution. This is false, since θ_1 and θ_2 are independent random variables while e_k and e_l are not. So $C^{\mathrm{c}}_p(E) > 1$ and thus $C^{\mathrm{c}}_\infty(E) > 1$.

(ii) Let $q_1,\ldots,q_m \in E$ be distinct and $\lambda_1,\ldots,\lambda_m \in \mathbb{T}$. By the multinomial formula for the power $p/2$ and Bessel–Parseval's formula, we get

$$(3)\qquad \begin{aligned}\Big\|\sum_{i=1}^m \lambda_i a_i \mathrm{e}_{q_i}\Big\|_p^p &= \int\Big|\sum_{\alpha\in A^m_{p/2}} \binom{p/2}{\alpha}\prod_{i=1}^m (\lambda_i a_i)^{\alpha_i}\mathrm{e}_{\sum\alpha_i q_i}\Big|^2 dm\\ &= \sum_{A\in\mathcal{R}_q}\Big|\sum_{\alpha\in A}\binom{p/2}{\alpha}\prod_{i=1}^m(\lambda_i a_i)^{\alpha_i}\Big|^2\end{aligned}$$

$$= \sum_{\alpha\in A^m_{p/2}} \binom{p/2}{\alpha}^2 \prod_{i=1}^m |a_i|^{2\alpha_i}$$

$$+ \sum_{A\in\mathcal{R}_q} \sum_{\alpha\neq\beta\in A} \binom{p/2}{\alpha}\binom{p/2}{\beta} \prod_{i=1}^m \lambda_i^{\alpha_i-\beta_i} a_i^{\alpha_i}\overline{a}_i^{\beta_i},$$

where $\mathcal{R}_q$ is the partition of $A^m_{p/2}$ induced by the equivalence relation $\alpha\sim\beta \Leftrightarrow \sum\alpha_i q_i = \sum\beta_i q_i$. If E is $p/2$-independent, the double sum in (3) is void and E is a 1-(ubs).

Furthermore, suppose E is not $p/2$-independent and let $q_1,\dots,q_m\in E$ be a minimal set of elements of E such that there are $\alpha,\beta\in A^m_{p/2}$ with $\alpha\sim\beta$. Then $m\le p$. Take $a_i=1$ in the former computation; then the clearly nonzero oscillation of (3) for $\lambda_1,\dots,\lambda_m\in\mathbb{T}$ does only depend on $\mathcal{R}_q$ and thus is finitely valued. This yields C_p. ∎

REMARK (1). In fact, (ii) holds with real $C^{\mathrm{r}}_p(E)$ instead of complex $C^{\mathrm{c}}_p(E)$: if we have some arithmetical relation $\alpha\sim\beta$, we may assume that $\alpha_i-\beta_i$ is odd for one i at least. Indeed, we may simplify all $\alpha_i-\beta_i$ by their greatest common divisor and this yields another arithmetical relation $\sum(\alpha'_i-\beta'_i)q_i=0$. But then the oscillation of (3) is again clearly nonzero for $\lambda_1,\dots,\lambda_m\in\mathbb{D}$.

REMARK (2). We shall see later that (i) also holds in the real setting. This is a property of $\mathbb{T}$ and fails for the Cantor group $\mathbb{D}^\infty$: the Rademacher sequence forms a real 1-(ubs) in $\mathcal{C}(\mathbb{D}^\infty)$ but is clearly not complex 1-unconditional in any $L^p(\mathbb{D}^\infty)$, $p\neq 2$ (see Section 11).

As $C^{\mathrm{c}}_p(E)=1$ is thus a quite exceptional situation and distinguishes so harshly between even integers and all other reals, one may wonder what its almost isometric counterpart will bring about. In the proof of Proposition 2.5(i), we used the fact that the e_n, seen as random variables, are dependent: the L^p norm for even integer p is just somewhat blind to this because it keeps the interaction of the random variables down to a finite number of arithmetical relations. The contrast with the other L^p norms becomes clear when we try to compute explicitly an expression of the type $\|\sum\lambda_q a_q \mathrm{e}_q\|_p$ for any $p\in[1,\infty[$. This sort of seemingly brutal computation has been applied successfully in [15, Prop. 2] and [47, Th. 1.4] to study isometric operators on L^p, p not an even integer.

We now undertake this tedious computation as preparatory work for Theorem 2.9, Lemma 7.3 and Proposition 7.7. Let us fix some more notation: for $x\in\mathbb{R}$ and $\alpha\in A_n$, put

$$\binom{x}{\alpha} = \binom{x}{n}\binom{n}{\alpha}.$$

This generalized multinomial coefficient is nonzero if and only if $x\ge n$ or $x\notin\mathbb{N}$.

COMPUTATIONAL LEMMA 2.6. *Let $\mathbb{U}=\mathbb{T}$ or $\mathbb{U}=\mathbb{D}$ in the complex and real case respectively. Let $1\le p<\infty$ and $m\ge 1$. Put*

$$\varphi_q(\lambda,z,t) = \Big|1+\sum_{i=1}^m \lambda_i z_i \mathrm{e}_{q_i}(t)\Big|^p, \qquad \Phi_q(\lambda,z) = \int \varphi_q(\lambda,z,t)\,dm(t),$$

for $q=(q_1,\dots,q_m)\in\mathbb{Z}^m$, $\lambda=(\lambda_1,\dots,\lambda_m)\in\mathbb{U}^m$ and $z=(z_1,\dots,z_m)\in D^m$, where D is the disc $\{|w|\le\varrho\}\subseteq\mathbb{C}$ for some $0<\varrho<1/m$. Let $\mathcal{R}_q$ be the partition of $\mathbb{N}^m$ induced by the equivalence relation $\alpha\sim\beta\Leftrightarrow\sum\alpha_i q_i=\sum\beta_i q_i$. Then

$$\Phi_q(\lambda,z) = \sum_{\alpha\in\mathbb{N}^m}\binom{p/2}{\alpha}^2\prod|z_i|^{2\alpha_i} \tag{4}$$

$$+ \sum_{A\in\mathcal{R}_q}\sum_{\alpha\neq\beta\in A}\binom{p/2}{\alpha}\binom{p/2}{\beta}\prod z_i^{\alpha_i}\overline{z}_i^{\beta_i}\lambda_i^{\alpha_i-\beta_i}.$$

Furthermore, $\{\Phi_q : q\in\mathbb{Z}^m\}$ is a relatively compact subset of $C^\infty(\mathbb{U}^m\times D^m)$.

Proof. The function Φ_q is infinitely differentiable on the compact set $\mathbb{U}^m\times D^m$. Furthermore, the family $\{\Phi_q : q_1,\dots,q_m\in\mathbb{Z}\}$ is bounded in $C^\infty(\mathbb{U}^m\times D^m)$ and hence relatively compact by Montel's theorem. Let us compute φ_q. By the expansion of the function $(1+w)^{p/2}$, analytic on the unit disc, and the multinomial formula, we have

$$\varphi_q(\lambda,z) = \Big|\sum_{a\ge 0}\binom{p/2}{a}\Big(\sum_{i=1}^m\lambda_i z_i \mathrm{e}_{q_i}\Big)^a\Big|^2$$

$$= \Big|\sum_{a\ge 0}\binom{p/2}{a}\sum_{\alpha\in A^m_a}\binom{a}{\alpha}\prod(\lambda_i z_i)^{\alpha_i}\mathrm{e}_{\sum\alpha_i q_i}\Big|^2$$

$$= \Big|\sum_{\alpha\in\mathbb{N}^m}\binom{p/2}{\alpha}\prod(\lambda_i z_i)^{\alpha_i}\mathrm{e}_{\sum\alpha_i q_i}\Big|^2.$$

Then, by Bessel–Parseval's formula,

$$\Phi_q(\lambda,z) = \sum_{A\in\mathcal{R}_q}\Big|\sum_{\alpha\in A}\binom{p/2}{\alpha}\prod(\lambda_i z_i)^{\alpha_i}\Big|^2,$$

and this gives (4) by expanding the modulus. ∎

REMARK (3). This expansion has a finite number of terms if and only if p is an even integer: then and only then $\binom{p/2}{\alpha}=0$ for $\sum\alpha_i>p/2$, whereas $\mathcal{R}_q$ clearly contains some class with two elements and thus an infinity thereof.

For example, we have the following arithmetical relation on q_1, q_2 or q_1, q_2, 0 respectively:

$$\overbrace{q_1+\ldots+q_1}^{|q_2|}=\overbrace{q_2+\ldots+q_2}^{|q_1|}\quad \text{if } \operatorname{sgn} q_1=\operatorname{sgn} q_2,$$
$$\overbrace{q_1+\ldots+q_1}^{|q_2|}+\overbrace{q_2+\ldots+q_2}^{|q_1|}=0\quad \text{if not.}$$

REMARK (4). This shows that Proposition 2.5(i) holds also in the real setting: we may suppose that $0\in E$; take $m=2$ and choose distinct $q_1,q_2\in E$. One of the two relations in Remark (3) yields an arithmetical relation on E with at least one odd coefficient, as done in Remark (1). But then (4) contains terms nonconstant in λ_1 or λ_2 and thus $C_p^{\mathrm{r}}(E)>1$.

We return to our computation.

COMPUTATIONAL LEMMA 2.7. *Let $r=(r_0,\ldots,r_m)\in E^{m+1}$ and put $q_i=r_i-r_0$ $(1\le i\le m)$. Define*

$$\Theta_r(\lambda,z)=\int\Big|\mathrm{e}_{r_0}+\sum_{i=1}^m\lambda_i z_i\mathrm{e}_{r_i}\Big|^p=\Phi_q(\lambda,z). \tag{5}$$

Let $\xi_0,\ldots,\xi_m\in\mathbb{Z}^*$ and

$$(\gamma_i,\delta_i)=(-\xi_i\vee 0,\xi_i\vee 0)\quad(1\le i\le m). \tag{6}$$

If the arithmetical relation

$$\xi_0r_0+\ldots+\xi_mr_m=0\quad\textit{while}\quad \xi_0+\ldots+\xi_m=0 \tag{7}$$

holds, then the coefficient of $\prod z_i^{\gamma_i}\overline{z}_i^{\delta_i}\lambda_i^{\gamma_i-\delta_i}$ in (4) is $\binom{p/2}{\gamma}\binom{p/2}{\delta}$ and thus independent of r. If $\sum|\xi_i|\le p$ or p is not an even integer, this coefficient is nonzero.

Proof. We have $\delta_i-\gamma_i=\xi_i$, $\sum\gamma_i-\sum\delta_i=\xi_0$ and $\sum\gamma_i+\sum\delta_i=|\xi_1|+\ldots+|\xi_m|$, so that $\sum\gamma_i\vee\sum\delta_i=\frac12\sum|\xi_i|$. Moreover, $\sum(\delta_i-\gamma_i)q_i=\sum\xi_ir_i=0$, so that $\gamma\sim\delta$. ■

The computational lemmas suggest the following definition.

DEFINITION 2.8. Let $E\subseteq\mathbb{Z}$.

(i) E enjoys *property* $(\mathcal{I}_n)$ of *almost n-independence* provided there is a finite subset $F\subseteq E$ such that $E\setminus F$ is n-independent, i.e. $\xi_1r_1+\ldots+\xi_mr_m\ne0$ for all $\xi\in\Xi_n^m$ and distinct $r_1,\ldots,r_m\in E\setminus F$.

(ii) E enjoys $(\mathcal{I}_\infty)$ if it enjoys $(\mathcal{I}_n)$ for all n, i.e. for any $\xi\in\Xi^m$ there is a finite set F such that $\xi_1r_1+\ldots+\xi_mr_m\ne0$ for distinct $r_1,\ldots,r_m\in E\setminus F$.

Note that property $(\mathcal{I}_1)$ is void and that $(\mathcal{I}_{n+1})\Rightarrow(\mathcal{I}_n)$. This property is also stable under unions with finite sets. The preceding computations yield

THEOREM 2.9. *Let $E=\{n_k\}\subseteq\mathbb{Z}$ and $1\le p<\infty$.*

(i) *Suppose p is an even integer. Then E is a real, and at the same times complex, (umbs) in $L^p(\mathbb{T})$ if and only if E enjoys $(\mathcal{I}_{p/2})$. If $(\mathcal{I}_{p/2})$ holds, there is in fact a finite $F\subseteq E$ such that $E\setminus F$ is a 1-(ubs) in $L^p(\mathbb{T})$.*

(ii) *If p is not an even integer and E is a real or complex (umbs) in $L^p(\mathbb{T})$, then E enjoys $(\mathcal{I}_\infty)$.*

Proof. Sufficiency in (i) follows directly from Proposition 2.5: if $E\setminus F$ is $p/2$-independent, then $C_p^{\mathrm{c}}(E\setminus F)=C_p^{\mathrm{r}}(E\setminus F)=1$.

Let us prove the necessity of the arithmetical property. We keep the notation of computational lemmas 2.6 and 2.7. Assume E fails $(\mathcal{I}_n)$ and let $\xi_0,\ldots,\xi_m\in\mathbb{Z}^*$ with $\sum\xi_i=0$ and $\sum|\xi_i|\le 2n$ such that for each $l\ge1$ there are distinct $r_0^l,\ldots,r_m^l\in E\setminus\{n_1,\ldots,n_l\}$ with $\xi_0r_0^l+\ldots+\xi_mr_m^l=0$. One may furthermore assume that at least one of the ξ_i is not even.

Assume E is a (umbs) in $L^p(\mathbb{T})$. Then the oscillation of Θ_r in (5) satisfies

$$\operatorname*{osc}_{\lambda\in\mathbb{U}^m}\Theta_{r^l}(\lambda,z)\xrightarrow[l\to\infty]{}0 \tag{8}$$

for each $z\in D^m$. We may assume that the sequence of functions Θ_{r^l} converges in $\mathcal{C}^\infty(\mathbb{U}^m\times D^m)$ to a function Θ. Then by (8), $\Theta(\lambda,z)$ is constant in λ for each $z\in D^m$; in particular, its coefficient of $\prod z_i^{\gamma_i}\overline{z}_i^{\delta_i}\lambda_i^{\gamma_i-\delta_i}$ is zero. (Note that at least one of the $\gamma_i-\delta_i$ is not even.) This is impossible by computational lemma 2.7 if p is either not an even integer or if $p\ge 2n$. ■

COROLLARY 2.10. *Let $E\subseteq\mathbb{Z}$. If E is a (umbs) in $\mathcal{C}(\mathbb{T})$, that is, E's Sidon constant is asymptotically 1, then E enjoys $(\mathcal{I}_\infty)$. The converse does not hold.*

Proof. Necessity follows from Theorem 2.9 and Proposition 2.2(ii). There is a counterexample to the converse in [52, Th. 4.11]: Rudin constructs a set E that enjoys $(\mathcal{I}_\infty)$ while E is not even a Sidon set. ■

For p an even integer, Sections 3 and 10 will provide various examples of (umbs) in $L^p(\mathbb{T})$. Proposition 9.2 gives a general growth condition on E under which it is a (umbs).

As we do not know any partial converse to Theorem 2.9(ii) and Corollary 2.10, the sole known examples of (umbs) in $L^p(\mathbb{T})$, p not an even integer, and $\mathcal{C}(\mathbb{T})$ are those given by Theorem 9.3. This theorem will therefore provide us with Sidon sets of constant asymptotically 1. Note, however, that Li [31, Th. 4] already constructed implicitly such a Sidon set by using Kronecker's theorem.

3. Examples for (umbs). After a general study of the arithmetical property $(\mathcal{I}_n)$ of almost independence, we shall investigate three classes of subsets of $\mathbb{Z}$: integer geometric sequences, more generally integer parts of real geometric sequences, and polynomial sequences.

The quantity

$$\langle\xi,E\rangle=\sup_{F\subseteq E\text{ finite}}\inf\{|\xi_1p_1+\ldots+\xi_mp_m|:p_1,\ldots,p_m\in E\setminus F\text{ distinct}\}$$
$$=\lim_{l\to\infty}\inf\{|\xi_1p_1+\ldots+\xi_mp_m|:p_1,\ldots,p_m\in\{n_l,n_{l+1},\ldots\}\text{ distinct}\},$$

where $\{n_k\}=E$, plays a key rôle. We have

PROPOSITION 3.1. *Let* $E=\{n_k\}\subseteq\mathbb{Z}$.

(i) *E enjoys $(\mathcal{I}_n)$ if and only if $\langle\xi,E\rangle\neq 0$ for all $\xi\in\Xi_n^m$. If $\langle\xi,E\rangle<\infty$ for some $\xi_1,\ldots,\xi_m\in\mathbb{Z}^*$, then E fails $(\mathcal{I}_{|\xi_1|+\ldots+|\xi_m|})$. Thus E enjoys $(\mathcal{I}_\infty)$ if and only if $\langle\xi,E\rangle=\infty$ for all $\xi_1,\ldots,\xi_m\in\mathbb{Z}^*$.*

(ii) *Suppose E is an increasing sequence. If E enjoys $(\mathcal{I}_2)$, then the pace $n_{k+1}-n_k$ of E tends to infinity.*

(iii) *Suppose $jF+s,kF+t\subseteq E$ for an infinite F, $j\neq k\in\mathbb{Z}^*$ and $s,t\in\mathbb{Z}$. Then E fails $(\mathcal{I}_{|j|+|k|})$.*

Proof. (i) Suppose $\langle\xi,E\rangle<\infty$. Then there is an $h\in\mathbb{Z}$ such that there are sequences $p_1^l,\ldots,p_m^l\in\{n_k\}_{k\geq l}$ with $\sum\xi_ip_i^l=h$ and $\{p_1^l,\ldots,p_m^l\}\neq\{p_1^{l+1},\ldots,p_m^{l+1}\}$ for all $l\geq 1$. As $\sum\xi_ip_i^l-\sum\xi_ip_i^{l+1}=0$ for $l\geq 1$, E fails $(\mathcal{I}_{|\xi_1|+\ldots+|\xi_m|})$.

(ii) Indeed, $\langle(1,-1),E\rangle=\infty$.

(iii) Put $\xi=(j,-k)$. Then $\langle\xi,E\rangle<\infty$. ■

Geometric sequences. Let $G=\{j^k\}_{k\geq 0}$ with $j\in\mathbb{Z}\setminus\{-1,0,1\}$. Then $G,jG\subseteq G$; so G fails $(\mathcal{I}_{|j|+1})$. In order to check $(\mathcal{I}_{|j|})$ for G, let us study more carefully the following Diophantine equation:

$$\xi_1j^{k_1}+\ldots+\xi_mj^{k_m}=0\quad\text{with }|\xi_1|+\ldots+|\xi_m|\leq 2|j|\ \&\ k_1<\ldots<k_m.\tag{9}$$

Suppose (9) holds. Then necessarily $m\geq 2$ and $\xi_1+\sum_{i=2}^m\xi_ij^{k_i-k_1}=0$. Hence $j\,|\,\xi_1$ and $|\xi_1|\geq|j|$. As $\xi_1<2|j|$, we may suppose $\xi_1=j$. Then $1+\sum_{i=2}^m\xi_ij^{k_i-k_1-1}=0$. Hence $k_2=k_1+1$ and $j\,|\,1+\xi_2$. As $|\xi_2|\leq|j|$, $\xi_2\in\{-1,|j|-1\}$. If $\xi_2=|j|-1$, then $m=3$, $k_3=k_1+2$ and $\xi_3=-\operatorname{sgn}j$. If $\xi_2=-1$, then $m=2$: otherwise, $j\,|\,\xi_3$ as before and $|\xi_1|+|\xi_2|+|\xi_3|>2|j|$. Thus (9) has only two solutions:

$$j\cdot j^k+(-1)\cdot j^{k+1}=0,\qquad j\cdot j^k+(|j|-1)\cdot j^{k+1}+(-\operatorname{sgn}j)\cdot j^{k+2}=0.\tag{10}$$

If j is positive, this shows that G enjoys $(\mathcal{I}_j)$: both solutions yield $\sum\xi_i\neq 0$. If j is negative, G enjoys $(\mathcal{I}_{|j|-1})$, but the second solution of (9) shows that G fails $(\mathcal{I}_{|j|})$.

Algebraic and transcendental numbers. An interesting feature of property $(\mathcal{I}_\infty)$ is that it distinguishes between algebraic and transcendental numbers. A similar fact has already been noticed by Murai [42, Prop. 26, Cor. 28].

PROPOSITION 3.2. *Let* $E=\{n_k\}\subseteq\mathbb{Z}$.

(i) *If $n_{k+1}/n_k\to\sigma$ where $\sigma>1$ is transcendental, then $\langle\xi,E\rangle=\infty$ for any $\xi_1,\ldots,\xi_m\in\mathbb{Z}^*$. Thus E enjoys $(\mathcal{I}_\infty)$.*

(ii) *Write $[x]$ for the integer part of a real x. Let $n_k=[\sigma^k]$ with $\sigma>1$ algebraic. Let $P(x)=\xi_0+\ldots+\xi_dx^d$ be the corresponding polynomial of minimal degree. Then $\langle\xi,E\rangle<\infty$ and E fails $(\mathcal{I}_{|\xi_0|+\ldots+|\xi_d|})$.*

Proof. (i) Suppose on the contrary that we have ξ and sequences $p_1^l<\ldots<p_m^l$ in E that tend to infinity such that $\xi_1p_1^l+\ldots+\xi_mp_m^l=0$. As the sequences $\{p_i^l/p_m^l\}_l$ $(1\leq i\leq m)$ are bounded, we may assume they are convergent—and by hypothesis, they converge either to 0, say for $i<j$, or to σ^{-d_i} for $d_i\in\mathbb{N}$ and $i\geq j$. But then $\xi_j\sigma^{-d_j}+\ldots+\xi_m\sigma^{-d_m}=0$ and σ is algebraic.

(ii) Apply Proposition 3.1(i) with ξ:

$$|\xi_0[\sigma^k]+\ldots+\xi_d[\sigma^{k+d}]|=|\xi_0([\sigma^k]-\sigma^k)+\ldots+\xi_d([\sigma^{k+d}]-\sigma^{k+d})|\leq\sum|\xi_i|.\ \blacksquare$$

Polynomial sequences. Let $E=\{P(k)\}$ for a polynomial P of degree d.

As usual, the investigation becomes more number-theoretical, but certain large arithmetical relations necessarily hold. Recall that

$$\Delta^jP(k)=\sum_{i=0}^j\binom{j}{i}(-1)^iP(k-i),$$
$$\sum_{i=0}^j\binom{j}{i}(-1)^i=0,\qquad\sum_{i=0}^j\binom{j}{i}=2^j.\tag{11}$$

As $\Delta^{d+1}P(k)=0$, we see that E fails $(\mathcal{I}_{2^d})$. However, this results is much too coarse. For example, the set of squares, the set of cubes and the set of biquadrates fail $(\mathcal{I}_2)$: we know the solutions $7^2+1^2=2\cdot 5^2$ (and $18^2+1^2=15^2+10^2$ [9, Book II, Problem 9]), $12^3+1^3=10^3+9^3$ [6, due to Frénicle] and $158^4+59^4=134^4+133^4$ (and $12231^4+2903^4=10381^4+10203^4$ [12])—and thus arbitrarily large solutions of $a^k+b^k=c^k+d^k$ for $k\in\{2,3,4\}$. Furthermore, $\{k^5\}$ and $\{k^6\}$ fail $(\mathcal{I}_3)$: we know the solutions $67^5+28^5+24^5=62^5+54^5+3^5$ (and $107^5+75^5+49^5=100^5+92^5+39^5$ [40]) and $23^6+15^6+10^6=22^6+19^6+3^6$ [49]. Finally, $\{k^7\}$ fails $(\mathcal{I}_4)$: we know the solution $149^7+123^7+14^7+10^7=146^7+129^7+90^7+15^7$ [10].

Note, however, that a positive answer to Euler's conjecture—for $k\geq 5$ $a^k+b^k=c^k+d^k$ has only trivial solutions in integers—would imply that the set of kth powers has $(\mathcal{I}_2)$. This conjecture has been neither proved nor disproved for any value of $k\geq 5$ (see [56]).

Conclusion. By Theorem 2.9, property $(\mathcal{I}_n)$ yields directly (umbs) in $L^{2p}(\mathbb{T})$, $p\leq n$ integer. But we do not know whether $(\mathcal{I}_\infty)$ ensures (umbs) in $L^p(\mathbb{T})$, p not an even integer.

4. Metric unconditional approximation property (umap). As we investigate simultaneously real and complex (umap), it is convenient to introduce a subgroup $\mathbb{U}$ of $\mathbb{T}$. Thus, if $\mathbb{U} = \mathbb{D}$, then the following applies to real (umap). If $\mathbb{U} = \mathbb{T}$, it applies to complex (umap).

If the reader is first and foremost interested in the application to harmonic analysis, he may concentrate on the equivalence (ii)$\Leftrightarrow$(iv) in Theorem 4.2.4 and then pass on to Section 6.

4.1. *Definition.* We start with defining the metric unconditional approximation property ((umap) for short). Recall that $\Delta T_k = T_k - T_{k-1}$ (where $T_0 = 0$).

DEFINITION 4.1.1. Let X be a separable Banach space.

(i) A sequence $\{T_k\}$ of operators on X is an *approximating sequence* (*a. s.*) if each T_k has finite rank and $\|T_k x - x\| \to 0$ for every $x \in X$. If X admits an a. s., it has the *bounded approximation property*. An a. s. of commuting projections is called a *finite-dimensional decomposition* (*fdd*).

(ii) (See [14].) X has the *unconditional approximation property* (*uap*) if there are an a. s. $\{T_k\}$ and a constant C such that

$$\Big\|\sum_{k=1}^{n} \lambda_k \Delta T_k\Big\| \le C \quad \text{for all } n \text{ and } \lambda_k \in \mathbb{U}. \tag{12}$$

The (*uap*) *constant* is the least such C.

(iii) (See [7, §3].) X has the *metric unconditional approximation property* (*umap*) if it has (uap) with constant $1+\varepsilon$ for any $\varepsilon > 0$.

Property (ii) is the approximation property which most appropriately generalizes the unconditional basis property. It has first been introduced by Pełczyński and Wojtaszczyk [44]. They showed that it holds if and only if X is a complemented subspace of a space with an unconditional (fdd). By [32, Th. 1.g.5], this implies that X is a subspace of a space with an unconditional basis. Thus, neither $L^1([0,1])$ nor $\mathcal{C}([0,1])$ share (uap).

Property (iii) has been introduced by Casazza and Kalton as an extreme form of metric approximation. It is now rather well understood: see [7, §3], [19, §8, 9], [18] and [17, §IV].

There is a simple and very useful criterion for (umap):

PROPOSITION 4.1.2 ([7, Th. 3.8] and [19, Lemma 8.1]). *Let X be a separable Banach space. X has (umap) if and only if there is an a. s. $\{T_k\}$ such that*

$$\sup_{\lambda\in\mathbb{U}} \|(I - T_k) + \lambda T_k\| \xrightarrow[k\to\infty]{} 1. \tag{13}$$

A careful reading of the above mentioned proof also gives the following results for a. s. that satisfy $T_{n+1}T_n = T_n$.

PROPOSITION 4.1.3. *Let X be a separable Banach space.*

(i) *Let $\{T_k\}$ be an a. s. for X such that $T_{n+1}T_n = T_n$. A subsequence $\{T'_k\}$ of $\{T_k\}$ realizes 1-(uap) in X if and only if for all $k \ge 1$,*

$$\sup_{\lambda\in\mathbb{U}} \|I - (1+\lambda)T'_k\| = 1.$$

(ii) *X has an unconditional metric (fdd) if and only if there is an (fdd) $\{T_k\}$ such that (13) holds.*

4.2. *A characterization of* (*umap*). We want to characterize (umap) in an even simpler way than Proposition 4.1.2. Relation (13) and the method of [28, Th. 4.2] suggest considering some unconditionality condition between a certain "beginning" and a certain "tail" of X. We propose two such notions.

DEFINITION 4.2.1. Let X be a separable Banach space.

(i) Let τ be a vector space topology on X. Then X has the property ($u(\tau)$) of *τ-unconditionality* if for all $u \in X$ and norm bounded sequences $\{v_j\} \subseteq X$ such that $v_j \xrightarrow{\tau} 0$,

$$\operatorname*{osc}_{\lambda\in\mathbb{U}} \|\lambda u + v_j\| \to 0. \tag{14}$$

(ii) Let $\{T_k\}$ be a commuting a. s. Then X has the property ($u(T_k)$) of *commuting block unconditionality* if for all $\varepsilon > 0$ and $n \ge 1$ we may choose $m \ge n$ such that for all $x \in T_n B_X$ and $y \in (I - T_m)B_X$,

$$\operatorname*{osc}_{\lambda\in\mathbb{U}} \|\lambda x + y\| \le \varepsilon. \tag{15}$$

Thus, given a commuting a. s. $\{T_k\}$, $T_n X$ is the "beginning" and $(I - T_m)X$ the "tail" of X. If the sequence is not commuting, there are more complex notions of beginning and tail: see the definition of A_n in the proof of [28, Th. 4.2].

We have

LEMMA 4.2.2. *Let X be a separable Banach space and $\{T_k\}$ a commuting a. s. for X. The following are equivalent.*

(i) *X enjoys ($u(\tau)$) for some vector space topology τ such that $T_n x \xrightarrow{\tau} x$ uniformly for $x \in B_X$;*

(ii) *X enjoys ($u(T_k)$).*

Proof. (i)$\Rightarrow$(ii). Suppose that (ii) fails: there are $n \ge 1$ and $\varepsilon > 0$ such that for each $m \ge n$, there are $x_m \in T_n B_X$ and $y_m \in (I - T_m)B_X$ such that

$$\operatorname*{osc}_{\lambda\in\mathbb{U}} \|\lambda x_m + y_m\| > \varepsilon.$$

As $T_n B_X$ is compact, we may suppose by extracting a convergent subsequence that $x_m = x$. Let τ be as in (i); then $y_m \xrightarrow{\tau} 0$ and ($u(\tau)$) must fail.

(ii)⇒(i). Define a vector space topology τ by

$$x_n \xrightarrow{\tau} 0 \Leftrightarrow \forall k\ \|T_k x_n\| \to 0.$$

Then $T_n x \xrightarrow{\tau} x$ uniformly on B_X. Indeed, $T_k(T_n x - x) = (T_n - I)T_k x$ and $T_n - I$ converges uniformly to 0 on $T_k B_X$, which is norm compact.

Let us check $(u(\tau))$. Let $u \in B_X$ and $\{v_j\} \subseteq B_X$ be such that $v_j \xrightarrow{\tau} 0$. Let $\varepsilon > 0$. There is $n \ge 1$ such that $\|T_n u - u\| \le \varepsilon$. Choose m such that (15) holds for $x \in T_n B_X$ and $y \in (I - T_m)B_X$. Then choose $k \ge 1$ such that $\|T_m v_j\| \le \varepsilon$ for $j \ge k$. We have, for any $\lambda \in \mathbb{U}$,

$$\begin{aligned}\|\lambda u + v_j\| &\le \|\lambda T_n u + (I - T_m)v_j\| + \|T_n u - u\| + \|T_m v_j\| \\ &\le \|T_n u + (I - T_m)v_j\| + 3\varepsilon \le \|u + v_j\| + 5\varepsilon.\end{aligned}$$

Thus we have (14). ■

In order to obtain (umap) from block independence, we have to construct unconditional skipped blocking decompositions.

DEFINITION 4.2.3. Let X be a separable Banach space. X admits *unconditional skipped blocking decompositions* if for each $\varepsilon > 0$, there is an unconditional a.s. $\{S_k\}$ such that for all $0 \le a_1 < b_1 < a_2 < b_2 < \ldots$ and $x_k \in (S_{b_k} - S_{a_k})X$,

$$\sup_{\lambda_k \in \mathbb{U}} \Big\| \sum \lambda_k x_k \Big\| \le (1+\varepsilon) \Big\| \sum x_k \Big\|.$$

We have

THEOREM 4.2.4. *Consider the following properties for a separable Banach space X.*

(i) *There are an unconditional commuting a.s. $\{T_k\}$ and a vector space topology τ such that X enjoys $(u(\tau))$ and $T_k x \xrightarrow{\tau} x$ uniformly for $x \in B_X$;*

(ii) *X enjoys $(u(T_k))$ for an unconditional commuting a.s. $\{T_k\}$;*

(iii) *X admits unconditional skipped blocking decompositions;*

(iv) *X has (umap).*

Then (iv)⇒(i)⇔(ii)⇒(iii). *If X has finite cotype, then* (iii)⇒(iv).

Proof. (i)⇔(ii) holds by Lemma 4.2.2.

(iv)⇒(ii). By [17, Th. IV.1], there is in fact an a.s. $\{T_k\}$ that satisfies (13) such that $T_k T_l = T_{\min(k,l)}$ if $k \ne l$.

Let C be a uniform bound for $\|T_k\|$. Let $\varepsilon > 0$ and $n \ge 1$. There is $m \ge n+2$ such that

$$\sup_{\lambda \in \mathbb{U}} \|\lambda T_{m-1} + (I - T_{m-1})\| \le 1 + \varepsilon/(2C).$$

Let $x \in T_n B_X$ and $y \in (I - T_m)B_X$. As $x - T_{m-1}x = 0$ and $T_{m-1}y = 0$,

$$\lambda x + y = \lambda T_{m-1}(x+y) + (I - T_{m-1})(x+y),$$

and, for all $\lambda \in \mathbb{U}$,

$$\|\lambda x + y\| \le (1 + \varepsilon/(2C))\|x + y\| \le \|x + y\| + \varepsilon.$$

(ii)⇒(iii). By [55, proof of Lemma III.9.2], we may suppose that $T_k T_l = T_{\min(k,l)}$ if $k \ne l$. Let $\varepsilon > 0$ and choose a sequence of $\eta_j > 0$ such that $1 + \varepsilon_j = \prod_{i \le j}(1 + \eta_i) < 1 + \varepsilon$ for all j. By (ii), there is a subsequence $\{S_j = T_{k_j}\}$ such that $k_0 = 0$ and thus $S_0 = 0$, and

$$\sup_{\lambda \in \mathbb{U}} \|x + \lambda y\| \le (1 + \eta_j)\|x + y\| \tag{16}$$

for $x \in (I - S_j)X$ and $y \in S_{j-1}X$. Let us show that it is an unconditional skipped blocking decomposition: we will prove by induction that

$$(H_j)\quad \begin{cases} \sup_{\lambda_i \in \mathbb{U}} \Big\| x + \sum_{i=1}^n \lambda_i x_i \Big\| \le (1+\varepsilon_j) \Big\| x + \sum_{i=1}^n x_i \Big\| \text{ for } x \in (I - S_j)X \\ \text{and } x_i \in (S_{b_i} - S_{a_i})X\ (0 \le a_1 < b_1 < \ldots < a_n < b_n \le j-1). \end{cases}$$

- (H_1) trivially holds.
- Assume (H_i) holds for $i < j$. Let x and x_i be as in (H_j). Let $\lambda_i \in \mathbb{U}$. Then

$$\Big\| x + \sum_{i=1}^n \lambda_i x_i \Big\| \le (1+\eta_j) \Big\| x + \overline{\lambda}_n \sum_{i=1}^n \lambda_i x_i \Big\| = (1+\eta_j) \Big\| x + x_n + \sum_{i=1}^{n-1} \overline{\lambda}_n \lambda_i x_i \Big\|$$

by (16). Note that $x + x_n \in (I - S_{a_n})X$; an application of (H_{a_n}) yields (H_j).

(iii)⇒(iv). Let $\varepsilon > 0$ and $r > 1$. There is an unconditional skipped blocking decomposition $\{T_k\}$ for ε. Let C_u be the (uap) constant of $\{T_k\}$. Let

$$V_{i,j} = T_{ir+j-1} - T_{(i-1)r+j} \quad \text{for } 1 \le j \le r \text{ and } i \ge 0.$$

The jth skipped blocks are

$$U_j = I - \Big(\sum_i V_{i,j} \Big) = \sum_i \Delta T_{ir+j};$$

then $\sum_{j=1}^r U_j = I$. Let

$$R_i = \frac{1}{r-1} \sum_{j=1}^r V_{i,j};$$

then R_i has finite rank and

$$R_0 + R_1 + \ldots = (rI - I)/(r-1) = I.$$

Thus $W_j = \sum_{i \le j} R_i$ defines an a.s. We can bound its (uap) constant. First, since $\{T_k\}$ is a skipped blocking decomposition,

$$\forall x\in B_X\quad \sup_{\lambda_i\in\mathbb{U}}\Big\|\sum\lambda_iR_ix\Big\|\le\frac{1}{r-1}\sum_{j=1}^{r}\sup_{\lambda_i\in\mathbb{U}}\Big\|\sum_i\lambda_iV_{i,j}x\Big\|\le\frac{1+\varepsilon}{r-1}\sum_{j=1}^{r}\|x-U_jx\|\le\frac{1+\varepsilon}{r-1}\Big(r+\sum_{j=1}^{r}\|U_jx\|\Big).$$

Let us bound $\sum_{j=1}^{r}\|U_jx\|$. Let $q<\infty$ be the cotype of X and C_c its cotype constant. Then by Hölder's inequality we have

$$(17)\qquad \forall x\in B_X\quad \sum\|U_jx\|\le r^{1-1/q}\Big(\sum\|U_jx\|^q\Big)^{1/q}\le r^{1-1/q}C_c\cdot\text{average}\Big\|\sum\pm U_jx\Big\|\le r^{1-1/q}C_cC_u.$$

Thus the (uap) constant of $\{W_j\}$ is at most $(1+\varepsilon)(r+C_cC_ur^{1-1/q})/(r-1)$. As ε is arbitrarily small and r arbitrarily large, X has (umap). ∎

We may remove the cotype assumption in Theorem 4.2.4(iii)⇒(iv) if the space has the properties of commuting ℓ_1-(ap) or ℓ_q-(fdd) for $q<\infty$, which will be introduced in Section 5:

THEOREM 4.2.5. *Consider the following properties for a separable Banach space X.*

(i) *There are a commuting ℓ_1-a. s. or an ℓ_q-(fdd) $\{T_k\}$, $q<\infty$, and a vector space topology τ such that X enjoys $(u(\tau))$ and $T_kx\xrightarrow{\tau}x$ uniformly for $x\in B_X$;*

(ii) *X enjoys $(u(T_k))$ for a commuting ℓ_1-a. s. or an ℓ_q-(fdd) $\{T_k\}$, $q<\infty$;*

(iii) *X admits unconditional skipped blocking decompositions and one may in fact take an ℓ_1-a. s. or an ℓ_q-(fdd) $\{T_k\}$, $q<\infty$, in its definition* 4.2.3;

(iv) *X has (umap).*

Then (i)⇔(ii)⇒(iii)⇒(iv).

Proof. Part (i)⇔(ii)⇒(iii) goes as before. To prove (iii)⇒(iv), note that in the proof of Theorem 4.2.4(iii)⇒(iv), one may replace the estimate in (17) by

$$\forall x\in B_X\quad \sum\|U_jx\|\le r^{1-1/q}\Big(\sum\|U_jx\|^q\Big)^{1/q}\le r^{1-1/q}C_\ell,$$

where C_ℓ is the ℓ_1-(ap) or the ℓ_q-(fdd) constant. ∎

5. The p-power approximation property ℓ_p-(ap)

5.1. *Definition*

DEFINITION 5.1.1. Let X be a separable Banach space.

(i) X has the *p-power approximation property ℓ_p-(ap)* if there are an a. s. $\{T_k\}$ and a constant C such that

$$(18)\qquad C^{-1}\|x\|\le\Big(\sum\|\Delta T_kx\|^p\Big)^{1/p}\le C\|x\|$$

for all $x\in X$. The *ℓ_p-(ap) constant* is the least such C.

(ii) (See [7, §3].) X has the *metric p-power approximation property ℓ_p-(map)* if it has ℓ_p-(ap) with constant $1+\varepsilon$ for any $\varepsilon>0$.

Note that ℓ_p-(ap) implies (uap) and ℓ_p-(map) implies (umap). Note also that in (18), the left inequality is trivial if $p=1$; the right inequality is trivial if $p=\infty$.

Property (ii) is implicit in Kalton's and Werner's [28] investigation of subspaces of L^p that are almost isometric to subspaces of ℓ_p.

The proof of Proposition 4.1.2 can be adapted to yield

PROPOSITION 5.1.2. *Let X be a separable Banach space.*

(i) *If there is an a. s. $\{T_k\}$ such that*

$$(19)\qquad (\|x-T_kx\|^p+\|T_kx\|^p)^{1/p}\xrightarrow[k\to\infty]{}1$$

uniformly on the unit sphere, then X has ℓ_p-(map). The converse holds if $p=1$.

(ii) *X has a metric ℓ_p-(fdd) if and only if there is an (fdd) $\{T_k\}$ such that* (19) *holds.*

We shall say that $\{T_k\}$ *realizes ℓ_p-(map)* if it satisfies (19).

Proof. Let $\{T_k\}$ be an a. s. that satisfies (19) and $\varepsilon>0$. By [25, Lemma 2.4], we may suppose that $T_{k+1}T_k=T_k$. Choose a sequence of $\eta_j>0$ such that $1+\varepsilon_k=\prod_{j\le k}(1+\eta_j)\le1+\varepsilon$ for each k. We may assume by taking a subsequence of the T_k's that for all k and $x\in X$,

$$(20)\qquad (1+\eta_k)^{-1}\|x\|\le(\|x-T_kx\|^p+\|T_kx\|^p)^{1/p}\le(1+\eta_k)\|x\|.$$

We then prove by induction the hypothesis (H_k):

$$\forall x\in X\quad (1+\varepsilon_k)^{-1}\|x\|\le\Big(\|x-T_kx\|^p+\sum_{j=1}^{k}\|\Delta T_jx\|^p\Big)^{1/p}\le(1+\varepsilon_k)\|x\|.$$

- (H_1) is true.
- Suppose (H_{k-1}) is true. Let $x\in X$. Note that

$$x-T_kx=(I-T_k)(x-T_{k-1}x),\qquad \Delta T_kx=T_k(x-T_{k-1}x).$$

By (20), we get

$$(\|x - T_k x\|^p + \|\Delta T_k x\|^p)^{1/p} \le (1+\eta_k)\|x - T_{k-1}x\|.$$

Hence

$$\Big(\|x - T_k x\|^p + \sum_{j=1}^{k} \|\Delta T_j x\|^p\Big)^{1/p}$$
$$\le (1+\eta_k)\Big(\|x - T_{k-1}x\|^p + \sum_{j=1}^{k-1} \|\Delta T_j x\|^p\Big)^{1/p} \le (1+\varepsilon_k)\|x\|$$

by (H_{k-1}).

• We obtain the lower bound in the same way. Thus the induction is complete.

Hence $\{T_k\}$ realizes ℓ_p-(ap) with constant $1+\varepsilon$. As ε is arbitrary, X has ℓ_p-(map).

If X has ℓ_1-(map), then for each $\varepsilon > 0$, there is a sequence $\{S_k\}$ such that

$$\|x\| \le \|x - S_k x\| + \|S_k x\| \le \sum \|\Delta S_k x\| \le (1+\varepsilon)\|x\|$$

for all $x \in X$. By a diagonal argument, this gives an a.s. $\{T_k\}$ satisfying (19).

(iii) If X has a metric ℓ_p-(fdd), then for each $\varepsilon > 0$ there is an (fdd) $\{T_k\}$ such that (18) holds with $C = 1+\varepsilon$. Then, for all $k \ge 1$,

$$(1-\varepsilon)\|T_k x\| \le \Big(\sum_{j=1}^{k} \|\Delta T_j x\|^p\Big)^{1/p} \le (1+\varepsilon)\|T_k x\|,$$

$$(1-\varepsilon)\|x - T_k x\| \le \Big(\sum_{j=k+1}^{\infty} \|\Delta T_j x\|^p\Big)^{1/p} \le (1+\varepsilon)\|x - T_k x\|.$$

Thus

$$((1-\varepsilon)/(1+\varepsilon))\|x\| \le (\|x - T_k x\|^p + \|T_k x\|^p)^{1/p} \le ((1+\varepsilon)/(1-\varepsilon))\|x\|.$$

By a diagonal argument, this gives an (fdd) $\{T_k\}$ satisfying (19). ∎

5.2. *Some consequences of ℓ_p-(ap).* We start with the simple

PROPOSITION 5.2.1. *Let X be a separable Banach space.*

(i) *If X has ℓ_p-(ap) with constant C, then X is C-isomorphic to a subspace of an ℓ_p-sum of finite-dimensional subspaces of X.*

(ii) *If furthermore X is a subspace of L^q, then X is $(C+\varepsilon)$-isomorphic to a subspace of $(\bigoplus \ell_q^n)_p$ for any given $\varepsilon > 0$.*

(iii) *In particular, if a subspace of L^p has ℓ_p-(ap) with constant C, then it is $(C+\varepsilon)$-isomorphic to a subspace of ℓ_p for any given $\varepsilon > 0$. If a subspace of L^p has ℓ_p-(map), then it is almost isometric to a subspace of ℓ_p.*

Proof. (i) Indeed, $\Phi : X \hookrightarrow (\bigoplus \operatorname{im} \Delta T_i)_p$, $x \mapsto \{\Delta T_i x\}_{i\ge1}$, is an embedding: for all $x \in X$,

$$C^{-1}\|x\|_X \le \|\Phi x\| = \Big(\sum \|\Delta T_i x\|_X^p\Big)^{1/p} \le C\|x\|_X.$$

(ii & iii) Recall that, given $\varepsilon > 0$, a finite-dimensional subspace of L^q is $(1+\varepsilon)$-isomorphic to a subspace of ℓ_q^n for some $n \ge 1$. ∎

We have in particular (see [24, §VIII, Def. 7] for the definition of Hilbert sets)

COROLLARY 5.2.2. *Let $E \subseteq \mathbb{Z}$ be infinite.*

(i) *No $L^q_E(\mathbb{T})$ $(1 \le q < \infty)$ has ℓ_p-(ap) for $p \ne 2$.*

(ii) *No $\mathcal{C}_E(\mathbb{T})$ has ℓ_q-(ap) for $q \ne 1$. If E is a Hilbert set, then $\mathcal{C}_E(\mathbb{T})$ fails ℓ_1-(ap).*

Proof. This is a consequence of Proposition 5.2.1(i): every infinite E contains a Sidon set and thus a $\Lambda(2 \vee p)$ set. So $L^p_E(\mathbb{T})$ contains ℓ_2. Also, if E is a Hilbert set, then $\mathcal{C}_E(\mathbb{T})$ contains c_0 by [30, Th. 2]. ∎

However, there is a Hilbert set E such that $\mathcal{C}_E(\mathbb{T})$ has complex (umap): see [31, Th. 10]. The class of sets E such that $\mathcal{C}_E(\mathbb{T})$ has ℓ_1-(ap) contains the Sidon sets and Blei's sup-norm-partitioned sets.

5.3. *A characterization of ℓ_p-(map).* Recall [28, Def. 4.1]:

DEFINITION 5.3.1. Let X be a separable Banach space.

(i) Let τ be a vector space topology on X. X enjoys *property* $(m_p(\tau))$ if for all $x \in X$ and norm bounded sequences $\{y_j\}$ such that $y_j \xrightarrow{\tau} 0$,

$$\big|\|x + y_j\| - (\|x\|^p + \|y_j\|^p)^{1/p}\big| \to 0.$$

(ii) X enjoys *property* $(m_p(T_k))$ for a commuting a.s. $\{T_k\}$ if for all $\varepsilon > 0$ and $n \ge 1$ we may choose $m \ge n$ such that for all $x \in B_X$,

$$\big|\|T_n x + (I - T_m)x\| - (\|T_n x\|^p + \|(I - T_m)x\|^p)^{1/p}\big| \le \varepsilon.$$

Then [28, Th. 4.2] may be read as follows:

THEOREM 5.3.2. *Let $1 \le p < \infty$ and consider the following properties for a separable Banach space X.*

(i) *There are an unconditional commuting a.s. $\{T_k\}$ and a vector space topology τ such that X enjoys $(m_p(\tau))$ and $T_k x \xrightarrow{\tau} x$ uniformly for $x \in B_X$;*

(ii) *X enjoys $(m_p(T_k))$ for an unconditional commuting a.s. $\{T_k\}$;*

(iii) *X has ℓ_p-(map).*

Then (i)⇔(ii). *If X has finite cotype, then* (ii)⇒(iii).

As for Theorem 4.2.4, we may remove the cotype assumption if X has commuting ℓ_1-(ap) or ℓ_p-(fdd), $p < \infty$:

THEOREM 5.3.3. *Let $1 \le p < \infty$. Consider the following properties for a separable Banach space X.*

(i) *There are an ℓ_p-(fdd) (or just a commuting ℓ_1-a. s. in the case $p = 1$) $\{T_k\}$ and a vector space topology τ such that X enjoys $(m_p(\tau))$ and $T_k x \xrightarrow{\tau} x$ uniformly for $x \in B_X$;*

(ii) *X enjoys $(m_p(T_k))$ for an ℓ_p-(fdd) (or just a commuting ℓ_1-a. s. in the case $p = 1$) $\{T_k\}$;*

(iii) *X has ℓ_p-(map).*

Then (i)⇔(ii)⇒(iii).

5.4. *Subspaces of L^p with ℓ_p-(map).* Although no translation invariant subspace of $L^p(\mathbb{T})$ has ℓ_p-(map) for $p \ne 2$, Proposition 5.2.1(iii) is not void. By the work of Godefroy, Kalton, Li and Werner [28], [18], we get examples of subspaces of L^p with ℓ_p-(map) and even a characterization of such spaces.

Let us treat the case $p = 1$. Recall first that a space X has the 1-strong Schur property when, given $\delta \in]0, 2]$ and $\varepsilon > 0$, any normalized δ-separated sequence in X contains a subsequence that is $(2/\delta+\varepsilon)$-equivalent to the unit vector basis of ℓ_1 (see [51]). In particular, a gliding hump argument shows that any subspace of ℓ_1 shares this property. By Proposition 5.2.1(iii), a space X with ℓ_1-(map) also does. Now recall the main theorem of [18]:

THEOREM. *Let X be a subspace of L^1 with the approximation property. Then the following properties are equivalent:*

(i) *The unit ball of X is compact and locally convex in measure;*

(ii) *X has (umap) and the 1-strong Schur property;*

(iii) *X is $(1 + \varepsilon)$-isomorphic to a w^*-closed subspace X_ε of ℓ_1 for any $\varepsilon > 0$.*

We may then add to these the fourth equivalent property

(iv) *X has ℓ_1-(map).*

Proof. We just showed that (ii) holds when X has ℓ_1-(map). Now suppose we have (iii) and let $\varepsilon > 0$. Thus there is a quotient Z of c_0 such that Z^* has the approximation property and Z^* is $(1+\varepsilon)$-isomorphic to X.

Let us show that any such Z^* has ℓ_1-(map). Z has the metric approximation property, with say $\{R_n\}$, because Z^* has it as a dual separable space. By [20, Th. 2.2], $\{R_n^*\}$ is a metric a. s. in Z^*. Let Q be the canonical quotient map from c_0 onto Z. Let $\{P_n\}$ be the sequence of projections associated with the natural basis of c_0. Then $\{P_n^*\}$ is also an a. s. in ℓ_1. Thus

$$\|P_n^* Q^* x^* - Q^* R_n^* x^*\|_{\ell_1} \to 0 \quad \text{for any } x^* \in Z^*.$$

By Lebesgue's dominated convergence theorem (see [27, Th. 1]), $QP_n - R_n Q \to 0$ weakly in the space $\mathcal{K}(c_0, Z)$ of compact operators from c_0 to Z.

By Mazur's theorem, there are convex combinations $\{C_n\}$ of $\{P_n\}$ and $\{D_n\}$ of $\{R_n\}$ such that $\|QC_n - D_n Q\|_{\mathcal{L}(c_0,Z)} \to 0$. Thus

$$\|C_n^* Q^* - Q^* D_n^*\|_{\mathcal{L}(Z^*,\ell_1)} \to 0. \tag{21}$$

Furthermore, $C_n^* : \ell_1 \to \ell_1$ has the form $C_n^*(x_1, x_2, \ldots) = (t_1 x_1, t_2 x_2, \ldots)$ with $0 \le t_i \le 1$. Therefore, defining $Q^* a = (a_1, a_2, \ldots)$, we get

$$\begin{aligned} \|C_n^* Q^* a\|_1 + \|Q^* a - C_n^* Q^* a\|_1 &= \|(t_1 a_1, t_2 a_2, \ldots)\|_1 + \|((1-t_1)a_1, (1-t_2)a_2, \ldots)\|_1 \\ &= \sum (|t_i| + |1-t_i|)|a_i| = \sum |a_i| = \|Q^* a\|_1. \end{aligned} \tag{22}$$

As $\{D_n^*\}$ is still an a. s. for Z^*, $\{D_n^*\}$ realizes ℓ_1-(map) in Z^* by (22), (21) and Proposition 5.1.2(i).

Thus X has ℓ_1-(ap) with constant $1 + 2\varepsilon$. As ε is arbitrary, X has ℓ_1-(map). ∎

For $1 < p < \infty$, we have similarly by [28, Th. 4.2]

PROPOSITION 5.4.1. *Let $1 < p < \infty$ and X be a subspace of L^p with the approximation property. The following are equivalent:*

(i) *X is $(1 + \varepsilon)$-isomorphic to a subspace X_ε of ℓ_p for any $\varepsilon > 0$;*

(ii) *X has ℓ_p-(map).*

Proof. (ii)⇒(i) is in Proposition 5.2.1. For (i)⇒(ii), it suffices to prove that any subspace Z of ℓ_p with the approximation property has ℓ_p-(map).

As Z is reflexive, Z admits a commuting shrinking a. s. $\{R_n\}$. Let i be the injection of Z into ℓ_p. Let $\{P_n\}$ be the sequence of projections associated with the natural basis of ℓ_p. It is also an a. s. for $\ell_{p'}$. Thus

$$\|i^* P_n^* x^* - R_n^* i^* x^*\|_{Z^*} \to 0 \quad \text{for any } x^* \in \ell_{p'}.$$

As before, there are convex combinations $\{C_n\}$ of $\{P_n\}$ and $\{D_n\}$ of $\{R_n\}$ such that $\|C_n i - i D_n\| \to 0$. The convex combinations are finite and may be chosen not to overlap, so that for each $n \ge 1$ there is $m > n$ such that

$$\|C_n x + (I - C_m)x\| = (\|C_n x\|^p + \|(I - C_m)x\|^p)^{1/p}$$

for $x \in \ell_p$. Thus Z has property $(m_p(D_n))$. Following the lines of [14, Lemma 1], we observe that $\{D_n\}$ is a commuting unconditional a. s. since $\{P_n\}$ is. By Theorem 5.3.2, Z has ℓ_p-(map). ∎

6. (uap) and (umap) in translation invariant subspaces.

Recall that $\mathbb{U}$ is a subgroup of $\mathbb{T}$. If $\mathbb{U} = \mathbb{D}$, the following applies to real (umap). If $\mathbb{U} = \mathbb{T}$, it applies to complex (umap).

6.1. *Remarks on (uap).* $L^p(\mathbb{T})$ spaces $(1 < p < \infty)$ are known to have an unconditional basis; furthermore, they have an unconditional (fdd) in translation invariant subspaces $L^p_{\Lambda_k}(\mathbb{T})$: this is a corollary of the Littlewood–Paley theory [33]. One may choose $\Lambda_0 = \{0\}$ and $\Lambda_k =]-2^k, -2^{k-1}] \cup [2^{k-1}, 2^k[$. Thus any $L^p_E(\mathbb{T})$ $(1 < p < \infty)$ has an unconditional (fdd) in translation invariant subspaces $L^p_{E\cap\Lambda_k}(\mathbb{T})$. The spaces $L^1(\mathbb{T})$ and $\mathcal{C}(\mathbb{T})$, however, do not even have (uap).

Li [31, Cor. 6, Th. 7] proves that in translation invariant subspaces, (umap) may as well be achieved with multipliers of finite rank. Modifications of his proof apply to (uap) and ℓ_p-(ap), ℓ_p-(map). Hence his result on (umap) may be generalized as follows:

PROPOSITION 6.1.1. *Let $E \subseteq \mathbb{Z}$ and $X \in \{\mathcal{C}(\mathbb{T}), L^p(\mathbb{T})\,(1 \le p < \infty)\}$.*

(i) *If X_E has (umap) (resp. (uap), ℓ_p-(ap) or ℓ_p-(map)) and $F \subseteq E$, then X_F also has (umap) (resp. (uap), ℓ_p-(ap) or ℓ_p-(map)).*

(ii) *If $\mathcal{C}_E(\mathbb{T})$ has (umap), then so do all $L^p_E(\mathbb{T})$ $(1 \le p < \infty)$. If $\mathcal{C}_E(\mathbb{T})$ has (uap), then so does $L^1_E(\mathbb{T})$.*

Note also that a. s. of multipliers commute and commute with one another.

Whereas (uap) is always satisfied for $L^p_E(\mathbb{T})$ $(1 < p < \infty)$, we have the following generalization of [31, remark after Th. 7, Prop. 9] for $L^1_E(\mathbb{T})$ and $\mathcal{C}_E(\mathbb{T})$:

LEMMA 6.1.2. *If X has (uap) with a commuting a. s. and $X \not\supseteq c_0$, then X is a dual space.*

Proof. Suppose $\{T_n\}$ is a commuting a. s. such that (12) holds. As $X \not\supseteq c_0$, $Px^{**} = \lim T_n^{**}x^{**}$ is well defined for each $x^{**} \in X^{**}$. As $\{T_n\}$ is an a. s., P is a projection onto X. Let us show that $\ker P$ is w^*-closed. Indeed, if $x^{**} \in \ker P$, then

$$\|T_n^{**}x^{**}\| = \lim_m \|T_m T_n^{**}x^{**}\| = \lim_m \|T_n T_m^{**}x^{**}\| = 0$$

and $T_n^{**}x^{**} = 0$. Thus

$$\ker P = \bigcap_n \ker T_n^{**}.$$

Let $M = (\ker P)_\perp$. Then $M^* = X$. ■

COROLLARY 6.1.3. *Let $E \subseteq \mathbb{Z}$.*

(i) *If $L^1_E(\mathbb{T})$ has (uap), then E is a Riesz set.*

(ii) *If $\mathcal{C}_E(\mathbb{T})$ has (uap) and $\mathcal{C}_E(\mathbb{T}) \not\supseteq c_0$, then E is a Rosenthal set.*

Proof. In both cases, Lemma 6.1.2 shows that the two spaces are separable dual spaces and thus have the Radon–Nikodym property. We may now apply Lust-Piquard's characterization [35]. ■

The converse of Proposition 6.1.1(ii) does not hold: $L^1_E(\mathbb{T})$ may have (uap) while $\mathcal{C}_E(\mathbb{T})$ fails this property. We have

PROPOSITION 6.1.4. *Let $E \subseteq \mathbb{Z}$.*

(i) *The Hardy space $H^1(\mathbb{T}) = L^1_{\mathbb{N}}(\mathbb{T})$ has (uap).*

(ii) *The disc algebra $A(\mathbb{T}) = \mathcal{C}_{\mathbb{N}}(\mathbb{T})$ fails (uap). More generally, if $\mathbb{Z} \setminus E$ is a Riesz set, then $\mathcal{C}_E(\mathbb{T})$ fails (uap).*

Proof. (i) Indeed, $H^1(\mathbb{T})$ has an unconditional basis [36]. Note that the first unconditional a. s. for $H^1(\mathbb{T})$ appears in [37, §II, introduction].

(ii) Let $\Delta \subset \mathbb{T}$ be the Cantor set. By Bishop's improvement [4] of Rudin–Carleson's interpolation theorem, every function in $\mathcal{C}(\Delta)$ extends to a function in $\mathcal{C}_E(\mathbb{T})$ if $\mathbb{Z} \setminus E$ is a Riesz set. By [43, main theorem], this implies that $\mathcal{C}(\Delta)$ embeds in $\mathcal{C}_E(\mathbb{T})$. Therefore $\mathcal{C}_E(\mathbb{T})$ cannot have (uap); otherwise $\mathcal{C}(\Delta)$ would embed in a space with an unconditional basis, which is false. ■

6.2. *Characterization of (umap).* Let us introduce

DEFINITION 6.2.1. Let $E \subseteq \mathbb{Z}$ and $X \in \{\mathcal{C}(\mathbb{T}), L^p(\mathbb{T})\,(1 \le p < \infty)\}$.

E enjoys the *Fourier block unconditionality property* $(\mathcal{U})$ in X whenever, for any $\varepsilon > 0$ and finite $F \subseteq E$, there is a finite $G \subseteq E$ such that for $f \in B_{X_F}$ and $g \in B_{X_{E\setminus G}}$,

$$\operatorname*{osc}_{\lambda\in\mathbb{U}} \|\lambda f + g\|_X \le \varepsilon. \tag{23}$$

LEMMA 6.2.2. *Let $E \subseteq \mathbb{Z}$ and $X \in \{\mathcal{C}(\mathbb{T}), L^p(\mathbb{T})\,(1 \le p < \infty)\}$. The following are equivalent.*

(i) *X_E has $(u(\tau_f))$, where τ_f is the topology of pointwise convergence of the Fourier coefficients*:

$$x_n \xrightarrow{\tau_f} 0 \Leftrightarrow \forall k\ \widehat{x}_n(k) \to 0;$$

(ii) *E enjoys $(\mathcal{U})$ in X;*

(iii) *X_E enjoys the property of block unconditionality for any, or equivalently for some, a. s. $\{T_k\}$ of multipliers.*

Proof. (i)⇒(ii). Suppose that (ii) fails: there are $\varepsilon > 0$ and a finite F such that for each finite G, there are $x_G \in B_{X_F}$ and $y_G \in B_{X_{E\setminus G}}$ such that

$$\operatorname*{osc}_{\lambda\in\mathbb{U}} \|\lambda x_G + y_G\| > \varepsilon.$$

As B_{X_F} is compact, we may suppose $x_G = x$. As $y_G \xrightarrow{\tau_f} 0$, $(u(\tau_f))$ fails.

(ii)⇒(iii). Let C be a uniform bound for $\|T_k\|$. Let $n \ge 1$ and $\varepsilon > 0$. Let F be the finite spectrum of T_n. Let G be such that (23) holds for all $f \in B_{X_F}$ and $g \in B_{X_{E\setminus G}}$. Let V be the element of de la Vallée-Poussin's a. s. such that $V|_{X_G} = I|_{X_G}$. Then $\|V\|_{\mathcal{L}(X_E)} \le 3$. As V has finite rank, we may choose $m > n$ such that $\|(I - T_m)V\|_{\mathcal{L}(X_E)} = \|V(I - T_m)\|_{\mathcal{L}(X_E)} \le \varepsilon$. Let then $x \in T_n B_{X_E}$ and $y \in (I - T_m) B_{X_E}$. We have

$$\begin{aligned}\|\lambda x + y\| &\le \|\lambda x + (I - V)y\| + \varepsilon \overset{(23)}{\le} \|x + (I - V)y\| + 4(C+1)\varepsilon + \varepsilon \\ &\le \|x + y\| + (4C + 6)\varepsilon.\end{aligned}$$

(iii)⇒(i) is proved as Lemma 4.2.2(ii)⇒(i): just note that if $y_j \xrightarrow{\tau_f} 0$, then $\|Ty_j\| \to 0$ for any finite rank multiplier T. ∎

THEOREM 6.2.3. *Let $E \subseteq \mathbb{Z}$ and $X \in \{\mathcal{C}(\mathbb{T}), L^p(\mathbb{T})\ (1 \le p < \infty)\}$. If X_E has (umap), then E enjoys $(\mathcal{U})$ in X. Conversely,*

(i) *If E enjoys $(\mathcal{U})$ in $L^p_E(\mathbb{T})$ $(1 < p < \infty)$, then $L^p_E(\mathbb{T})$ has (umap).*

(ii) *If E enjoys $(\mathcal{U})$ in $L^1_E(\mathbb{T})$ and $L^1_E(\mathbb{T})$ has (uap), then $L^1_E(\mathbb{T})$ has (umap).*

(iii) *If E enjoys $(\mathcal{U})$ in $\mathcal{C}_E(\mathbb{T})$ and $\mathcal{C}_E(\mathbb{T})$ has ℓ_1-(ap), in particular if E is a Sidon set, then $\mathcal{C}_E(\mathbb{T})$ has (umap).*

Proof. Notice first that in the three cases, (umap) implies $(\mathcal{U})$ by Lemma 6.2.2(iii)⇒(ii).

(i) Notice that $L^p_E(\mathbb{T})$ $(1 < p < \infty)$ has an unconditional (fdd) $\{\pi_{E \cap \Lambda_k}\}$ of multipliers. Thus $(\mathcal{U})$ implies (umap) by Theorem 4.2.5(ii)⇒(iv).

By Lemma 6.2.2, parts (ii) and (iii) follow from Theorem 4.2.4(ii)⇒(iv) and Theorem 4.2.5(ii)⇒(iv) respectively. ∎

7. Property (umap) and arithmetical block independence. We may now apply the technique used in the investigation of (umbs) in order to obtain arithmetical conditions analogous to $(\mathcal{I}_n)$ (see Def. 2.8) for (umap). According to Theorem 6.2.3, it suffices to investigate the property $(\mathcal{U})$ of block unconditionality: we have to compute an expression of type $\|f + \lambda g\|_p$, where the spectra of f and g are far apart and $\lambda \in \mathbb{U}$. As before, $\mathbb{U} = \mathbb{T}$ (resp. $\mathbb{U} = \mathbb{D}$) is the complex (resp. real) choice of signs. To this end, we return to the notation of computational lemmas 2.6 and 2.7. Define

$$\begin{aligned}\Psi_r(\lambda, z) &= \Theta_r((\overbrace{1, \ldots, 1}^{j}, \overbrace{\lambda, \ldots, \lambda}^{m-j}), z) \\ &= \int \Big| e_{r_0}(t) + \sum_{i=1}^{j} z_i e_{r_i}(t) + \lambda \sum_{i=j+1}^{m} z_i e_{r_i}(t) \Big|^p dm(t) \\ &= \sum_{\alpha \in \mathbb{N}^m} \binom{p/2}{\alpha}^2 \prod |z_i|^{2\alpha_i} \\ &\quad + \sum_{A \in \mathcal{R}_q} \sum_{\alpha \ne \beta \in A} \binom{p/2}{\alpha} \binom{p/2}{\beta} \lambda^{\sum_{i>j} \alpha_i - \beta_i} \prod z_i^{\alpha_i} \overline{z}_i^{\beta_i}.\end{aligned} \tag{24}$$

As in computational lemma 2.7, we make the following observation:

COMPUTATIONAL LEMMA 7.1. *Let $\xi_0, \ldots, \xi_m \in \mathbb{Z}^*$ and γ, δ be as in (6). If the arithmetic relation (7) holds, then the coefficient of the term $\lambda^{\sum_{i>j} \gamma_i - \delta_i} \prod z_i^{\gamma_i} \overline{z}_i^{\delta_i}$ in (24) is $\binom{p/2}{\gamma}\binom{p/2}{\delta}$ and thus independent of r. If $\sum |\xi_i| \le p$ or p is not an even integer, this coefficient is nonzero. If $\xi_0 + \ldots + \xi_j$ is nonzero* (resp. *odd*), *then this term is nonconstant in $\lambda \in \mathbb{U}$.*

Thus the following arithmetical property shows up. It is somewhat similar to property $(\mathcal{I}_n)$ of almost independence.

DEFINITION 7.2. Let $E \subseteq \mathbb{Z}$ and $n \ge 1$.

(i) E enjoys the *complex* (resp. *real*) *property* $(\mathcal{J}_n)$ of *block independence* if for any $\xi \in \Xi^m_n$ with $\xi_1 + \ldots + \xi_j$ nonzero (resp. odd) and given $p_1, \ldots, p_j \in E$, there is a finite $F \subseteq E$ such that $\xi_1 p_1 + \ldots + \xi_m p_m \ne 0$ for all $p_{j+1}, \ldots, p_m \in E \setminus F$.

(ii) E enjoys *complex* (resp. *real*) $(\mathcal{J}_\infty)$ if it enjoys complex (resp. real) $(\mathcal{J}_n)$ for all $n \ge 1$.

Thus property $(\mathcal{J}_n)$ has, unlike $(\mathcal{I}_n)$, a complex and a real version. Real $(\mathcal{J}_n)$ is strictly weaker than complex $(\mathcal{J}_n)$: see Section 8. Notice that $(\mathcal{J}_1)$ is void and $(\mathcal{J}_{n+1})$⇒$(\mathcal{J}_n)$ in both complex and real cases. Also $(\mathcal{I}_n) \not\Rightarrow (\mathcal{J}_n)$: we shall see in the following section that $E = \{0\} \cup \{n^k\}_{k \ge 0}$ provides a counterexample. The property $(\mathcal{J}_2)$ of real block independence appears implicitly in [31, Lemma 12].

REMARK. In spite of the intricate form of this arithmetical property, $(\mathcal{J}_n)$ is the "simplest" candidate, in some sense, that reflects the features of $(\mathcal{U})$:

- it must hold for a set E if and only if it holds for a translate $E + k$; this explains $\sum \xi_i = 0$ in Definition 7.2(i);
- as for the property $(\mathcal{U})$ of block independence, it must connect the beginning of E with its tail;
- Li gives an example of a set E whose pace does not tend to infinity while $\mathcal{C}_E(\mathbb{T})$ has ℓ_1-(map). Thus no property $(\mathcal{J}_n)$ should forbid parallelogram relations of the type $p_2 - p_1 = p_4 - p_3$, where p_1, p_2 are in the beginning of E and p_3, p_4 in its tail. This explains the condition that $\xi_1 + \ldots + \xi_j$ be nonzero (resp. odd) in Definition 7.2(i).

We now repeat the argument of Theorem 2.9 to obtain an analogous statement which relates the property $(\mathcal{U})$ of Definition 6.2.1 with our new arithmetical conditions:

LEMMA 7.3. *Let $E = \{n_k\} \subseteq \mathbb{Z}$ and $1 \le p < \infty$.*

(i) *Suppose p is an even integer. Then E enjoys the complex* (*resp. real*) *Fourier block unconditionality property $(\mathcal{U})$ in $L^p(\mathbb{T})$ if and only if E enjoys complex* (*resp. real*) $(\mathcal{J}_{p/2})$.

(ii) *If p is not an even integer and E enjoys complex (resp. real) $(\mathcal{U})$ in $L^p(\mathbb{T})$, then it enjoys complex (resp. real) $(\mathcal{J}_\infty)$.*

Proof. Let us first prove the necessity of the arithmetical property and assume E fails $(\mathcal{J}_n)$; then there are $\xi_0,\ldots,\xi_m \in \mathbb{Z}^*$ with $\sum \xi_i = 0$, $\sum|\xi_i| \le 2n$ and $\xi_0+\ldots+\xi_j$ nonzero (resp. odd); and there are $r_0,\ldots,r_j \in E$ and sequences $r^l_{j+1},\ldots,r^l_m \in E\setminus\{n_1,\ldots,n_l\}$ such that

$$\xi_0 r_0 + \ldots + \xi_j r_j + \xi_{j+1} r^l_{j+1} + \ldots + \xi_m r^l_m = 0.$$

Assume E enjoys $(\mathcal{U})$ in $L^p(\mathbb{T})$. Then the oscillation of Ψ_r in (24) satisfies

$$\operatorname*{osc}_{\lambda\in\mathbb{U}} \Psi_{r^l}(\lambda,z) \xrightarrow[l\to\infty]{} 0 \tag{25}$$

for each $z \in D^m$. The argument is now exactly the same as in Theorem 2.9: we may assume that Ψ_{r^l} converges in $\mathcal{C}^\infty(\mathbb{U}\times D^m)$ to a function Ψ. Then by (25), $\Psi(\lambda,z)$ is constant in λ for each $z\in D^m$, and this is impossible by computational lemma 7.1 if either p is not an even integer or $p \ge 2n$.

Let us now prove the sufficiency of $(\mathcal{J}_{p/2})$ when p is an even integer. First, let $A^{k,l}_n = \{\alpha \in A_n : \alpha_i = 0 \text{ for } k < i \le l\}$ (A_n is defined before Proposition 2.5), and convince yourself that $(\mathcal{J}_{p/2})$ is equivalent to

$$\forall k\ \exists l \ge k\ \forall \alpha,\beta \in A^{k,l}_{p/2} \tag{26}$$

$$\sum \alpha_i n_i = \sum \beta_i n_i \Rightarrow \sum_{i\le k} \alpha_i = \sum_{i\le k}\beta_i \text{ (resp. mod 2)}.$$

Let $f = \sum a_i e_{n_i} \in \mathcal{P}_E(\mathbb{T})$. Let $k \ge 1$ and $\lambda \in \mathbb{U}$. By the multinomial formula,

$$\|\lambda\pi_k f + (f-\pi_l f)\|^p_p = \int\Big|\sum_{\alpha\in A^{k,l}_{p/2}} \binom{p/2}{\alpha} \lambda^{\sum_{i\le k}\alpha_i}\Big(\prod a_i^{\alpha_i}\Big) e_{\sum\alpha_i n_i}\Big|^2 dm$$

$$= \int\Big|\sum_{j=0}^{n}\lambda^j \sum_{\substack{\alpha\in A^{k,l}_{p/2}\\ \alpha_1+\ldots+\alpha_k=j}} \binom{p/2}{\alpha}\Big(\prod a_i^{\alpha_i}\Big) e_{\sum\alpha_i n_i}\Big|^2 dm.$$

(26) now signifies that we may choose $l \ge k$ such that the terms of the above sum over j (resp. the terms with j odd and those with j even) have disjoint spectrum. But then $\|\lambda\pi_k f + (f-\pi_l f)\|_p$ is constant for $\lambda\in\mathbb{U}$ and E enjoys $(\mathcal{U})$ in $L^p(\mathbb{T})$. ■

Note that for even p, we have as in Proposition 2.5 a constant $C_p > 1$ such that either (23) holds for $\varepsilon = 0$ or fails for any $\varepsilon \le C_p$. We thus get

COROLLARY 7.4. *Let $E\subseteq\mathbb{Z}$ and p be an even integer. If E enjoys complex (resp. real) $(\mathcal{U})$ in $L^p(\mathbb{T})$, then there is a partition $E = \bigcup E_k$ into finite sets such that for any coarser partition $E = \bigcup E'_k$,*

$$\forall f \in \mathcal{P}_E(\mathbb{T}) \qquad \operatorname*{osc}_{\lambda_k\in\mathbb{U}} \Big\|\sum \lambda_k \pi_{E'_{2k}} f\Big\|_p = 0.$$

Among other things, $E = E_1 \cup E_2$ where the $L^p_{E_i}(\mathbb{T})$ have a complex (resp. real) 1-unconditional (fdd).

Lemma 7.3 and Theorem 6.2.3 yield the main result of this section.

THEOREM 7.5. *Let $E\subseteq\mathbb{Z}$ and $1\le p<\infty$.*

(i) *Suppose p is an even integer. Then $L^p_E(\mathbb{T})$ has complex (resp. real) (umap) if and only if E enjoys complex (resp. real) $(\mathcal{J}_{p/2})$.*

(ii) *If p is not an even integer and $L^p_E(\mathbb{T})$ has complex (resp. real) (umap), then E enjoys complex (resp. real) $(\mathcal{J}_\infty)$.*

COROLLARY 7.6. *Let $E\subseteq\mathbb{Z}$.*

(i) *If $\mathcal{C}_E(\mathbb{T})$ has complex (resp. real) (umap), then E enjoys complex (resp. real) $(\mathcal{J}_\infty)$.*

(ii) *If any $L^p_E(\mathbb{T})$, p not an even integer, has complex (resp. real) (umap), then all $L^p_E(\mathbb{T})$ with p an even integer have complex (resp. real) (umap).*

Suppose p is an even integer. Then Section 8 gives various examples of sets such that $L^p_E(\mathbb{T})$ has complex or real (umap). Proposition 9.2 gives a general growth condition that ensures (umap).

For $X = L^p(\mathbb{T})$, p not an even integer, and $X = \mathcal{C}(\mathbb{T})$, however, we encounter the same obstacle as for (umbs). Section 8 only gives sets E such that X_E fails (umap). Thus, we have to prove this property by direct means. This yields four types of examples of sets E such that $\mathcal{C}_E(\mathbb{T})$—and thus by [31, Th. 7] all $L^p_E(\mathbb{T})$ $(1\le p<\infty)$ as well—have (umap):

- Sets found by Li [31]: Kronecker's theorem is used to construct a set containing arbitrarily long arithmetic sequences and a set whose pace does not tend to infinity. Meyer's [38, VIII] techniques are used to construct a Hilbert set;
- The sets that satisfy the growth condition of Theorem 9.3;
- Sequences $E = \{n_k\}\subseteq\mathbb{Z}$ such that n_{k+1}/n_k is an odd integer: see Proposition 9.1.

We know no example of a set E such that some $L^p_E(\mathbb{T})$, p not an even integer, has (umap) while $\mathcal{C}_E(\mathbb{T})$ fails it.

There is also a good arithmetical description of the case where $\{\pi_k\}$ or a subsequence thereof realizes (umap).

PROPOSITION 7.7. *Let $E = \{n_k\}\subseteq\mathbb{Z}$ and $X \in \{\mathcal{C}(\mathbb{T}), L^p(\mathbb{T})\ (1\le p<\infty)\}$. Consider an increasing sequence $\{E_k\}$ of finite sets such that $E = \bigcup E_k$.*

(i) *Suppose p is an even integer. Then $\{\pi_{E_k}\}$ realizes complex (resp. real) (umap) in $L^p_E(\mathbb{T})$ if and only if there is an $l \ge 1$ such that*

$$(27)\qquad \forall p_1,\ldots,p_m \in E \qquad \xi_1p_1+\ldots+\xi_mp_m = 0 \Rightarrow \forall k \ge l \sum_{p_j\in E_k} \xi_j = 0 \ (\textit{resp. is even})$$

for all $\xi \in \Xi^m_{p/2}$. Then $L^p_E(\mathbb{T})$ admits the 1-unconditional (fdd) $\{\pi_{E_k}\}_{k\ge l}$. In particular, if we choose $E_k = \{n_1,\ldots,n_k\}$, then $\pi_{E_k} = \pi_k$ realizes complex and real (umap) if and only if there is a finite F such that for $\xi \in \Xi^m_{p/2}$,

$$(28)\qquad \forall p_1,\ldots,p_m \in E \quad \xi_1p_1+\ldots+\xi_mp_m = 0 \Rightarrow p_1,\ldots,p_m \in F.$$

(ii) *Suppose p is not an even integer. If $\{\pi_{E_k}\}$ realizes complex (resp. real) (umap), then for each $\xi \in \Xi^m$ there is a $k \ge 1$ such that (27) holds. In particular, if $\{\pi_k\}$ realizes either complex or real (umap), then for all $\xi \in \Xi^m$ there is a finite F such that (28) holds.*

Proof. It is analogous to the proof of Lemma 7.3: suppose we have $\xi \in \Xi^m_n$ such that (27) fails for any $l \ge 1$. Then there are $\xi_0,\ldots,\xi_m \in \mathbb{Z}^*$ with $\sum \xi_i = 0$, $\sum|\xi_i| \le 2n$ and $\xi_0+\ldots+\xi_j$ nonzero (resp. odd) for some j; for each l, there are $r^l_0,\ldots,r^l_j \in E_l$ and $r^l_{j+1},\ldots,r^l_m \in E\setminus E_l$ such that $\xi_0r^l_0+\ldots+\xi_mr^l_m = 0$.

But then $\{\pi_{E_k}\}$ cannot realize complex (resp. real) (umap): the function Ψ_r in (24) would satisfy (25) and we would obtain a contradiction as in Theorem 2.9.

Sufficiency in (i) is proved exactly as in Lemma 7.3(i). ∎

In particular, suppose that the cardinality $\#[E_k]$ is uniformly bounded by M and $\{\pi_{E_k}\}$ realizes (umap) in $L^p_E(\mathbb{T})$. If $p \ne 2$ is an even integer, then E is a $\Lambda(p)$ set, being the union of a finite set and M $p/2$-independent sets (see Prop. 2.5 and [52, Th. 4.5(b)]). If p is not an even integer, then E is a $\Lambda(q)$ set for any q by the same argument.

Let p be an even integer. If (27) holds, then a tail of $\{U_k\}$ realizes 1-(fdd) in $L^p_E(\mathbb{T})$. Similarly, if (28) holds, then E is a (umbs) in $L^p(\mathbb{T})$.

8. Examples for (umap). The pairing $\langle\xi,E\rangle$ underlines the asymptotic nature of property $(\mathcal{J}_n)$. It has been defined before Proposition 3.1, whose proof can be adapted to give

PROPOSITION 8.1. *Let $E = \{n_k\} \subseteq \mathbb{Z}$.*

(i) *If $\langle\xi,E\rangle < \infty$ for $\xi_1,\ldots,\xi_m \in \mathbb{Z}^*$ with $\sum\xi_i$ nonzero (resp. odd), then E fails complex (resp. real) $(\mathcal{J}_{|\xi_1|+\ldots+|\xi_m|})$. Conversely, if E fails complex (resp. real) $(\mathcal{J}_n)$, then there are $\xi_1,\ldots,\xi_m \in \mathbb{Z}^*$ with $\sum\xi_i$ nonzero (resp. odd) and $\sum|\xi_i| \le 2n-1$ such that $\langle\xi,E\rangle < \infty$.*

(ii) *Thus E enjoys complex (resp. real) $(\mathcal{J}_\infty)$ if and only if $\langle\xi,E\rangle = \infty$ for all $\xi_1,\ldots,\xi_m \in \mathbb{Z}^*$ with $\sum\xi_i$ nonzero (resp. odd).*

Proof of the converse in (i). If E fails complex (resp. real) $(\mathcal{J}_n)$, then there are $\xi \in \Xi^m_n$ with $\xi_1+\ldots+\xi_j$ nonzero (resp. odd), $p_1,\ldots,p_j \in E$ and sequences $p^l_{j+1},\ldots,p^l_m \in \{n_k\}_{k\ge l}$ such that $\sum_{i>j}\xi_ip^l_i = -\sum_{i\le j}\xi_ip_i$. Let $\xi' = (\xi_{j+1},\ldots,\xi_m)$. Then $\sum|\xi'_i| \le 2n-1$ and $\langle\xi',E\rangle < \infty$. ∎

An immediate application is, as in Proposition 3.1,

PROPOSITION 8.2. *Let $E = \{n_k\} \subseteq \mathbb{Z}$.*

(i) *Suppose E enjoys $(\mathcal{I}_{2n-1})$. Then E enjoys complex $(\mathcal{J}_n)$ and actually there is a finite set F such that* (28) *holds for $\xi \in \Xi^m_n$.*

(ii) *Suppose E enjoys $(\mathcal{I}_\infty)$. Then E enjoys complex $(\mathcal{J}_\infty)$ and actually for all $\xi \in \Xi^m$ there is a finite F such that* (28) *holds.*

(iii) *Let $E' = \{n_k+m_k\}$ with $\{m_k\}$ bounded. Then $\langle\xi,E\rangle = \infty$ if and only if $\langle\xi,E'\rangle = \infty$. Thus $(\mathcal{I}_\infty)$ and complex and real $(\mathcal{J}_\infty)$ are stable under bounded perturbations of E.*

(iv) *Suppose there is $h \in \mathbb{Z}$ such that $E\cup\{h\}$ fails complex (resp. real) $(\mathcal{J}_n)$. Then E fails complex (resp. real) $(\mathcal{J}_{2n-1})$. Thus the complex and real properties $(\mathcal{J}_\infty)$ are stable under unions with an element: if E enjoys it, then so does $E\cup\{h\}$.*

(v) *Suppose $jF+s, kF+t \in E$ for an infinite F, $j \ne k \in \mathbb{Z}^*$ and $s,t \in \mathbb{Z}$. Then E fails complex $(\mathcal{J}_{|j|+|k|})$, and also real $(\mathcal{J}_{|j|+|k|})$ if j and k have different parity.*

We now turn to an investigation of various sets E in relation to their arithmetical properties.

Geometric sequences. Let $G = \{j^k\}_{k\ge0}$ with $j \in \mathbb{Z}\setminus\{-1,0,1\}$.

(1) As $G, jG \subseteq G$, G fails complex $(\mathcal{J}_{|j|+1})$, and also real $(\mathcal{J}_{|j|+1})$ if j is even. The solutions (10) to the Diophantine equation (9) show at once that G enjoys complex $(\mathcal{J}_{|j|})$, since there is no arithmetical relation $\xi \in \Xi^m_{|j|}$ between the beginning and the tail of G. If j is odd, then G enjoys in fact real $(\mathcal{J}_\infty)$. Indeed, let $\xi_1,\ldots,\xi_m \in \mathbb{Z}^*$ and $k_1 < \ldots < k_m$; then $\sum\xi_ij^{k_i} \in j^{k_1}\mathbb{Z}$ and either $|\sum\xi_ij^{k_i}| \ge j^{k_1}$ or $\sum\xi_ij^{k_i} = 0$. Thus, if $\langle\xi,E\rangle < \infty$ then $\langle\xi,E\rangle = 0$ and $\sum\xi_i$ is even since j is odd. Now apply Proposition 8.1(iii). The same argument yields that even $G\cup -G\cup\{0\}$ enjoys real $(\mathcal{J}_\infty)$. Actually, more is true: see Proposition 9.1.

(2) $G\cup\{0\}$ may behave differently than G with respect to property $(\mathcal{J}_n)$, thus this property is not stable under unions with an element. Indeed, the first solution in (10) may be written as $(-j+1)\cdot 0 + j\cdot j^k + (-1)\cdot j^{k+1} = 0$. If j is positive, then $(-j+1)+j+(-1) \le 2j$ and $G\cup\{0\}$ fails complex $(\mathcal{J}_j)$. A look at (10) shows that it nevertheless enjoys complex $(\mathcal{J}_{j-1})$. On

the other hand, $G \cup \{0\}$ still enjoys complex $(\mathcal{J}_{|j|})$ if j is negative. In the real setting, our arguments yield the same if j is even, but we already saw that $G \cup \{0\}$ still enjoys real $(\mathcal{J}_\infty)$ if j is odd.

Symmetric sets. By Propositions 3.1(iii) and 8.2(vi), they enjoy neither $(\mathcal{I}_2)$ nor complex $(\mathcal{J}_2)$. They may nevertheless enjoy real $(\mathcal{J}_n)$. Introduce property $(\mathcal{J}_n^{\mathrm{sym}})$ for E: it holds if for all $p_1, \dots, p_j \in E$ and $\eta \in \mathbb{Z}^{*m}$ with $\sum_{i=1}^m \eta_i$ even and $\sum_{i=1}^m |\eta_i| \le 2n$ and $\eta_1 + \dots + \eta_j$ odd, there is a finite set F such that $\eta_1 p_1 + \dots + \eta_m p_m \neq 0$ for any $p_{j+1}, \dots, p_m \in E \setminus F$. Then we obtain

PROPOSITION 8.3. *$E \cup -E$ has real $(\mathcal{J}_n)$ if and only if E has $(\mathcal{J}_n^{\mathrm{sym}})$.*

Proof. By definition, $E \cup -E$ enjoys real $(\mathcal{J}_n)$ if and only if for all $p_1, \dots, p_j \in E$ and $\xi, \zeta \in \mathbb{Z}^m$ with $\xi + \zeta \in \Xi_n^m$ and $\sum_{i \le k}(\xi_i - \zeta_i)$ odd, there is a finite set F such that $\sum(\xi_i - \zeta_i)p_i \neq 0$ for any $p_{j+1}, \dots, p_m \in E \setminus F$—and thus if and only if E enjoys $(\mathcal{J}_n^{\mathrm{sym}})$: just consider the mappings between arithmetical relations $(\xi, \zeta) \mapsto \eta = \xi - \zeta$ and $\eta \mapsto (\xi, \zeta)$ such that $\eta = \xi - \zeta$, where $\xi_i = \eta_i/2$ if η_i is even and, noting that the number of odd η_i's must be even, $\xi_i = (\eta_i - 1)/2$ and $\xi_i = (\eta_i + 1)/2$ respectively for each half of them. ■

Consider again a geometric sequence $G = \{j^k\}$. If j is odd, we saw before that $G \cup -G$ and $G \cup -G \cup \{0\}$ enjoy real $(\mathcal{J}_\infty)$. If j is even, then $G \cup -G$ fails real $(\mathcal{J}_{j+1})$ since G does. $G \cup -G \cup \{0\}$ fails real $(\mathcal{J}_{j/2+1})$ by the arithmetical relation $1 \cdot 0 + j \cdot j^k + (-1) \cdot j^{k+1} = 0$ and Proposition 8.3. $G \cup -G$ enjoys real $(\mathcal{J}_j)$ and $G \cup -G \cup \{0\}$ enjoys real $(\mathcal{J}_{j/2})$ as the solutions in (10) show by a simple checking.

Algebraic and transcendental numbers. The proof of Proposition 3.2 adapts to

PROPOSITION 8.4. *Let $E = \{n_k\} \subseteq \mathbb{Z}$.*

(i) *If $n_{k+1}/n_k \to \sigma$ where $\sigma > 1$ is transcendental, then E enjoys complex $(\mathcal{J}_\infty)$.*

(ii) *Let $n_k = [\sigma^k]$ with $\sigma > 1$ algebraic. Let $P(x) = \xi_0 + \dots + \xi_d x^d$ be the corresponding polynomial of minimal degree. Then E fails complex $(\mathcal{J}_{|\xi_0| + \dots + |\xi_d|})$, and also real $(\mathcal{J}_{|\xi_0| + \dots + |\xi_d|})$ if $P(1)$ is odd.*

Polynomial sequences. Let $E = \{P(k)\}$ for a polynomial P of degree d. The arithmetical relation (11) cannot be adapted to property $(\mathcal{J}_n)$. Notice, though, that $\{\Delta^j P\}_{j=1}^d$ is a basis for the space of polynomials of degree less than d and that $2^d P(k) - P(2k)$ is a polynomial of degree at most $d-1$. Writing it in the basis $\{\Delta^j P\}_{j=1}^d$ yields an arithmetical relation $2^d \cdot P(k) - 1 \cdot P(2k) + \sum_{j=0}^d \xi_j \cdot P(k-j) = 0$ such that $2^d - 1 + \sum \xi_j$ is odd. By Proposition 8.1(ii), E fails real $(\mathcal{J}_n)$ for a certain n. This n may be bounded in certain cases:

- The set of squares fails real $(\mathcal{J}_2)$: let F_n be the Fibonacci sequence defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. As $\{F_{n+1}/F_n\}$ is the sequence of convergents of the continued fraction associated with an irrational (the golden ratio), $F_n \to \infty$ and $F_n F_{n+2} - F_{n+1}^2 = (-1)^n$ (see [13]). Inspired by [41, p. 15], we observe that

$$(F_n F_{n+2} + F_{n+1}^2)^2 + (F_{n+1}^2)^2 = (F_n F_{n+1} + F_{n+1} F_{n+2})^2 + 1^2.$$

- The set of cubes fails real $(\mathcal{J}_2)$: starting from Binet's [3] simplified solution of Euler's equation [11], we observe that $p_n = 9n^4$, $q_n = 1 + 9n^3$, $r_n = 3n(1 + 3n^3)$ satisfy $p_n^3 + q_n^3 = r_n^3 + 1^3$ and tend to infinity.
- The set of biquadrates fails real $(\mathcal{J}_3)$: by an equality of Ramanujan (see [48, p. 386]),

$$(4n^5 - 5n)^4 + (6n^4 - 3)^4 + (4n^4 + 1)^4 = (4n^5 + n)^4 + (2n^4 - 1)^4 + 3^4.$$

As for $(\mathcal{I}_n)$, a positive answer to Euler's conjecture would imply that the set of kth powers has complex $(\mathcal{J}_2)$ for $k \ge 5$.

Conclusion. By Theorem 7.5, property $(\mathcal{J}_n)$ yields directly (umap) in $L^{2p}(\mathbb{T})$, $p \le n$ integer. But we do not know whether $(\mathcal{J}_\infty)$ ensures (umap) in $L^p(\mathbb{T})$, p not an even integer, or $\mathcal{C}(\mathbb{T})$.

Nevertheless, the study of property $(\mathcal{J}_3)$ permits us to determine the density of sets such that X_E enjoys (umap) for some $X \neq L^2(\mathbb{T}), L^4(\mathbb{T})$: see Proposition 10.2. Other applications are given in Section 12.

9. The positive results: parity and a sufficient growth condition.

In the real case, parity plays an unexpected rôle.

PROPOSITION 9.1. *Let $E = \{n_k\} \subseteq \mathbb{Z}$ and suppose that n_{k+1}/n_k is an odd integer for all sufficiently large k. Then $\mathcal{C}_E(\mathbb{T})$ has real (umap).*

Proof. Let us verify that real $(\mathcal{U})$ holds. Let $\varepsilon > 0$ and $F \subseteq E \cap [-n, n]$. Let l, to be chosen later, be such that n_{k+1}/n_k is an odd integer for $k \ge l$. Take $G \supseteq \{n_1, \dots, n_l\}$ finite. Let $f \in B_{\mathcal{C}_F}$ and $g \in B_{\mathcal{C}_{E \setminus G}}$. Then $g(u \exp i\pi/n_l) = -g(u)$ and

$$|f(u \exp i\pi/n_l) - f(u)| \le (\pi/|n_l|)\|f'\|_\infty \le \pi n/|n_l| \le \varepsilon$$

by Bernstein's inequality and for l large enough. Thus, for some $u \in \mathbb{T}$,

$$\begin{aligned}\|f - g\|_\infty = |f(u) + g(u \exp i\pi/n_l)| &\le |f(u \exp i\pi/n_l) + g(u \exp i\pi/n_l)| + \varepsilon \\ &\le \|f + g\|_\infty + \varepsilon.\end{aligned}$$

As E is a Sidon set, we may apply Theorem 6.2.3(iii). ■

Furthermore, if E satisfies the hypothesis of Proposition 9.1, so does $E\cup -E=\{n_1,-n_1,n_2,-n_2,\ldots\}$. But $E\cup -E$ fails even complex $(\mathcal{J}_2)$ and no $X_{E\cup -E}\neq L^2_{E\cup -E}(\mathbb{T})$ has complex (umap). On the other hand, if there is an even integer h such that $n_{k+1}/n_k=h$ infinitely often, then E fails real $(\mathcal{J}_{|h|+1})$ by Proposition 8.2(vi).

For $X=L^p(\mathbb{T})$ with p an even integer, a look at $(\mathcal{I}_n)$ and $(\mathcal{J}_n)$ gives by Theorems 2.9 and 7.5 the following general growth condition:

PROPOSITION 9.2. *Let $E=\{n_k\}\subseteq\mathbb{Z}$ and $p\geq 1$ an integer. If*

$$\liminf|n_{k+1}/n_k|\geq p+1, \tag{29}$$

then $L^{2p}_E(\mathbb{T})$ has complex (umap) and E is a (umbs) in $L^{2p}(\mathbb{T})$.

Proof. Suppose we have an arithmetical relation

$$\xi_1 n_{k_1}+\ldots+\xi_m n_{k_m}=0 \quad \text{with } \xi\in\Xi_p^m \text{ and } |n_{k_1}|<\ldots<|n_{k_m}|. \tag{30}$$

Then $|\xi_m n_{k_m}|\leq|\xi_1 n_{k_1}|+\ldots+|\xi_{m-1}n_{k_{m-1}}|$. The left hand side is smallest when $|\xi_m|=1$. As $|\xi_1|+\ldots+|\xi_m|\leq 2p$ and necessarily $|\xi_i|\leq p$, the right hand side is largest when $|\xi_{m-1}|=p$ and $|\xi_{m-2}|=p-1$. Furthermore, it is largest when $k_m=k_{m-1}+1=k_{m-2}+2$. Thus, if (30) holds, then $|n_{k_m}|\leq p|n_{k_{m-1}}|+(p-1)|n_{k_{m-2}}|$. By (29), this is impossible as soon as m is chosen sufficiently large, because $p+1>p+(p-1)/(p+1)$. ■

Note that Proposition 9.2 is best possible: if j is negative, then $\{j^k\}$ fails $(\mathcal{I}_{|j|})$. If j is positive, then $\{j^k\}\cup\{0\}$ fails complex $(\mathcal{J}_j)$.

Although we could prove that E enjoys $(\mathcal{I}_\infty)$ and $(\mathcal{J}_\infty)$ when $n_{k+1}/n_k\to\infty$, we need a direct argument in order to get the corresponding functional properties: we have

THEOREM 9.3. *Let $E=\{n_k\}\subseteq\mathbb{Z}$ be such that $n_{k+1}/n_k\to\infty$. Then $\mathcal{C}_E(\mathbb{T})$ has ℓ_1-(map) with $\{\pi_k\}$ and E is a (umbs) in $\mathcal{C}(\mathbb{T})$. If the ratios n_{k+1}/n_k are all integers, then the converse holds.*

Note that by Proposition 2.2(ii), E is a (umbs) in $L^p(\mathbb{T})$ for all $1\leq p<\infty$ as soon as it is a (umbs) in $\mathcal{C}(\mathbb{T})$. Recall further that, by [31, Th. 7], $L^p_E(\mathbb{T})$ has complex (umap) as soon as $\mathcal{C}_E(\mathbb{T})$ has ℓ_1-(map) (and hence complex (umap)).

Proof (of Theorem 9.3). Suppose $|n_{j+1}/n_j|\geq q$ for $j\geq l$ and some $q>1$ to be fixed later. Let $f=\sum a_j e_{n_j}\in\mathcal{P}_E(\mathbb{T})$ and $k\geq l$. We show by induction that for all $p\geq k$,

$$\|\pi_p f\|_\infty\geq\Big(1-\frac{\pi^2}{2}\frac{1-q^{2(k-p)}}{q^2-1}\Big)\|\pi_k f\|_\infty + \sum_{j=k+1}^{p}\Big(1-\frac{\pi^2}{2}\frac{1-q^{2(j-p)}}{q^2-1}\Big)|a_j|. \tag{31}$$

• There is nothing to show for $p=k$.

• Suppose (31) holds. Let $u\in\mathbb{T}$ be such that $\|\pi_p f\|_\infty=|\pi_p f(u)|$. Then there is a $v\in\mathbb{T}$ such that

$$\begin{cases}|\arg u/v|\leq\pi/|n_{p+1}|,\\ |\pi_p f(u)+a_{p+1}e_{n_{p+1}}(v)|=\|\pi_p f\|_\infty+|a_{p+1}|.\end{cases} \tag{32}$$

Indeed, there are $|n_{p+1}|$ equidistant points $v\in\mathbb{T}$ such that $a_{p+1}e_{n_{p+1}}(v)$ and $\pi_p f(u)$ have the same argument: there is necessarily one such point v at distance at most $\frac12 2\pi/|n_{p+1}|=\pi/|n_{p+1}|$ from u.

By [38, §I.4, Lemma 8], by Bernstein's inequality applied to $\pi_k f''$ and separately to each $a_j e''_{n_j}$, $j>k$,

$$\begin{aligned}|\pi_p f(u)-\pi_p f(v)|&\leq\tfrac12|\arg u/v|^2\|\pi_p f''\|_\infty\\ &\leq\frac{\pi^2}{2|n_{p+1}|^2}|n_k|^2\|\pi_k f\|_\infty+\sum_{j=k+1}^{p}\frac{\pi^2}{2|n_{p+1}|^2}|n_j|^2|a_j|\\ &\leq\frac{\pi^2}{2}q^{2(k-p-1)}\|\pi_k f\|_\infty+\sum_{j=k+1}^{p}\frac{\pi^2}{2}q^{2(j-p-1)}|a_j|.\end{aligned} \tag{33}$$

Thus we get successively

$$\begin{aligned}\|\pi_{p+1}f\|_\infty&\geq|\pi_{p+1}f(v)|\geq|\pi_p f(u)+a_{p+1}e_{n_{p+1}}(v)|-|\pi_p f(u)-\pi_p f(v)|\\ &\overset{(32\&33)}{\geq}\|\pi_p f\|_\infty+|a_{p+1}|-\frac{\pi^2}{2}q^{2(k-p-1)}\|\pi_k f\|_\infty\\ &\quad-\sum_{j=k+1}^{p}\frac{\pi^2}{2}q^{2(j-p-1)}|a_j|\\ &\overset{(31)}{\geq}\Big(1-\frac{\pi^2}{2}\frac{1-q^{2(k-p-1)}}{q^2-1}\Big)\|\pi_k f\|_\infty\\ &\quad+\sum_{j=k+1}^{p+1}\Big(1-\frac{\pi^2}{2}\frac{1-q^{2(j-p-1)}}{q^2-1}\Big)|a_j|.\end{aligned}$$

• So (31) is true for all $p\geq k$. Finally,

$$\|f\|_\infty=\lim_{p\to\infty}\|\pi_p f\|_\infty\geq\Big(1-\frac{\pi^2}{2}\frac{1}{q^2-1}\Big)\Big(\|\pi_k f\|_\infty+\sum_{j=k+1}^{\infty}|a_j|\Big). \tag{34}$$

Thus $\{\pi_j\}_{j\geq k}$ realizes ℓ_1-(ap) with constant $1+\pi^2/(2q^2-2-\pi^2)$. As q may be chosen arbitrarily large, E has ℓ_1-(map) with $\{\pi_j\}$. Additionally, (34) shows by choosing $\pi_k f=0$ that E is a (umbs) in $\mathcal{C}(\mathbb{T})$.

Finally, the converse holds by Proposition 8.2(vi): if n_{k+1}/n_k does not tend to infinity while being integer, then there are $h\in\mathbb{Z}\setminus\{0,1\}$ and an infinite F such that $F,hF\subseteq E$. ■

REMARK. Suppose still that $E = \{n_k\} \subseteq \mathbb{Z}$ with $n_{k+1}/n_k \to \infty$. A variation of the above argument yields that the space of *real* functions with spectrum in $E \cup -E$ has ℓ_1-(ap).

Recall that $E = \{n_k\} \subseteq \mathbb{Z}$ is a *Hadamard set* if there is a $q > 1$ such that $n_{k+1}/n_k \geq q$ for all k. It is a classical fact that then E is a Sidon set; Riesz products (see [34, Chapter 2] even yield effective bounds for its Sidon constant. In particular, $C^{\mathrm{c}}_\infty(E) \leq 2$ if $q \geq 3$. Our computations provide an alternative proof for $q > \sqrt{\pi^2/2+1}$ and give a better bound for $q \geq \sqrt{\pi^2+1}$: as we may suppose that $n_1 = 0$ in the preceding proof, (34) yields for $k = 1$ the following

COROLLARY 9.4. *Let $E = \{n_k\} \subseteq \mathbb{Z}$ and $q > \sqrt{\pi^2/2+1}$. If $|n_{k+1}| \geq q|n_k|$, then the Sidon constant of E satisfies $C^{\mathrm{c}}_\infty(E) \leq 1 + \pi^2/(2q^2-2-\pi^2)$.*

This estimate is optimal for large q in the following sense: Let $E = \{0, 1, q\}$ with $q \geq 2$ an integer. Then

$$|1 + \mathrm{e}_1(t) - \mathrm{e}_q(t)|^2 = 3 + 2\cos t - 2\cos(q-1)t - 2\cos qt$$

and, as $\cos(q-1)t + \cos qt \geq 0$ for $|t| \leq \pi/(2q)$ and $\cos t \leq \cos \pi/(2q)$ for $|t| \in [\pi/(2q), \pi]$,

$$\begin{aligned} \|1 + \mathrm{e}_1 - \mathrm{e}_q\|^2_\infty &\leq 5 \vee \left(3 + 2\cos\frac{\pi}{2q} + 2 + 2\right) = 7 + 2\cos\frac{\pi}{2q} \\ &\leq 7 + 2\left(1 - \frac{16-8\sqrt{2}}{\pi^2}\left(\frac{\pi}{2q}\right)^2\right) = 9 - \frac{8-4\sqrt{2}}{q^2} \end{aligned}$$

since $\cos t \leq 1 - ((16-8\sqrt{2})/\pi^2)t^2$ for $|t| \leq \pi/4$. So

$$C^{\mathrm{c}}_\infty(E) \geq 3\left(9 - \frac{8-4\sqrt{2}}{q^2}\right)^{-1/2} \geq 1 + \frac{4-2\sqrt{2}}{9q^2}$$

since $(1-t)^{-1/2} \geq 1 + t/2$ for $t < 1$. In fact, one even obtains $C^{\mathrm{c}}_\infty(E) \geq 1 + (\pi^2/12)q^{-2} + o(q^{-2})$ and there is no hope to improve Corollary 9.4.

Note, however, that there are sets E that satisfy $n_{k+1}/n_k \to 1$ and nevertheless enjoy $(\mathcal{I}_\infty)$ (see end of Section 10): they might be (umbs) in $\mathcal{C}(\mathbb{T})$, but this is unknown.

10. Density conditions.

We apply combinatorial tools to find out how "large" a set E may be while enjoying $(\mathcal{I}_n)$ or $(\mathcal{J}_n)$, and how "small" it must be.

The coarsest notion of largeness is that of density. Recall that the *maximal density* of $E \subseteq \mathbb{Z}$ is defined by

$$d^*(E) = \lim_{h\to\infty} \max_{a\in\mathbb{Z}} \frac{\#[E \cap \{a+1, \ldots, a+h\}]}{h}.$$

Suppose E enjoys $(\mathcal{I}_n)$ with $n \geq 2$. Then E is a $\Lambda(2n)$ set by Theorem 2.9(i). By [39, §1, Cor. 2], $d^*(E) = 0$. Now suppose E enjoys complex or real $(\mathcal{J}_n)$ with $n \geq 2$. As Li [31, Th. 2] shows, there are sets E such that $\mathcal{C}_E(\mathbb{T})$ has ℓ_1-(map) while E contains arbitrarily long arithmetic sequences, and we cannot apply Mikheev's argument.

Kazhdan (see [23, Th. 3.1]) proved that if $d^*(E) > 1/n$, then there is a $t \in \{1, \ldots, n-1\}$ such that $d^*(E \cap (E+t)) > 0$. One might hope that it should in fact suffice to choose t in any interval of length n. However, Hindman [23, Th. 3.2] exhibits a counterexample: given $s \in \mathbb{Z}$ and positive ε, there is a set E with $d^*(E) > 1/2 - \varepsilon$ and there are arbitrarily large a such that $E \cap (E-t) = \emptyset$ for all $t \in \{a+1, \ldots, a+s\}$. Thus, we have to be satisfied with

LEMMA 10.1. *Let $E \subseteq \mathbb{Z}$ with positive maximal density. Then there is a $t \geq 1$ such that the following holds: for any $s \in \mathbb{Z}$ we have some a, $|a| \leq t$, such that $d^*((E+a) \cap (E+s)) > 0$.*

Proof. By a result of Erdős (see [23, Th. 3.8]), there is a $t \geq 1$ such that $F = E + 1 \cup \ldots \cup E + t$ satisfies $d^*(F) > 1/2$. But then, by [23, Th. 3.4], $d^*(F \cap (F+s)) > 0$ for any $s \in \mathbb{Z}$. This means that for any s there are $1 \leq a, b \leq t$ such that $d^*((E+a) \cap (E+s+b)) > 0$. ■

We are now able to prove

PROPOSITION 10.2. *Let $E \subseteq \mathbb{Z}$.*

(i) *If E has positive maximal density, then there is an $a \in \mathbb{Z}$ such that $E \cup \{a\}$ fails real $(\mathcal{J}_2)$. Therefore E fails real $(\mathcal{J}_3)$.*

(ii) *If $d^*(E) > 1/2$, then E fails real $(\mathcal{J}_2)$.*

Proof. (ii) is proven in [31, Prop. 14]. (i) is a consequence of Lemma 10.1: indeed, if E has positive maximal density, then the lemma yields some $a \in \mathbb{Z}$ and an infinite $F \subseteq E$ such that for all $s \in F$ there are arbitrarily large $k, l \in E$ such that $k + a = l + s$. Thus $E \cup \{a\}$ fails real $(\mathcal{J}_2)$. Furthermore, E fails real $(\mathcal{J}_3)$ by Proposition 8.2(iv). ■

We may reformulate the remaining open case of $(\mathcal{J}_2)$. Let us introduce the *infinite difference set* of E: $\Delta E = \{t : \#[E \cap (E-t)] = \infty\}$ (see [57] and [54]). Then E has real $(\mathcal{J}_2)$ if and only if, for any $a \in E$, ΔE meets $E - a$ finitely many times only. Thus our question is: are there sets with positive maximal density such that $(E-a) \cap \Delta E$ is finite for all $a \in E$?

Proposition 9.2 and Theorem 9.3 show that there is only one general condition of lacunarity on E that ensures properties $(\mathcal{I}_n)$, $(\mathcal{J}_n)$ or $(\mathcal{I}_\infty)$, $(\mathcal{J}_\infty)$: E must grow exponentially or superexponentially. One may nevertheless construct inductively "large" sets that enjoy these properties: they must only be sufficiently irregular to avoid all arithmetical relations. Thus there are sequences with growth slower than k^{2n-1} which nevertheless enjoy both $(\mathcal{I}_n)$ and complex and real $(\mathcal{J}_n)$. See [21, §II, (3.52)] for a proof in the case $n=2$: it can be easily adapted to $n \geq 2$ and shows also the way to construct sets that satisfy $(\mathcal{I}_\infty)$ and $(\mathcal{J}_\infty)$ and grow more slowly than k^{n_k} for any sequence $n_k \to \infty$.

11. Unconditionality and probabilistic independence. Let us first show how simple the problems of (umbs) and (umap) become when considered for independent uniformly distributed random variables and their span in some space.

Let $\mathbb{D}^\infty$ be the Cantor group and Γ its dual group of Walsh functions. Consider the set $R=\{r_i\} \subseteq \Gamma$ of Rademacher functions, i.e. the coordinate functions on $\mathbb{D}^\infty$; they form a family of independent random variables that take values -1 and 1 with equal probability $1/2$. Thus $\|\sum \lambda_i a_i r_i\|_X$ does not depend on the choice of signs $\lambda_i = \pm 1$ for any space $X \in \{\mathcal{C}(\mathbb{D}^\infty), L^p(\mathbb{D}^\infty)\ (1 \le p < \infty)\}$ and R is a real 1-(ubs) in them.

Clearly, R is also a complex (ubs) in all such X. But no space X_Λ has complex (umap) for any $X \neq L^2(\mathbb{D}^\infty)$ and any infinite $\Lambda = \{\gamma_i\} \subseteq \Gamma$. Indeed, Λ would have an analogue property $(\mathcal{U})$ of block unconditionality in X: for any $\varepsilon > 0$ there would be n such that

$$\max_{\lambda\in\mathbb{T}} \|\lambda a\gamma_1 + b\gamma_n\|_p \le (1+\varepsilon)\|a\gamma_1 + b\gamma_n\|_p.$$

But this is false: for $1 \le p < 2$, take $a=b=1$, $\lambda = \mathrm{i}$:

$$\max_{\lambda\in\mathbb{T}} \|\lambda\gamma_1 + \gamma_n\|_p \ge \big(\tfrac12(|\mathrm{i}+1|^p + |\mathrm{i}-1|^p)\big)^{1/p} = \sqrt2 > \|\gamma_1+\gamma_n\|_p = 2^{1-1/p};$$

for $2 < p \le \infty$, take $a=1$, $b=\mathrm{i}$, $\lambda=\mathrm{i}$:

$$\max_{\lambda\in\mathbb{T}} \|\lambda\gamma_1 + \mathrm{i}\gamma_n\|_p \ge \big(\tfrac12(|\mathrm{i}+\mathrm{i}|^p + |\mathrm{i}-\mathrm{i}|^p)\big)^{1/p} = 2^{1-1/p} > \|\gamma_1+\mathrm{i}\gamma_n\|_p = \sqrt2.$$

This is simply due to the fact that the image domain of the characters on $\mathbb{D}^\infty$ is too small. Take now the infinite torus $\mathbb{T}^\infty$ and consider the set $S=\{s_i\}$ of Steinhaus functions, i.e. the coordinate functions on $\mathbb{T}^\infty$; they form again a family of independent random variables with values uniformly distributed in $\mathbb{T}$. Then S is clearly a complex 1-(ubs) in any $X \in \{\mathcal{C}(\mathbb{T}^\infty), L^p(\mathbb{T}^\infty)\ (1 \le p < \infty)\}$.

As the random variables $\{\mathrm{e}_n\}$ also have values uniformly distributed in $\mathbb{T}$, some sort of approximate independence should suffice to draw the same conclusions as in the case of S.

A first possibility is to look at the joint distribution of $(\mathrm{e}_{p_1},\dots,\mathrm{e}_{p_n})$ with $p_1,\dots,p_n \in E$ and to require it to be close to the product of the distributions of the e_{p_i}. For example, Pisier [46, Lemma 2.7] gives the following characterization: E is a Sidon set if and only if there are a neighbourhood V of 1 in $\mathbb{T}$ and $\beta>0$ such that for any finite $F \subseteq E$,

$$m[\mathrm{e}_p \in V : p \in F] \le 2^{-\beta\#[F]}. \tag{35}$$

Murai [42, §4.2] calls $E \subseteq \mathbb{Z}$ *pseudo-independent* if for all $A_1,\dots,A_n \subseteq \mathbb{T}$,

$$m[\mathrm{e}_{p_i} \in A_i : 1 \le i \le n] \xrightarrow[p_i\to\infty]{p_i\in E} \prod_{i=1}^n m[\mathrm{e}_{p_i} \in A_i] = \prod_{i=1}^n m[A_i]. \tag{36}$$

His theorem [42, Lemma 30] gives together with Proposition 8.1(iii) the following

PROPOSITION 11.1. *Let $E \subseteq \mathbb{Z}$. Then E is pseudo-independent if and only if E enjoys $(\mathcal{I}_\infty)$.*

Note that by Corollary 2.10, (36) does not imply (35).

Another possibility is to define some notion of almost independence. Berkes [1] introduces the following notion: let us call a sequence of random variables $\{X_n\}$ *almost i.i.d.* (independent and identically distributed) if, after enlarging the probability space, there is an i.i.d. sequence $\{Y_n\}$ such that $\|X_n - Y_n\|_\infty \to 0$. We have the straightforward

PROPOSITION 11.2. *Let $E=\{n_k\} \subseteq \mathbb{Z}$. If E is almost i.i.d., then E is a (umbs) in $C(\mathbb{T})$.*

Proof. Let $\{Y_j\}$ be an i.i.d. sequence and suppose $\|\mathrm{e}_{n_j} - Y_j\|_\infty \le \varepsilon$ for $j \ge k$. Then

$$\sum_{j\ge k}|a_j| = \Big\|\sum_{j\ge k} a_j Y_j\Big\|_\infty \le \Big\|\sum_{j\ge k} a_j \mathrm{e}_{n_j}\Big\|_\infty + \varepsilon\sum_{j\ge k}|a_j|$$

and the unconditionality constant of $\{n_k, n_{k+1}, \dots\}$ is less than $(1-\varepsilon)^{-1}$. ■

Suppose $E=\{n_k\} \subseteq \mathbb{Z}$ is such that n_{k+1}/n_k is an integer for all k. In that case, Berkes [1] proves that E is almost i.i.d. if and only if $n_{k+1}/n_k \to \infty$. We thus recover a part of Theorem 9.3.

12. Summary of results. Remarks and questions. For the convenience of the reader, we now reorder our results by putting together those which are relevant to a given class of Banach spaces.

12.1. *The case $X = L^p(\mathbb{T})$ with p an even integer.* Let p be an even integer. We observed the following facts.

- Real and complex (umap) differ in any space $L^p(\mathbb{T})$: consider Proposition 9.1 or $E = \{\pm(p/2)^k\}$.
- By Theorem 7.5, $L^p_E(\mathbb{T})$ has complex (resp. real) (umap) if so does $L^{p+2}_E(\mathbb{T})$.

• The study of geometric sequences in Section 8 shows that the converse is false for any p. In the complex case, $E = \{(p/2)^k\}$ is a counterexample. In the real case, take $E = \{0\} \cup \{\pm p^k\}$.

• (umap) is not stable under unions with an element: for each p, there is a set E such that $L^p_E(\mathbb{T})$ has complex (resp. real) (umap), but $L^p_{E\cup\{0\}}(\mathbb{T})$ does not. In the complex case, consider $E = \{(p/2)^k\}$. In the real case, consider $E = \{\pm(p/2)^k\}$.

• If E is a symmetric set and $p \neq 2$, then $L^p_E(\mathbb{T})$ fails complex (umap). Proposition 8.3 gives a criterion for real (umap).

What is the relationship between (umbs) and complex (umap)? By Proposition 8.2(i) and 7.7(i) we have

PROPOSITION 12.1.1. *Let $E = \{n_k\} \subseteq \mathbb{Z}$ and $n \geq 1$.*

(i) *If E is a (umbs) in $L^{4n-2}(\mathbb{T})$, then $L^{2n}_E(\mathbb{T})$ has complex (umap).*

(ii) *If $\{\pi_k\}$ realizes complex (umap) in $L^{2n}_E(\mathbb{T})$, then E is a (umbs) in $L^{2n}(\mathbb{T})$.*

By Proposition 10.2(i) we also have

PROPOSITION 12.1.2. *Let $E \subseteq \mathbb{Z}$ and $p \neq 2, 4$ an even integer. If $L^p_E(\mathbb{T})$ has real (umap), then $d^*(E) = 0$.*

Note also this consequence of Propositions 3.2, 8.4, 11.1 and Theorems 2.9, 7.5:

PROPOSITION 12.1.3. *Let $\sigma > 1$ and $E = \{[\sigma^k]\}$. Then the following properties are equivalent:*

(i) *σ is transcendental;*

(ii) *$L^p_E(\mathbb{T})$ has complex (umap) for any even integer p;*

(iii) *E is a (umbs) in any $L^p(\mathbb{T})$, p an even integer;*

(iv) *E is pseudo-independent.*

12.2. *The cases $X = L^p(\mathbb{T})$ with p not an even integer and $X = \mathcal{C}(\mathbb{T})$.* In this section, X denotes either $L^p(\mathbb{T})$, p not an even integer, or $\mathcal{C}(\mathbb{T})$.

Theorems 2.9 and 7.5 only permit us to use the negative results of Section 8: thus, we can just gather negative results about the functional properties of E. For example, we know by Proposition 8.2(iv) that $(\mathcal{I}_\infty)$ and $(\mathcal{J}_\infty)$ are stable under unions with an element. Nevertheless, we cannot conclude that the same holds for (umap). The negative results are (by Section 8):

• for any infinite $E \subseteq \mathbb{Z}$, $X_{E\cup 2E}$ fails real (umap). Thus (umap) is not stable under unions;

• if E is a polynomial sequence (see Section 8), then E is not a (umbs) in X and X_E fails real (umap);

• if E is a symmetric set, then E is not a (umbs) in X and X_E fails complex (umap). Proposition 8.3 gives a criterion for real (umap);

• if $E = \{[\sigma^k]\}$ with $\sigma > 1$ an algebraic number (in particular, if E is a geometric sequence), then E is not a (umbs) in X and X_E fails complex (umap).

Furthermore, by Proposition 9.1, real and complex (umap) differ in X.

Theorem 9.3 is the only but general positive result on (umbs) and complex (umap) in X. Proposition 9.1 yields further examples for real (umap).

What about the sets that satisfy $(\mathcal{I}_\infty)$ or $(\mathcal{J}_\infty)$? We only know that $(\mathcal{I}_\infty)$ does not even ensure sidonicity by Corollary 2.10.

One might wonder whether for some reasonable class of sets E, E is a finite union of sets that enjoy $(\mathcal{I}_\infty)$ or $(\mathcal{J}_\infty)$. This is false even for Sidon sets: for example, let E be the geometric sequence $\{j^k\}_{k\geq 0}$ with $j \in \mathbb{Z}\setminus\{-1,0,1\}$ and suppose $E = E_1 \cup \ldots \cup E_n$. Then $E_i = \{j^k\}_{k\in A_i}$, where the A_i's are a partition of the set of positive integers. But then one of the A_i contains arbitrarily large a and b such that $|a-b| \leq n$. This means that there is an infinite subset $B \subseteq A_i$ and an h, $1 \leq h \leq n$, such that $h + B \subseteq A_i$. We may apply Proposition 8.2(vi): E_i enjoys neither $(\mathcal{I}_{j^h+1})$ nor complex $(\mathcal{J}_{j^h+1})$—nor real $(\mathcal{J}_{j^h+1})$ if furthermore j is even.

Is there a result corresponding to Proposition 12.1.1(ii)? We do not know. But suppose that $1+\varepsilon_n = \|I-(1+\lambda)\pi_n\|_{\mathcal{L}(X)}$ converges sufficiently rapidly to 1: suppose that not only $\varepsilon_n \to 0$ but also $\sum \varepsilon_n < \infty$. As

$$\max_{\lambda_i\in\mathbb{T}} \Big\| \sum_{i=l}^{k} \lambda_i a_i e_{n_i} + \sum_{i>k} a_i e_{n_i} \Big\|_X \leq (1+\varepsilon_k) \max_{\lambda_i\in\mathbb{T}} \Big\| \sum_{i=l}^{k-1} \lambda_i a_i e_{n_i} + \sum_{i\geq k} a_i e_{n_i} \Big\|_X,$$

we get

$$\max_{\lambda_i\in\mathbb{T}} \Big\| \sum_{i\geq l} \lambda_i a_i e_{n_i} \Big\|_X \leq \prod_{i\geq l}(1+\varepsilon_i) \Big\| \sum_{i\geq l} a_i e_{n_i} \Big\|_X,$$

and if $\sum \varepsilon_n < \infty$, then $(1+\varepsilon_l)(1+\varepsilon_{l+1})\ldots \to 1$ as $l \to \infty$, i.e. E is a (umbs).

Let us finally state

PROPOSITION 12.2.1. *Let $E \subseteq \mathbb{Z}$. If X_E has real (umap), then $d^*(E) = 0$.*

12.3. *Questions.* The following questions remain open:

Combinatorics. Regarding Proposition 10.2(i), is there a set E enjoying $(\mathcal{J}_2)$ with positive maximal density, or even with a uniformly bounded pace? Furthermore, may a set E with positive maximal density admit a partition $E = \bigcup E_i$ into finite sets such that all $E_i + E_j$, $i \leq j$, are pairwise disjoint? Then $L^4_E(\mathbb{T})$ would admit a 1-unconditional (fdd) by Proposition 7.7(i).

Functional analysis. Let $X \in \{L^1(\mathbb{T}), \mathcal{C}(\mathbb{T})\}$ and consider Theorem 6.2.3. Is $(\mathcal{U})$ sufficient for X_E to have (umap)? Is there a set $E \subseteq \mathbb{Z}$ such that some space $L^p_E(\mathbb{T})$, p not an even integer, has (umap), while $\mathcal{C}_E(\mathbb{T})$ fails it?

Harmonic analysis. Is there a Sidon set $E = \{n_k\} \subseteq \mathbb{Z}$ with constant asymptotically 1 such that n_{k+1}/n_k is uniformly bounded? What about the case $E = [\sigma^k]$ for a transcendental $\sigma > 1$? If E enjoys $(\mathcal{I}_\infty)$, is E a (umbs) in $L^p(\mathbb{T})$ $(1 \le p < \infty)$? What about $(\mathcal{J}_\infty)$?

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On operators satisfying the Rockland condition

by

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Abstract. Let G be a homogeneous Lie group. We prove that for every closed, homogeneous subset Γ of G^* which is invariant under the coadjoint action, there exists a regular kernel P such that P goes to 0 in any representation from Γ and P satisfies the Rockland condition outside Γ. We prove a subelliptic estimate as an application.

Introduction. The purpose of this paper is to construct operators which satisfy the Rockland condition in a given set Γ of representations, and are equal to 0 outside Γ. Rockland operators satisfy remarkable subelliptic estimates ([11], [7], [9], [10], [14]; see also [15]) making them a good substitute for elliptic operators on homogeneous groups. Christ *et al.* [2] gave a calculus for pseudodifferential operators on homogeneous groups: the formulas for products and adjoints and criteria for existence of left or right parametrices (generalizing results of [8]). However, one should note that the great flexibility of the classical calculus of pseudodifferential operators is in large part due to the ease of constructing scalar functions (cutoffs and partitions of unity). In the homogeneous group case we want to pre-specify operators in a set of representations and still have regular kernels; this is not straightforward, in fact not always possible. Our kernels may serve as cutoffs on spectral side (for the spatial cutoffs one simply uses multiplications with smooth functions). The conditions we impose seem to be necessary. We present also a simple application in which we derive some L^p estimates.

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Preliminaries. We consider a homogeneous group G, that is, a nilpotent Lie group equipped with a family of automorphisms (*dilations*) $\{\delta_t\}_{t>0}$ such that $\delta_t\delta_s = \delta_{ts}$ and for all $x \in G$ we have $\delta_t x \to e$ as $t \to 0$. The reader may wish to consult [6] (our definition is a bit more general). We identify G with

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