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Quasiconformal mappings and Sobolev spaces

by

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Abstract. We examine how Poincaré inequalities change under quasiconformal maps between appropriate metric spaces having the same Hausdorff dimension. We also show that for many metric spaces the Sobolev functions can be identified with functions satisfying Poincaré inequalities, and this allows us to extend to the metric space setting the fact that quasiconformal maps from $\mathbb{R}^Q$ onto $\mathbb{R}^Q$ preserve the Sobolev space $L^{1,Q}(\mathbb{R}^Q)$.

Introduction. Quasiconformal mappings can be defined on any metric space by requiring that they distort infinitesimal balls by a bounded amount. In order for this definition to be useful one needs to be able to deduce global properties of the mapping from this infinitesimal condition. In a recent paper ([HeK2], see also [HeK1]), Heinonen and Koskela showed that many of the classical results on quasiconformal self-maps of Euclidean space can be extended to quasiconformal maps between more general metric spaces of the same Hausdorff dimension. For example, such quasiconformal maps are quasisymmetric, absolutely continuous, and have Jacobians in A_∞ (see Theorem 1.2).

The key assumption needed is that the space where the map is defined should be highly connected, meaning that there are "many" paths joining any part of the space to any other part of the space. No such assumption is made on the target space. The connectivity can be precisely defined using moduli of path families (see the Loewner condition in [HeK2]). However, it turns out that it can be expressed more usefully in terms of an analytic condition; namely, that the space support a $(1,p)$-Poincaré inequality for some suitable p. In the classical case this means that there are constants C, λ such

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that for any continuously differentiable function u and any ball B we have

$$(1)\qquad \fint_B |u-u_B|\,d\mu \le Cr\Big(\fint_{\lambda B} |\nabla u|^p\,d\mu\Big)^{1/p}$$

where r is the radius of B, u_B is the average value of u on B, and $\fint$ denotes the average value of the integral. This is satisfied by $\mathbb{R}^Q$ with $p=1$ (and hence for all $p\ge 1$). Of course, on more general metric spaces one needs to avoid talking about "continuously differentiable functions". This is where the notion of an upper gradient comes in.

The importance of the Poincaré inequality for obtaining global results from the definition of quasiconformality gives rise to the following natural question. If a space supports a Poincaré inequality, and there is a quasiconformal map from this space onto another space of the same dimension, must the image also satisfy a Poincaré inequality? We will prove that this is the case when p is less than Q, the dimension of the space, but that it is false when p is greater than Q. With some additional assumptions either on the map or on the space, we show that if a space supports a $(1,Q)$-Poincaré inequality, then so does the image of the space. The positive results are proven in Section 2, while Section 3 contains the counter-example for $p>Q$.

The question of how Poincaré inequalities behave under quasiconformal mappings is intimately related to how quasiconformal maps affect the spaces of Sobolev functions. Indeed, we will demonstrate in Section 4 that for many metric spaces the space of functions that satisfy a $(1,p)$-Poincaré inequality with "gradient" in L^p is exactly the same as the Sobolev space $L^{1,p}$ either of Korevaar and Schoen (see [KS]) or of Hajłasz (see [Ha]). The result alluded to above, that quasiconformal maps preserve a $(1,Q)$-Poincaré inequality, then becomes a generalization of the well known fact that quasiconformal maps from $\mathbb{R}^Q$ onto $\mathbb{R}^Q$ preserve the Sobolev spaces $L^{1,Q}_{\mathrm{loc}}(\mathbb{R}^Q)$ and $L^{1,Q}(\mathbb{R}^Q)$.

For a more extensive treatment of the equivalence between Sobolev spaces and Poincaré spaces see [HaK2]. Results on the preservation of classical Sobolev spaces can be found in [GR] (Theorem 4.2, Chapter 5), [L] (Theorem 5.3), and [Z] (Remark 4.2).

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NOTE. We have just recently received a preprint by Jeremy Tyson [T] in which he shows that quasiconformal maps preserve the Loewner condition and so, by a result in [HeK2], they also preserve the $(1,Q)$-Poincaré inequality. Thus the extra conditions that we need to treat this case are redundant.

1. Preliminaries. Our standing assumptions on the spaces we are considering are that they are locally compact, metric spaces equipped with a regular, Borel measure, and that the space is regular with respect to this measure. Regularity of a space X (or Q-regularity if we want to emphasize the parameter Q) with respect to a measure μ means that there are constants C and Q for which

$$(2)\qquad C^{-1}R^Q \le \mu(B(x,R)) \le CR^Q$$

for all $x\in X$ and all $R\le \operatorname{diam} X$. Whenever we want to emphasize that μ is a measure on X (and not on some other space) we write μ_X. We will generally use $|x-y|$ to denote the distance, in the appropriate metric, between points x and y.

L^p_{loc} will denote functions that are p-integrable on *all* balls. This will be more convenient for us than the usual definition, which requires p-integrability only on some neighbourhood of every point. The two definitions agree when the space is *proper*, i.e., when every closed ball is compact.

Throughout the paper, functions will take values either in $\mathbb{R}$ or in $[0,\infty]$, whichever is appropriate.

By a path we mean a continuous image of a closed interval.

A function g is said to be an *upper gradient* of another function u if $|u(\alpha(a))-u(\alpha(b))|\le \int_a^b g(\alpha(t))\,dt$ whenever $a,b\in\mathbb{R}$ and $\alpha:[a,b]\to X$ is 1-Lipschitz. This concept was introduced in [HeK2], where the name very weak gradient was used instead. See also [S]. A rephrasing of the definition is that for every rectifiable path γ we have $|u(a_\gamma)-u(b_\gamma)|\le\int_\gamma g$ where a_γ, b_γ are the endpoints of γ. The function α of the original definition is simply the arc-length parameterization of γ.

A homeomorphism f from a metric space X onto another metric space Y is called *quasiconformal* if there is a finite constant H for which

$$\limsup_{r\to 0}\frac{\sup_y\{|f(x)-f(y)| : |x-y|\le r\}}{\inf_y\{|f(x)-f(y)| : |x-y|\ge r\}}\le H$$

at all points $x\in X$.

A homeomorphism f from a metric space X onto another metric space Y is called *quasisymmetric* if there exists a homeomorphism $\eta:[0,\infty)\to[0,\infty)$ so that whenever $|x-a|\le t|x-b|$ for some $t>0$ and for some three points x,a,b in X, then the inequality $|f(x)-f(a)|\le\eta(t)|f(x)-f(b)|$ holds in Y. The inverse of a quasisymmetric homeomorphism is also a quasisymmetric homeomorphism. For a discussion of quasisymmetric maps see [TV].

1.1. *Poincaré inequalities and quasiconformal maps.* A pair of measurable functions (u,g) *satisfies a (q,p)-Poincaré inequality* if there are constants C and $1\le\lambda$ such that for each ball B, of radius r, there is a real number a_B such that

$$\left(\fint_B |u - a_B|^q \, d\mu \right)^{1/q} \le Cr \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}. \tag{3}$$

Here $\fint$ denotes the average value of the integral. Note that there are no a priori assumptions on the integrability of u or g. Both sides of the inequality may be infinite. When u is locally integrable we can choose a_B to be u_B, the average value of u over B.

$P^{1,p}(X)$ consists of all functions, u, on X for which there exists $g \in L^p(X)$ such that the pair (u, g) satisfies a $(1,p)$-Poincaré inequality. $P^{1,p}_{\mathrm{loc}}(X)$ is defined similarly.

There is a Sobolev-type embedding theorem for functions satisfying Poincaré inequalities.

THEOREM 1.1 (Hajłasz, Koskela). *Suppose that X is Q-regular and that (u,g) satisfies a $(1,p)$-Poincaré inequality for some $1 \le p \le Q$. Then (u,g) satisfies a (q,p)-Poincaré inequality for $1 \le q < pQ/(Q-p)$, where the right-hand term is ∞ when $p = Q$.*

This follows from Theorem 5.1 in [HaK2] and the proof thereof. The proof of the theorem makes use of pointwise estimates of the type in Lemma 4.3 of this paper. See also [HaK1].

A space *supports a (q,p)-Poincaré inequality* if each pair consisting of a continuous function and an upper gradient of that function satisfies a (q,p)-Poincaré inequality. These are good spaces for quasiconformal maps, as the next theorem illustrates (see [HeK2]). Say that X, Y, and f satisfy (‡) if:

(i) X and Y are Q-regular spaces with $Q > 1$.
(ii) X is proper and quasiconvex.
(iii) Y is linearly locally connected.
(iv) f is a quasiconformal map from X onto Y that maps bounded sets to bounded sets.

We say that a space is *quasiconvex* if there is a constant $C > 0$ so that every pair of points x and y in the space can be joined by a curve γ whose length is bounded by $C|x-y|$. *Linearly locally connected* means that there is a constant $C \ge 1$ so that for each x in the space and each $r > 0$ the following two conditions hold:

(1) any pair of points in $B(x,r)$ can be joined in $B(x,Cr)$;
(2) any pair of points in the complement of $B(x,r)$ can be joined in the complement of $B(x,r/C)$.

By *joining*, we mean joining by a continuum.

THEOREM 1.2 (Heinonen, Koskela). *Suppose that X, Y, and f satisfy (‡). If X supports a $(1,Q)$-Poincaré inequality, then f is quasisymmetric. If X supports a $(1,p)$-Poincaré inequality for some $1 \le p < Q$, then in addition to being quasisymmetric, f is also absolutely continuous, and the pullback measure is A_∞-related to μ_X.*

The *pullback measure* ν_f is that measure on X that assigns to a set E the measure $\mu_Y(f(E))$. We say that f is *absolutely continuous* if ν_f is absolutely continuous. We will use J_f to denote the Radon–Nikodym (or volume) derivative of ν_f with respect to the measure μ_X. When ν_f is absolutely continuous, then $\nu_f(E) = \int_E J_f \, d\mu_X$. A measure σ is said to be *A_∞-related* to a measure τ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\tau(E) < \delta\tau(B)$ implies that $\sigma(E) < \varepsilon\sigma(B)$ whenever B is a ball and E is a measurable subset of B. If both measures are doubling then the A_∞ relation is equivalent to several other conditions. For example, $d\sigma = w\, d\tau$ and w satisfies a reverse Hölder or A_p condition. See Corollary 14, Chapter 1 of [ST].

2. Preserving Poincaré inequalities. As noted in the introduction, we want to see what happens to Poincaré inequalities under quasiconformal maps. In *this* section, X and Y will be Q-regular spaces, and f will be a quasisymmetric map from X onto Y whose pullback measure is A_∞-related to μ, the measure on X. These hypotheses on f are satisfied, for example, when X, Y, and f satisfy (‡) and X supports a $(1,p)$-Poincaré inequality for some $1 \le p < Q$.

Given a pair (u,g), we define $f(u,g)$ to be the pair

$$(u \circ f^{-1}, (g \circ f^{-1})(J_f \circ f^{-1})^{-1/Q}).$$

The explanation for this definition is that if $g = |\nabla u|$, then $f(u,g)$ is basically $(u \circ f^{-1}, |\nabla(u \circ f^{-1})|)$.

THEOREM 2.1. *If (u,g) satisfies a $(1,p)$-Poincaré inequality in X for some $1 \le p \le Q$, then $f(u,g)$ satisfies a $(1,p')$-Poincaré inequality in Y, where $p' = p$ if $p = Q$, and $1 \le p' < Q$ if $1 \le p < Q$.*

Proof. Write (v,h) for the pair $f(u,g)$. The measure of a set E, whether in X or Y, is denoted by $|E|$. Because a $(1,p)$-Poincaré inequality implies a $(1,q)$-Poincaré inequality for $q \ge p$, we can assume that p is as close to Q as is necessary for the following proof to work.

Because f is quasisymmetric, we can treat pre-images in X of balls in Y as "balls" in X. The various Poincaré inequalities can be expressed using these sets instead of balls, with minor modifications. To wit, for $A = f^{-1}(B')$, λA is defined to be $f^{-1}(\lambda B')$, and the radius term is replaced by $\operatorname{diam} A$.

The map f is absolutely continuous, so $|f(A)| = \int_A J_f$ for any measurable set A. The pullback measure is A_∞-related to μ and it is also doubling. This last fact is a consequence of f being quasisymmetric and Y being regular. Thus J_f satisfies a reverse Hölder condition and also an A_p condition. So there exist constants C and $0 < \varepsilon < 1$ such that for every ball $B \subseteq X$ we have

$$(4)\qquad \Big(\fint_B J_f^{1+\varepsilon}\Big) \le C\Big(\fint_B J_f\Big)^{1+\varepsilon} \quad\text{and}\quad \Big(\fint_B J_f^{-\varepsilon}\Big) \le C\Big(\fint_B J_f\Big)^{-\varepsilon}.$$

The second inequality is the A_p condition for $p = 1 + \varepsilon^{-1}$. It follows easily from quasisymmetry of f and regularity of Y that these inequalities remain valid when B is replaced by the pre-image in X of a ball in Y.

Denote the exponent conjugate to $1+\varepsilon$ by k. Recall that Theorem 1.1 states that (u,g) satisfies a (q,p)-Poincaré inequality for $q < Qp/(Q-p)$. In particular, (u,g) satisfies a (k,p)-Poincaré inequality.

We will prove that (v,h) satisfies a $(1,p')$-Poincaré inequality for $p' = Qp(1+\varepsilon)/(p+\varepsilon Q)$. Fix a ball B_0 in Y. Let $A = f^{-1}(B_0)$, $a = a_A$ and $A' = \lambda A$. Then

$$(5)\qquad \fint_{B_0} |v-a| = \frac{|A|}{|B_0|}\fint_A |u-a| J_f \le \frac{|A|}{|B_0|}\Big(\fint_A |u-a|^k\Big)^{1/k}\Big(\fint_A J_f^{1+\varepsilon}\Big)^{1/(1+\varepsilon)}$$
$$\le C\frac{|A|}{|B_0|}\Big(\fint_A |u-a|^k\Big)^{1/k}\fint_A J_f \le C(\operatorname{diam} A)\Big(\fint_{A'} g^p\Big)^{1/p}.$$

The equality at the start comes from a change of variables, while the last two inequalities follow from reverse Hölder, the (k,p)-Poincaré inequality, and the fact that

$$\frac{|A|}{|B_0|}\fint_A J_f = 1.$$

We will convert the g integral into an h integral by playing around with exponents. Set

$$\delta = \frac{\varepsilon(Q-p)}{Q(1+\varepsilon)}, \qquad s = \frac{Q(1+\varepsilon)}{p+\varepsilon Q},$$

and let t be the conjugate exponent of s. Note that $ps = p'$, $\delta t = \varepsilon$, and $1 - \delta s = p'/Q$. We now have

$$(6)\qquad \Big(\fint_{A'} g^p\Big)^{1/p} = \Big(\fint_{A'} g^p J_f^\delta J_f^{-\delta}\Big)^{1/p} \le \Big(\fint_{A'} g^{ps} J_f^{\delta s}\Big)^{1/(ps)}\Big(\fint_{A'} J_f^{-\delta t}\Big)^{1/(pt)}$$
$$= \Big(\fint_{A'} g^{p'} J_f^{1-p'/Q}\Big)^{1/p'}\Big(\fint_{A'} J_f^{-\varepsilon}\Big)^{1/p-1/p'}$$
$$= \Big(\frac{|\lambda B_0|}{|A|}\fint_{\lambda B_0} h^{p'}\Big)^{1/p'}\Big(\fint_{A'} J_f^{-\varepsilon}\Big)^{1/p-1/p'}.$$

The second-last equality comes from our choice of exponents, while the last equality comes from a change of variables in the first integral and the definition of h. Recalling property (4), we obtain

$$\Big(\fint_{A'} J_f^{-\varepsilon}\Big)^{1/p-1/p'} \le C\Big(\fint_{A'} J_f\Big)^{-\varepsilon(1/p-1/p')} = \Big(\frac{|\lambda B_0|}{|A|}\Big)^{1/Q-1/p'}.$$

Combining this last estimate with (5) and (6) we see that

$$\fint_{B_0} |v-a| \le C|B_0|^{1/Q}\Big(\fint_{B_0} h^{p'}\Big)^{1/p'} \le Cr\Big(\fint_{B_0} h^{p'}\Big)^{1/p'}$$

where r is the radius of B_0. This is the required Poincaré inequality. ■

Note that the function h in the above proof is Q-integrable if and only if g is Q-integrable. Thus we get the following corollary:

Corollary 2.2. *If $u \in P^{1,Q}(X)$ (resp. $u \in P^{1,Q}_{\mathrm{loc}}(X)$), then $u \circ f^{-1} \in P^{1,Q}(Y)$ (resp. $u \circ f^{-1} \in P^{1,Q}_{\mathrm{loc}}(Y)$).*

The next theorem is morally also a corollary; however, we need to deal with a technical problem involving upper gradients.

Theorem 2.3. *If X supports a $(1,p)$-Poincaré inequality for some $1 \le p \le Q$, then Y $(= f(X))$ supports a $(1,p')$-Poincaré inequality, where $p' = p$ if $p = Q$ and $1 \le p' < Q$ if $1 \le p < Q$.*

If X only supports a $(1,p)$-Poincaré inequality for some $p > Q$, then Y need not satisfy *any* Poincaré inequality. An example is given in the next section.

Suppose that v is continuous in Y and h is an upper gradient of v. We need to prove that (v,h) satisfies a $(1,p')$-Poincaré inequality. Let $u = v \circ f$ and $g = (h \circ f)|J_f|^{1/Q}$. Note that $(v,h) = f(u,g)$. If we knew that g were an upper gradient of u, then (u,g) would satisfy a $(1,p)$-Poincaré inequality and so, by Theorem 2.1, (v,h) would satisfy a $(1,p')$-Poincaré inequality, and we would be done. Unfortunately, g may not be an upper gradient. The problem is caused by the fact that f need not be absolutely continuous on all rectifiable paths in X. Nevertheless, f is absolutely continuous on (modulus) a.e. path and this suffices to show that g is as good as an upper gradient. Recall that $\varrho \ge 0$ is *admissible* for a family Γ of paths if $\int_\gamma \varrho \ge 1$ for all $\gamma \in \Gamma$, and Γ has *p-modulus zero* if the infimum over all admissible ϱ of $\int_X \varrho^p$ is zero.

Say that a measurable function τ is a *p-weak upper gradient* of a function w if for every rectifiable path γ, except for a family of p-modulus zero, the following estimate holds:

$$(7)\qquad |w(a_\gamma) - w(b_\gamma)| \le \int_\gamma \tau$$

where a_γ, b_γ are the endpoints of γ. When $p = Q$, or when we are not interested in the precise value of p, we simply refer to weak upper gradients.

The next lemma implies that weak upper gradients can be used instead of upper gradients in integral estimates.

LEMMA 2.4. *If τ is a p-weak upper gradient of w, then for all $q \le p$ there is a decreasing sequence $\{\tau_n\}$ of upper gradients of w for which* $\|\tau_n - \tau\|_q \to 0$.

We need another lemma first.

LEMMA 2.5. *If a path family Γ has p-modulus zero, and every path in Γ is bounded, then Γ has q-modulus zero for all $q \le p$.*

Proof. Choose a point x_0 in X. Set $B_k = B(x_0, k)$. Define Γ_k to be all those paths of Γ that lie in B_k. Then $\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k$. It suffices to show that Γ_k has q-modulus zero. This is trivial because the measure of B_k is finite. ∎

Proof of Lemma 2.4. Let Γ denote the exceptional path family in the definition of weak upper gradient. All the elements of Γ are rectifiable, and so bounded. The previous lemma says that Γ has q-modulus zero. Therefore there exist $f_j \ge 0$ such that $\int_\gamma f_j \ge 1$ for all $\gamma \in \Gamma$ and $\|f_j\|_q \le 2^{-j}$. Define τ_n to be $\tau + \sum_{j=n}^{\infty} f_j$. Clearly, the τ_n are decreasing and $\|\tau_n - \tau\|_q \to 0$. Note also that $\int_\gamma \tau_n = \infty$ for all $\gamma \in \Gamma$ and that $\int_\gamma \tau_n \ge \int_\gamma \tau$ for all rectifiable γ. Thus each τ_n is an upper gradient of w. ∎

Returning now to the proof of Theorem 2.3, we will show below that $g \sim \tau$ a.e., for some τ that is a weak upper gradient of u (we are using $\sim$ to indicate that the quantities are comparable with constants depending only on f and on the spaces). It follows from this that there is a decreasing sequence $\{\tau_n\}$ of upper gradients of u for which $\|\tau_n - \tau\|_Q \to 0$. Each pair (u, τ_n) satisfies a $(1,p)$-Poincaré inequality, thus (u, τ), and as a result (u, g), must also satisfy a $(1,p)$-Poincaré inequality. Theorem 2.1 now implies that $(v, h) = f(u, g)$ satisfies a $(1,p')$-Poincaré inequality, as required.

Define

$$L_f(x) = \lim_{r\to 0}\Big(\sup_{|y-x|\le r}\frac{|f(x)-f(y)|}{r}\Big) \quad \text{and} \quad \tau = (h \circ f)L_f.$$

Note that by regularity of the spaces, and the quasisymmetry and absolute continuity of f, we have $L_f \sim J_f^{1/Q}$ a.e. Thus $g \sim \tau$ a.e. It remains to check that τ is a weak upper gradient of u.

The proof of Theorem 8.1 of [HeK2] shows that our assumptions on f are enough to ensure that f is absolutely continuous on every rectifiable path in X, except for a family of Q-modulus zero. When we say that f is absolutely continuous on a path we mean that $f \circ \sigma$ is absolutely continuous, where σ is the arclength parameterization of the path. Consider a rectifiable path γ on which f is absolutely continuous. Then $\widetilde{\gamma} = f(\gamma)$ is a rectifiable path in Y. Denote the endpoints of γ by a and b. Because h is an upper gradient of v we have

$$|u(b) - u(a)| = |v(f(a)) - v(f(b))| \le \int_{\widetilde{\gamma}} h.$$

Since f is absolutely continuous on γ, we can do a change of variables and obtain

$$\int_{\widetilde{\gamma}} h = \int_\gamma (h \circ f)|f'|,$$

where $|f'|(\sigma(y))$ denotes $|(f\circ\sigma)'|(\sigma(y))$. It is clear that $|f'|(\sigma(y)) \le L_f(\sigma(y))$, and thus we have

$$|u(b) - u(a)| \le \int_\gamma (h \circ f)L_f = \int_\gamma \tau.$$

So τ is indeed a weak upper gradient of u. ∎

3. The Cantor diamond. We are going to construct subsets X_λ of $\mathbb{C}$ which are 2-regular and support a $(1, p(\lambda))$-Poincaré inequality, where $p(\lambda) > 2$ and $p(\lambda) \to 2$ as $\lambda \to 0$. We will then show that for each X_λ there is a quasisymmetric map from X_λ onto another 2-regular space, Y_λ, whose pullback measure is A_∞-related to the measure on X, yet Y_λ does not support *any* Poincaré inequality. In particular, this demonstrates that Theorem 8 is false in general for $p > Q$.

The restrictions of the Euclidean metric and Lebesgue measure furnish a metric and a measure for the various spaces considered here.

E_λ denotes the Cantor set in $[0,1]$ obtained in the usual way by first taking out an interval of length $1-\lambda$ and leaving two intervals of length $\lambda/2$ and then continuing inductively. The dimension d_λ of E_λ is $\log 2$ divided by $\log(2/\lambda)$. The space X_λ in which we are interested is obtained by replacing each of the complementary intervals of E_λ by a square having that interval as one of its diagonals. Thus we have a line of diamonds along the unit interval, and they are joined up by E_λ. X_λ satisfies our standard assumptions. In addition, it is compact, quasiconvex, and 2-regular.

THEOREM 3.1. *X_λ supports a $(1,p)$-Poincaré inequality for each*

$$p > \frac{2 - d_\lambda}{1 - d_\lambda}.$$

Proof. Fix a p in the indicated range. For any two points a and b in X_λ, we set $B_{ab} = B(a, |b-a|) \cup B(b, |b-a|)$.

Suppose that g is an upper gradient of a function u on X_λ. If we can show that for all a, b,

$$|u(b) - u(a)| \le C|b-a|^{1-2/p}\Big(\int_{B_{ab}} g^p\Big)^{1/p}, \tag{8}$$

then we easily get the Poincaré inequality. Inequality (8) holds whenever a and b lie in the same diamond, as each diamond is just like $\mathbb{R}^2$. We show below that the inequality also holds whenever $a, b \in E_\lambda$. The general case is a simple consequence of these two special cases.

The map $F(x,y) = (x, \delta(x)\tan(\pi y/4))$ is a Lipschitz map from $Q_0 = [0,1]\times[-1,1]$ onto X_λ. Here $\delta(x)$ is the distance from x to E_λ. On $E_\lambda \times [-1,1]$ the map F is simply vertical projection. On S_λ, the complement of $E_\lambda \times [-1,1]$, F is one-to-one and locally bi-Lipschitz.

Set $u' = u\circ F$ and $g' = g \circ F$. The map u' is constant on each of the vertical fibres of $E_\lambda\times[-1,1]$. By a fibre, we simply mean a line segment contained in Q_0. Clearly, g' is an upper gradient for u' on each of the horizontal fibres. On each horizontal fibre, F is 4-bi-Lipschitz. We can now conclude that if $a, b \in E_\lambda$, then

$$|u(b)-u(a)| = |u'(b)-u'(a)| \le C\int_{R'_{ab}} g' = C\int_{R'_{ab}\cap S_\lambda} g'$$

where R'_{ab} is $[a,b]\times[-1,1]$. Let $R_{ab} = F(R'_{ab})$ and $D_{ab} = F(R'_{ab}\cap S_\lambda)$. The volume derivative of F at a point $w = (x,y)$ in S_λ is essentially $\delta(x)$. This in turn is comparable to $\delta(F(w))$. Performing the change of variables in the inequality above, we obtain

$$|u(b)-u(a)| \le C\int_{D_{ab}} \frac{g(z)}{\delta(z)} \le C\Big(\int_{D_{ab}} g^p\Big)^{1/p}\Big(\int_{D_{ab}}\delta^{-q}\Big)^{1/q}.$$

The construction of E_λ guarantees that $\int_{D_{ab}}\delta^{-q} \le C|a-b|^{2-q}$. This yields the required estimate:

$$|u(b)-u(a)| \le C|a-b|^{1-2/p}\Big(\int_{D_{ab}} g^p\Big)^{1/p} \le C|a-b|^{1-2/p}\Big(\int_{B_{ab}} g^p\Big)^{1/p}. \ ■$$

REMARK. The usual chaining argument (Theorem 5.1 of [HaK2], for example) shows that if (u,g) satisfies a $(1,p)$-Poincaré inequality, with $p > Q$, on some Q-regular space X, then a Hölder estimate of the type (8) holds for a.e. a and b.

Fix λ. It is easy to see that there is a quasisymmetric map from $\mathbb{R}$ onto $\mathbb{R}$ that maps E_λ onto $E_{1/3}$, the usual Cantor set. This map can be extended to a quasiconformal (and hence quasisymmetric) map of the plane. Lemma 2.3 of [K2] shows that for any $1/2 < \alpha \le 1$ there is a quasiconformal map of the plane which satisfies $|f(x)-f(y)| \sim |x-y|^\alpha$ for all $x, y \in \mathbb{R}$. In particular, there is a map sending $E_{1/3}$ to a regular set of dimension β for some $\beta > 1$. By composition, we get a quasiconformal map f of the plane that maps E_λ onto a regular set of dimension $\beta > 1$. As f is quasisymmetric, it is easily seen that $Y_\lambda = F(X_\lambda)$ is 2-regular and satisfies our standard assumptions.

We can assume that f fixes 0 and 1. Define a function u on Y_λ by $u(y) = \inf_\gamma H_\beta(\gamma)$, where H_β denotes β-dimensional Hausdorff measure and the infimum is taken over all curves in Y_λ joining y and 0. Note that u is constant on the images of the diamonds and that $0 = u(0) \le u(y) \le u(1) = H_\beta(f(E_\lambda)) < \infty$. For any two points a, b in Y_λ we have

$$|u(b)-u(a)| \le C|b-a|^\beta.$$

This follows from the β-regularity of $f(E_\lambda)$ and the quasisymmetry of f. In particular, u is continuous. Because the exponent is greater than 1, u must be constant along any rectifiable path. Consequently, 0 is an upper gradient for u. But u is non-constant and so Y_λ cannot support any Poincaré inequality.

4. Sobolev spaces for $p > 1$. If X is a Riemannian manifold, the Sobolev space $L^{1,p}(X)$ is the space of locally integrable functions with weak derivatives in L^p. This definition makes no sense for more general X as derivatives will no longer be defined. We shall give two ways of defining the Sobolev spaces, for the case $p > 1$, on metric spaces. The first is due to Hajłasz [Ha], the second is a modification of a definition of Korevaar and Schoen [KS]. Both definitions involve some sort of modulus of continuity. We aim to show that these spaces are equal, for reasonable spaces, and that they are, in fact, the same as $P^{1,p}(X)$. This will then allow us to prove a general theorem on preservation of the Sobolev space $L^{1,Q}$ under quasiconformal maps.

In this section we do not need X to be a regular space. It is sufficient for the measure to be doubling on X, i.e., that μ is non-zero and $\mu(2B) \le C\mu(B)$ for every ball B. The reason is that the key estimate, Lemma 4.3(ii), holds with this hypothesis. Note that the doubling condition implies that every ball in X has positive measure.

Say that $u \in \mathcal{M}^{1,p}(X)$ if there exists $g \in L^p(X)$ such that for a.e. x, y the inequality

$$|u(x)-u(y)| \le |x-y|(g(x)+g(y)) \tag{9}$$

holds. We can define a seminorm on $\mathcal{M}^{1,p}(X)$ be setting $\|u\| = \inf\|g\|_{L^p}$ where the inf is taken over all g satisfying the above inequality. In $\mathbb{R}^Q$, this definition yields the usual Sobolev space and the seminorm is equivalent to the usual seminorm (see [Ha]).

For $\varepsilon > 0$, write

$$e_\varepsilon(x,y;u) = \frac{|u(x)-u(y)|}{\varepsilon} \quad\text{and}\quad e_\varepsilon^p(x;u) = \int_{B(x,\varepsilon)} e_\varepsilon^p(x,y;u)\,d\mu(y).$$

The Sobolev space of Korevaar and Schoen consists of those functions for which

$$\sup_f \Big(\limsup_{\varepsilon\to 0} \int_X f(x) e^p_\varepsilon(x;u)\, d\mu(x) \Big) < \infty, \tag{10}$$

where the sup is taken over functions in $C_c(X)$ with values in the range $[0,1]$. When X is a Riemannian manifold this definition yields the usual Sobolev space and the quantity in (10) is equivalent to the usual seminorm (see [KS]). We will change this definition slightly. Set

$$E^p(u,X) = \sup_B \Big(\limsup_{\varepsilon\to 0} \int_B e^p_\varepsilon(x;u)\, d\mu(x) \Big),$$

where the sup is taken over all balls. The function u is said to be in $\mathcal{L}^{1,p}(X)$ if $E^p(u,X)$ is finite. It is clear that this space is the same as that of Korevaar and Schoen when X is proper, i.e., when the closure of every ball is compact. In particular, we get the usual Sobolev space when X is Euclidean space.

THEOREM 4.1. *We have the following inclusions*:

$$\mathcal{M}^{1,p}(X) \subseteq P^{1,p}(X) \subseteq \mathcal{L}^{1,p}(X).$$

The first inclusion is trivial, while the second is proven below. Theorem 4.5 below gives a sufficient condition for the spaces $P^{1,p}(X), \mathcal{L}^{1,p}(X)$, and $\mathcal{M}^{1,p}(X)$ to be equal, while Proposition 4.4 considers the reverse of the first inclusion above.

COROLLARY 4.2. *All the above spaces coincide when* $X = \mathbb{R}^Q$.

For proving the theorem, we gather some facts about Poincaré inequalities and Riesz potentials together in a lemma. Set

$$J_p(g,r,x) = \sum_{k=0}^{\infty} 2^{-k} r \Big(\fint_{B(x,2^{-k}r)} |g|^p \, d\mu \Big)^{1/p}.$$

This is a minor variant of the generalized Riesz potentials defined in [HaK2].

LEMMA 4.3. (i) *If the pair* (u,g) *satisfies a* $(1,p)$*-Poincaré inequality, then for almost every* x,y *we have*

$$|u(x)-u(y)| \le C(J_p(g,r_{xy},x) + J_p(g,r_{xy},y)),$$

where $r_{xy} = 2|x-y|$.

(ii) *There is a constant* C, *independent of* ε, *such that for any* $x \in X$ *and any* $0 < \varepsilon \le 2 \operatorname{diam} X$ *we have*

$$\int_{B(x,\varepsilon)} (J_p(g,\varepsilon,y))^p \, d\mu(y) \le C\varepsilon^p \int_{B(x,2\varepsilon)} g^p \, d\mu.$$

(iii) *There is a constant* C, *independent of* ε, *such that for any* $0 < \varepsilon \le 2 \operatorname{diam} X$ *we have*

$$\int_X (J_p(g,\varepsilon,y))^p \, d\mu(y) \le C\varepsilon^p \int_X g^p \, d\mu.$$

Part (i) follows from the usual chaining argument (see the proof of Theorem 3.2 in [HaK2], for example). The second part follows from Theorem 5.3 of [HaK2] and its proof. The third part follows from (ii) and a covering argument.

Proof of Theorem 4.1. Suppose now that (u,g) satisfies a $(1,p)$-Poincaré inequality and that $g \in L^p(X)$. We abbreviate $J_p(g,r,x)$ to $J(r,x)$. Part (i) of Lemma 4.3 implies that for a.e. x and a.e. $y \in B(x,\varepsilon)$ the following inequality holds:

$$|u(x)-u(y)| \le C(J(2\varepsilon,x) + J(2\varepsilon,y)).$$

From this estimate and parts (ii) and (iii) of the lemma we have

$$\int_X e^p_\varepsilon(x;u)\, d\mu(x) \le C \int_X \Big(\varepsilon^{-p} J(2\varepsilon,x) + \fint_{B(x,4\varepsilon)} g^p \, d\mu \Big) d\mu(x) \le C \int_X g^p \, d\mu.$$

Consequently, $E^p(u,X) \le C \int_X g^p \, d\mu < \infty$ and so $u \in \mathcal{L}^{1,p}(X)$. ■

Lemma 4.3 also allows us to describe more precisely the relationship between $\mathcal{M}^{1,p}(X)$ and $P^{1,p}(X)$.

PROPOSITION 4.4. (i) *If* u *satisfies a* $(1,p)$*-Poincaré inequality, then there is a function* h *in weak* L^p *of* X *for which* $|u(x)-u(y)| \le |x-y|(h(x)+h(y))$ *for a.e.* x *and* y *in* X.

(ii) *If* $\operatorname{diam} X < \infty$, *then* $P^{1,p}(X) \subseteq \mathcal{M}^{1,q}(X)$ *for all* $1 < q < p$.

(iii) *If* (u,g) *satisfies a* $(1,q)$*-Poincaré inequality for some* $1 < q < p$, *and* $g \in L^p(X)$, *then* $u \in \mathcal{M}^{1,p}(X)$.

Proof. The potential $J_p(g,r,x)$ can be trivially estimated by $2r(Mg^p)^{1/p}(x)$, where M denotes the usual maximal function. Because g^p is in L^1, the corresponding maximal function is in weak L^1. Putting these facts together with part (i) of Lemma 4.3, we obtain (i) above.

Part (ii) follows easily from (i).

For (iii), note that the inequality of (i) is satisfied for $h = (Mg^q)^{1/q}$. But g^q is in $L^{p/q}$ and p/q is greater than 1, therefore Mg^q is in $L^{p/q}$. It follows that $h \in L^p$, and this implies that $u \in \mathcal{M}^{1,p}(X)$. ■

This proposition tells us that the reverse of the first inclusion in Theorem 4.1 is almost true. One of the questions raised by Hajłasz and Koskela in [HaK2] is whether (u,g) satisfying a $(1,p)$-Poincaré inequality with $g \in L^p$ actually implies that (u,g) satisfies a $(1,q)$-Poincaré inequality for some $q < p$. If this were the case, then part (iii) above would imply that there is actually equality in Theorem 4.1. This is a delicate question because it is known that there are spaces that support a $(1,p)$-Poincaré inequality but do not support a $(1,q)$-Poincaré inequality for any $q<p$; nevertheless, the answer to Hajłasz and Koskela's question is yes in these spaces (see [K1]).

We next present a sufficient condition for the various Sobolev spaces to be equal.

THEOREM 4.5. *If X supports a $(1,q)$-Poincaré inequality for some $1\le q<p$, then $P^{1,p}(X)=\mathcal{L}^{1,p}(X)=\mathcal{M}^{1,p}(X)$.*

Proof. By Theorem 4.1, it suffices to show that $\mathcal{L}^{1,p}(X)\subseteq\mathcal{M}^{1,p}(X)$. Suppose that $u\in\mathcal{L}^{1,p}(X)$. A priori, we do not even know that u has an upper gradient. To enable ourselves to apply our Poincaré inequality, we will approximate u by functions which have upper gradients, which we can estimate, and then take a limit in order to get the required information about u.

Let ε be some positive number. By the usual covering arguments there is a subcollection $\{B_i\}$ of the set $\{B(x,\varepsilon)\}_{x\in X}$ of balls which is at most countable and for which $X=\bigcup B_i$ and the balls $\frac15 B_i$ are disjoint. This last property implies, for example, that the number of B_i containing a given point is uniformly bounded. By following the standard construction of a partition of unity, we obtain non-negative functions ϕ_i that are $C\varepsilon^{-1}$-Lipschitz, and for which $\operatorname{supp}\phi_i\subseteq B_i$ and $\sum\phi_i=1$.

Next we check that u is integrable over any ball. A little bit of work combining the definition of $\mathcal{L}^{1,p}(X)$ and the fact that all balls have positive measure reveals that given any ball B there is some $\delta>0$ with the property that $e^p_\delta(x;u)$ is finite for all x in B. In particular, u is integrable over $B(x,\delta)$ for all $x\in B$. The covering property described in the previous paragraph, along with the doubling property of μ, can be used to show that B can be covered by a finite number of these balls $B(x,\delta)$. It follows immediately that u is integrable over B.

Now define h_ε to be $\sum\lambda_i\phi_i$, where λ_i is the average value of h over the ball B_i. The L^1 norm of h_ε is bounded (independently of ε) by a multiple of the L^1 norm of h, and when h is continuous one easily verifies that $\int_B|h-h_\varepsilon|\to0$ as $\varepsilon\to0$ for any ball B. It follows that this property holds whenever h is integrable over all balls.

LEMMA 4.6. *If $|b-a|<\varepsilon$, then*
$$|u_\varepsilon(b)-u_\varepsilon(a)|\le M|b-a|\fint_{B(a,2\varepsilon)}e^1_{5\varepsilon}(x;u)\,d\mu(x).$$

Proof. Label the B_i so that $a\in B_0$. Then
$$\begin{aligned}u_\varepsilon(b)-u_\varepsilon(a)&=\sum_i\lambda_i\phi_i(b)-\sum\lambda_i\phi_i(a)\\&=\sum_i(\lambda_i-\lambda_0)\phi_i(b)-\sum_i(\lambda_i-\lambda_0)\phi_i(a)\\&=\sum_i(\lambda_i-\lambda_0)(\phi_i(b)-\phi_i(a)).\end{aligned}$$

Now if $\{a,b\}\cap B_i=\emptyset$, then $\phi_i(b)=\phi_i(a)=0$. There are at most C of the B_i for which $\{a,b\}\cap B_i\ne\emptyset$, so the above sum has at most C terms. Consider one of the B_i for which $\{a,b\}\cap B_i\ne\emptyset$. We have
$$\lambda_i-\lambda_0=\fint_{B_i}u-\fint_{B_0}u=\fint_{B_0}\fint_{B_i}(u(y)-u(x))\,d\mu(y)\,d\mu(x).$$
Taking absolute values and taking into account some obvious inclusions we find that
$$|\lambda_i-\lambda_0|\le C\fint_{B(a,2\varepsilon)}\fint_{B(x,5\varepsilon)}|u(y)-u(x)|\,d\mu(y)\,d\mu(x).$$
Recalling our earlier definitions, we see that
$$\begin{aligned}|\lambda_i-\lambda_0|&\le C\varepsilon\fint_{B(a,2\varepsilon)}\fint_{B(x,5\varepsilon)}e^1_{5\varepsilon}(x,y;u)\,d\mu(y)\,d\mu(x)\\&=C\varepsilon\fint_{B(a,2\varepsilon)}e^1_{5\varepsilon}(x;u)\,d\mu(x).\end{aligned}$$
Combining the various inequalities we obtain the inequality we seek:
$$\begin{aligned}|u_\varepsilon(b)-u_\varepsilon(a)|&\le\sum|\lambda_i-\lambda_0|\cdot|\phi_i(b)-\phi_i(a)|\\&\le\sum|\lambda_i-\lambda_0|C\varepsilon^{-1}|b-a|\\&\le M|b-a|\fint_{B(a,2\varepsilon)}e^1_{5\varepsilon}(x;u)\,d\mu(x).\ \blacksquare\end{aligned}$$

Set
$$g_\varepsilon(a)=\fint_{B(a,2\varepsilon)}e^1_{5\varepsilon}(x;u)\,d\mu(x).$$
The lemma above says that $|u_\varepsilon(b)-u_\varepsilon(a)|\le C|b-a|g_\varepsilon(a)$ whenever $|b-a|\le\varepsilon$. Because u is integrable over any ball, the function g_ε is uniformly bounded on each ball and so u_ε is locally Lipschitz and has Mg_ε as an upper gradient.

We want to get estimates on the L^p norm of the g_ε. We first note that $e^1_{5\varepsilon}(x;u)$ is bounded by $(e^p_{5\varepsilon}(x;u))^{1/p}$. As a consequence, we obtain the estimate
$$g_\varepsilon(a)\le\Big(\fint_{B(a,2\varepsilon)}e^p_{5\varepsilon}(x;u)\,d\mu(x)\Big)^{1/p}.$$
By Fubini's theorem we deduce that $\int_B g^p_\varepsilon\le C\int_{2B}e^p_{5\varepsilon}(x;u)\,d\mu(x)$ whenever B is a ball of diameter at least 2ε. Now we make use of the definition of

$\mathcal{L}^{1,p}(X)$ to deduce that for any ball B,

$$\limsup_{\varepsilon\to 0}\int_B g_\varepsilon^p \le E^p(u,X) < \infty. \tag{11}$$

Next we want to find a convergent subsequence of the g_ε, because we will need to take limits later on. The bound (11) implies that for each natural number N there exists $\varepsilon_N > 0$ such that $\int_{B_N} g_{\varepsilon_N}^p \le 2E^p(u,X)$. Here B_N is the ball of radius N centred at some fixed point of X. Let $G_N = g_{\varepsilon_N}\chi_{B_N}$. Then $\|G_N\|_p^p \le 2E^p(u,X)$ for all N. This last can be rewritten as $\|G_N^q\|_{p/q}^{p/q} \le 2E^p(u,X)$ for all N. Some subsequence of the G_N^q converges weakly in $L^{p/q}$ to $H\in L^{p/q}$, as $p>q$. Write G for $H^{1/q}$. Then $G\in L^p$ and G_N^q converges weakly to G^q in $L^{p/q}$.

We are finally in a position to use the Poincaré inequality which X supports. The function u_ε is continuous and has g_ε as an upper gradient. Thus for any ball B of radius r we have

$$\fint_B |u_\varepsilon - (u_\varepsilon)_B|\,d\mu \le Cr\Big(\fint_{\lambda B} g_\varepsilon^q\,d\mu\Big)^{1/q}.$$

We can now switch to the cut-off subsequence G_N of the g_ε that we obtained above and take a limit to find that (u,G) satisfies a $(1,q)$-Poincaré inequality. But G lies in L^p, so by Proposition 4.4 the function u must lie in $\mathcal{M}^{1,p}(X)$. ■

THEOREM 4.7. *Suppose that f is a quasiconformal map from X onto Y, that X, Y, and f satisfy (‡), and that X supports a $(1,p)$-Poincaré inequality for some $1<p<Q$. Then $u\in\mathcal{L}^{1,Q}(X)$ (resp. $u\in\mathcal{M}^{1,Q}(X)$) if and only if $u\circ f^{-1}\in\mathcal{L}^{1,Q}(Y)$ (resp. $u\circ f^{-1}\in\mathcal{M}^{1,Q}(Y)$).*

Proof. Our first two assumptions, and Theorem 1.2, guarantee that f satisfies the standing hypotheses of Section 2, i.e., it is quasisymmetric and its pullback measure is A_∞-related to μ, the measure on X. The same is true for f^{-1}. Corollary 2.2 implies that $u\in P^{1,Q}(X)$ if and only if $u\circ f^{-1}\in P^{1,Q}(Y)$.

Theorem 4.5 implies that $P^{1,Q}(X)=\mathcal{L}^{1,Q}(X)=\mathcal{M}^{1,Q}(X)$, because X supports a $(1,p)$-Poincaré inequality for some $p<Q$. However, Theorem 2.3 says that Y supports a $(1,p')$-Poincaré inequality for some $p'<Q$, and so $P^{1,Q}(Y)=\mathcal{L}^{1,Q}(Y)=\mathcal{M}^{1,Q}(Y)$. The theorem follows. ■

Our results in this section can be refined somewhat by taking into account dependence on seminorms. For example, in Theorem 4.5 the spaces $\mathcal{L}^{1,p}(X)$ and $\mathcal{M}^{1,p}(X)$ are not only equal (as sets) but also the seminorms are comparable. In Theorem 4.7, the seminorms of u and $u\circ f^{-1}$ are also comparable. The details are left to the reader.

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