STUDIA MATHEMATICA 130 (3) (1998)

A density theorem for algebra representations on the space (s)

by

W. ŻELAZKO (Warszawa)

Abstract. We show that an arbitrary irreducible representation $\mathbf{T}$ of a real or complex algebra on the F-space (s), or, more generally, on an arbitrary infinite (topological) product of the field of scalars, is totally irreducible, provided its commutant is trivial. This provides an affirmative solution to a problem of Fell and Doran for representations on these spaces.

Let X be a real or complex locally convex space and $\mathcal{A}$ an algebra over the same field of scalars as X. A *representation* $\mathbf{T}$ of $\mathcal{A}$ on X is an algebra homomorphism $a \mapsto T_a$ of $\mathcal{A}$ into the algebra $L(X)$ of all continuous endomorphisms of X. We say that $\mathbf{T}$ is *irreducible* if no proper closed linear subspace of X is invariant for all the operators T_a, $a \in \mathcal{A}$. Thus $\mathbf{T}$ is irreducible if and only if for each non-zero x in X the orbit $\mathcal{O}(\mathbf{T}; x) = \{T_a x : a \in \mathcal{A}\}$ is dense in X.

A representation $\mathbf{T}$ is called *totally irreducible* (cf. [3]) if for each natural n and for each n-tuple $x_1, \ldots, x_n$ of linearly independent elements of X the multiple orbit $\mathcal{O}(\mathbf{T}; x_1, \ldots, x_n) = \{(T_a x_1, \ldots, T_a x_n) \in X^n : a \in \mathcal{A}\}$ is dense in X^n in the product topology. Thus $\mathbf{T}$ is totally irreducible if and only if the algebra $\mathbf{T}\mathcal{A} = \{T_a : a \in \mathcal{A}\}$ is dense in $L(X)$ in the strong operator topology (the topology of pointwise convergence on X).

The *commutant* of $\mathbf{T}$ is defined as $\mathbf{T}' = \{T \in L(X) : TT_a = T_a T$ for all $a \in \mathcal{A}\}$. We say that $\mathbf{T}'$ is *trivial* if it consists of the scalar multiples of the identity operator only.

Fell and Doran asked in [3] (Problem II, p. 329) the following question.

PROBLEM. Let X be a locally convex space and $\mathbf{T}$ an irreducible representation on X with trivial commutant of an algebra $\mathcal{A}$ over the same field of scalars as X. Does it follow that $\mathbf{T}$ is totally irreducible?

The above problem asks whether the topological version of the celebrated Jacobson Density Theorem (cf. below) holds true. Let X be a real or complex vector space. An *algebraic representation* of an algebra $\mathcal{A}$ on X

1991 *Mathematics Subject Classification*: 47A67, 16G30.

is a homomorphism of $\mathcal{A}$ into the algebra of all endomorphisms of X and it is called (*algebraically*) *irreducible* if the orbit of every non-zero $x \in X$ is equal to X, in particular no proper subspace of X can be invariant for all T_a. Generally speaking, an irreducible representation on a locally convex space does not need to be algebraically irreducible. The following result is a consequence of the Jacobson Density Theorem (cf. [2], p. 123, Theorem 10; [3], pp. 283–286; [4], pp. 271–274).

THEOREM J. *Let X be a real or complex vector space and let $\mathcal{A}$ be an algebra over the same field of scalars as X. Let $\mathbf{T}$ be an algebraically irreducible representation of $\mathcal{A}$ on X such that every endomorphism of X commuting with all the operators T_a is a scalar multiple of the identity operator. Then for each linearly independent n-tuple $x_1, \ldots, x_n$ in X the orbit $\mathcal{O}(\mathbf{T}; x_1, \ldots, x_n)$ coincides with X^n.*

The above result does not imply an affirmative answer to the above Problem even when $\mathbf{T}$ is algebraically irreducible, since the triviality of $\mathbf{T}'$ does not necessarily mean that the endomorphisms of X commuting with all T_a must coincide with the scalar multiples of the identity. For algebraically irreducible representations on spaces of type F such an affirmative answer was given in [8]. No example has been known of an infinite-dimensional Banach space X for which the answer to the Problem is in the affirmative, nor is there an example of a locally convex space for which the answer is negative, though the general belief is that such an example should exist.

In this paper we give an affirmative answer to the Problem in the case when X is the classical space (s) (cf. [1], example p. 10; we use Banach's notation which is also repeated in [6]), or, more generally when X is an infinite product of the field of scalars.

The space (s) consists of all scalar-valued sequences $x = (\xi_i)_{i=1}^{\infty}$ with the topology of coordinatewise convergence. It is a completely metrizable locally convex space (cf. [1] or [6]) and its topology can be given by means of the increasing sequence of seminorms

$$\|x\|_n = \max\{|\xi_i| : 1 \le i \le n\}. \tag{1}$$

Clearly (s) is topologically isomorphic to the countable product of the field of scalars. We shall need the following facts about (s), or, more generally, about any locally convex space X topologically isomorphic to an arbitrary product of the field of scalars with the product topology (see [5], pp. 287–292, also §18.5, p. 214).

(I) The space X is reflexive.

(II) The topology of the dual space X^* is the maximal locally convex topology $\tau_{\max}^{\mathrm{LC}}$, it is given by means of all the seminorms on X^*.

It is known (see e.g. [7], example on p. 56) that for every vector space X, the topology $\tau_{\max}^{\mathrm{LC}}$ is complete and under this topology all linear functionals and all endomorphisms of X are continuous, and also all linear subspaces of X are closed.

Now, let $\mathbf{T}$ be an arbitrary representation of $\mathcal{A}$ on a locally convex space X, and denote by $\mathbf{T}^*$ the map $a \mapsto T_a^*$, where T^* is the adjoint of a (continuous) linear endomorphism T on X. Since $T_{ab}^* = T_b^* T_a^*$, this map is a representation of $\mathcal{A}_{\mathrm{rev}}$ on X^*, where $\mathcal{A}_{\mathrm{rev}}$ is the reverse algebra of $\mathcal{A}$, i.e., $\mathcal{A}$ provided with the reverse multiplication $a.b = ba$.

We now can prove our result.

THEOREM. *Let $\mathbf{T}$ be an irreducible representation of an algebra $\mathcal{A}$ on the space (s) and assume that $\mathbf{T}$ has a trivial commutant. Then $\mathbf{T}$ is totally irreducible.*

Proof. Observe first that the representation $\mathbf{T}^*$ of $\mathcal{A}_{\mathrm{rev}}$ on $(s)^*$ is also irreducible and has a trivial commutant. If not, there is a proper closed linear subspace $Y \subset (s)^*$ invariant for all operators T_a^*, or an operator $S \in (\mathbf{T}^*)'$ which is not a scalar multiple of the identity. But then by (I) the linear space $Y^\perp = \{F \in (s)^{**} = (s) : Y \subset \ker F\}$ is a proper closed subspace of s which is invariant with respect to all operators $T_a = T_a^{**}$, or $S^* \in \mathbf{T}'$, while it is not a scalar multiple of the identity, both excluded by the assumption.

Since, by (II), all linear subspaces of $(s)^*$ are closed and all its endomorphisms are continuous, the representation $\mathbf{T}^*$ is algebraically irreducible and has a trivial commutant. Theorem J now implies that for any n-tuples $f_1, \ldots, f_n$, $g_1, \ldots, g_n$, of functionals in $(s)^*$, the g_i being linearly independent, there is a in $\mathcal{A}$ with

$$T_a^* g_i = f_i, \quad i = 1, \ldots, n. \tag{2}$$

Take as g_i the coordinate functionals $g_i(x) = \xi_i$ for $x = (\xi_i) \in (s)$. Clearly $g_i \in (s)^*$. Let now $x_1, \ldots, x_k$, $y_1, \ldots, y_k$ be arbitrary k-tuples of elements of (s), the x_i being linearly independent. We shall be done if for each natural n we find a in $\mathcal{A}$ with

$$\|T_a x_i - y_i\|_n = 0, \quad i = 1, \ldots, k. \tag{3}$$

First we take functionals $h_1, \ldots, h_k$ in $(s)^*$ biorthogonal to x_i, i.e. $h_i(x_j) = \delta_{i,j}$, $1 \le i, j \le k$. Put $\xi_j^{(m)} = g_j(y_m)$, $1 \le m \le k$, $j = 1, 2, \ldots$, and $f_j = \sum_{m=1}^{k} \xi_j^{(m)} h_m$, $j = 1, 2, \ldots$ Fix n and find a in $\mathcal{A}$ so that (2) holds true for f_i and g_i so defined. For all j we have $f_j(x_m) = \xi_j^{(m)} = g_j(y_m)$, $1 \le m \le k$. Thus, by (2) we obtain

$$g_j(T_a x_m) = f_j(x_m) = g_j(y_m), \quad 1 \le m \le k, \ 1 \le j \le n,$$

which, by (1), implies the formula (3). The conclusion follows.

The above reasoning can be applied to any product space $X = \mathbb{K}^J$, where $\mathbb{K} = \mathbb{C}$ or $\mathbf{R}$ and J is an arbitrary index set. We have distinguished here a countable J since only in this case is the space $\mathbb{K}^J = (s)$ infinite-dimensional and metrizable. The Problem of Fell and Doran can be trivially solved (by use of Theorem J) for any vector space equipped with the topology $\tau_{\max}^{\mathrm{LC}}$, or (as observed in [3], p. 327) with the topology $\tau_{\max}^{w}$, the maximal weak topology given by means of all seminorms of the form $x \mapsto |f(x)|$, where f is an arbitrary linear functional on the space this question. For $\tau_{\max}^{w}$ all linear subspaces are also closed and all endomorphisms are continuous (it is a minimal locally convex topology with these properties), so that all irreducible representations are algebraically irreducible.

It seems that the infinite products of the field of scalars are the only known locally convex spaces for which the Fell and Doran Problem has an affirmative answer and for which the irreducible representations do not coincide with algebraically irreducible ones. For more information on the Fell and Doran Problem for Banach spaces the reader is referred to [9].

The author wishes to express his thanks to the referee for many improvements of the paper.

References

[1] S. Banach, *Théorie des Opérations Linéaires*, Warszawa, 1932.
[2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, 1973.
[3] J. M. G. Fell and R. S. Doran, *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles*, Vol. I, Academic Press, 1988.
[4] N. Jacobson, *Lectures in Abstract Algebra*, Vol. II, van Nostrand, 1953.
[5] G. Köthe, *Topological Vector Spaces I*, Springer, 1969.
[6] S. Rolewicz, *Metric Linear Spaces*, PWN and Reidel, 1984.
[7] H. H. Schaefer, *Topological Vector Spaces*, Springer, 1971.
[8] W. Żelazko, *A density theorem for F-spaces*, Studia Math. 96 (1990), 159–166.
[9] —, *On a problem of Fell and Doran*, Colloq. Math. 62 (1991), 31–37.

Institute of Mathematics
Polish Academy of Sciences
P.O. Box 137
Śniadeckich 8
00-950 Warszawa, Poland
E-mail: zelazko@impan.impan.gov.pl

Received February 3, 1998
Revised version March 24, 1998
(4047)

STUDIA MATHEMATICA 130 (3) (1998)

Index of Volumes 121–130