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Received October 10, 1997 (3972)

STUDIA MATHEMATICA 130 (3) (1998)

Derivations with a hereditary domain, II

by

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Abstract. The nilpotency of the separating subspace of an everywhere defined derivation on a Banach algebra is an intriguing question which remains still unsolved, even for commutative Banach algebras. On the other hand, closability of partially defined derivations on Banach algebras is a fundamental problem motivated by the study of time evolution of quantum systems. We show that the separating subspace $\mathcal{S}(D)$ of a Jordan derivation defined on a subalgebra B of a complex Banach algebra A satisfies $B[B\cap\mathcal{S}(D)]B\subset \mathrm{Rad}_{\mathcal{B}}(A)$ provided that $BAB\subset A$ and $\dim(\mathrm{Rad}_{\mathcal{J}}(A)\cap\bigcap_{n=1}^{\infty}B^n)<\infty$, where $\mathrm{Rad}_{\mathcal{J}}(A)$ and $\mathrm{Rad}_{\mathcal{B}}(A)$ denote the Jacobson and the Baer radicals of A respectively. From this we deduce the closability of partially defined derivations on complex semiprime Banach algebras with appropriate domains. The result applies to several relevant classes of algebras.

0. Introduction. The study of partially defined derivations is motivated by the time evolution and spatial translation in quantum physics. The general theory of partially defined derivations on Banach algebras is mainly concerned with the theory of closability, generator properties and classification of closed derivations. For a thorough treatment of this topic we refer the reader to [1] and [20]. On the other hand, any everywhere defined derivation on a nonassociative complete normed algebra A yields a meaningful partially defined derivation on the Banach algebra $L(A)$ of all continuous linear operator on A (see [23, 24] for more details).

It is appropriate to point out that there are examples of nonclosable partially defined derivations on C*-algebras (see [1; Example 1.4.4]). In contrast, the case where derivations are everywhere defined is by far more satisfactory. B. E. Johnson and A. M. Sinclair [14] showed that everywhere defined derivations on semisimple Banach algebras are automatically continuous. However, at present the answers to the following equivalent questions remain open, even for commutative Banach algebras (see [3, 7, 8, 16, 19]).

1. Is the separating subspace of any everywhere defined derivation on a Banach algebra contained in the Baer radical of the algebra?

1991 *Mathematics Subject Classification*: Primary 46H40.

2. Are everywhere defined derivations continuous on semiprime Banach algebras?

3. Are everywhere defined derivations continuous on prime Banach algebras?

R. V. Garimella [7, 8] and V. Runde [19] give some affirmative answers to the preceding questions provided that suitable additional requirements on the algebra are fulfilled. J. Cusack [3] showed that if the answer to any of the preceding question is negative, then there exists a topologically simple radical Banach algebra, no example of which is currently known.

In this paper we continue the work started in [25, 26] by investigating the questions listed above for a Jordan derivation D defined on a subalgebra B, which is not assumed to be closed or dense, of a complex Banach algebra A, satisfying $BAB \subset A$. We prove that $B[B \cap \mathcal{S}(D)]B \subset \mathrm{Rad}_{\mathcal{B}}(A)$ if $\dim(\mathrm{Rad}_{\mathcal{J}}(A) \cap \bigcap_{n=1}^{\infty} B^n) < \infty$, where $\mathrm{Rad}_{\mathcal{J}}(A)$ and $\mathrm{Rad}_{\mathcal{B}}(A)$ denote the Jacobson and the Baer radicals of A respectively. From this we deduce that D is automatically closable in each of the following situations.

1. A is semiprime, B is dense in A, and $\dim(\mathrm{Rad}_{\mathcal{J}}(A) \cap \bigcap_{n=1}^{\infty} B^n) < \infty$.
2. A is prime, B is dense, and $\dim \bigcap_{n=1}^{\infty} [bB \cap \mathrm{Rad}_{\mathcal{J}}(A)]^n < \infty$ for some $b \in B$ with $b^2 \neq 0$.
3. A is an integral domain and $\dim \bigcap_{n=1}^{\infty} [bB \cap \mathrm{Rad}_{\mathcal{J}}(A)]^n < \infty$ for some nonzero $b \in B$.

1. Algebraic preliminaries. We call a subalgebra B of an algebra A *hereditary* if it satisfies

$$BAB \subset B.$$

For a historical account of this concept we refer the reader to [17], where several remarkable properties of hereditary subalgebras can be found.

The important point to note here is the following result stated in [26; Lemma 1].

LEMMA 1. *Let B be a hereditary subalgebra of a complex Banach algebra A acting irreducibly on a complex Banach space X such that $B^3X \neq 0$. Then B is strictly dense on the quotient complex normed space BX/M, where $M = \{y \in BX : By = 0\}$, with the natural action $b(y+M) = by + M$.*

EXAMPLES 1. 1. Pedersen [18; 1.5.1] defines the hereditary subalgebras of a C*-algebra A as those $*$-subalgebras B of A for which $0 \leq a \leq b$, $b \in B$ and $a \in A$ implies $a \in B$. Assume that B is a hereditary subalgebra of a C*-algebra A in the sense of Pedersen. Let $b \in B$. If $a \in A$ with $0 \leq a$, then [18; Proposition 1.3.5] shows that $b^*ab \leq \|a\| b^*b \in B$ and therefore b^*ab lies in B. For an arbitrary $a \in A$ we can write $a = (a_1 - a_2) + i(a_3 - a_4)$ for suitable $a_k \in A$ with $0 \leq a_k$, for $k = 1, 2, 3, 4$. Hence $b^*ab = b^*a_1b - b^*a_2b + i(b^*a_3b - b^*a_4b) \in B$. If $b_1, b_2 \in B$ and $a \in A$ we have $b_1ab_2 = \frac{1}{4}\sum_{k=0}^{3}(-i)^k(b_1^* + i^kb_2)^*a(b_1^* + i^kb_2) \in B$. Consequently, $BAB \subset B$. Actually, it is known [6] that for a closed $*$-subalgebra B of A, Pedersen's condition and the property $BAB \subset B$ are equivalent.

2. The one-sided ideals of an algebra A are easily checked to be hereditary subalgebras of A.

3. For any elements a, b in an algebra A, aAb is a hereditary subalgebra of A.

4. The intersection of an arbitrary family of hereditary subalgebras of an algebra A is a hereditary subalgebra of A. Accordingly, the intersection of a left ideal and a right ideal of A is a hereditary subalgebra of A. However, arbitrary hereditary subalgebras of A may be far from having this form. We illustrate this fact in the following example.

5. Let A be the commutative Banach algebra of all bounded complex-valued functions on the interval $[0, 1]$. The set B of those f in A which are differentiable at 0 and satisfy $f(0) = 0$ is easily seen to be a hereditary subalgebra of A which is not an ideal of A.

Let A be an algebra. A linear map D from a subalgebra B of A to A is said to be a *derivation* if it satisfies

$$D(ab) = D(a)b + aD(b) \quad \forall a, b \in B.$$

D is said to be a *Jordan derivation* if

$$D(a^2) = D(a)a + aD(a) \quad \forall a \in B.$$

The Jordan derivation identity is equivalent to assuming that

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b) \quad \forall a, b \in B,$$

where

$$a \cdot b = \tfrac{1}{2}(ab + ba)$$

is the Jordan product of a and b in A. It should be noted that every derivation is a Jordan derivation. I. N. Herstein [12] showed that any everywhere defined Jordan derivation on a prime ring of characteristic other than 2 is a derivation, and J. M. Cusack [2] extended this result to the case of any semiprime ring in which $2a = 0$ implies $a = 0$.

A linear subspace I of A is said to be a *Jordan ideal* of A if

$$I \cdot A + A \cdot I \subset I.$$

We also define

$$U_a(b) = 2a \cdot (a \cdot b) - a^2 \cdot b = aba$$

for all $a,b \in A$. For an arbitrary subset C of A we define

$$C^{(1)} = \bigcap_{k=1}^{\infty} U_C \overset{k}{\ldots} U_C(C) \quad \text{and} \quad C^{(n+1)} = (C^{(n)})^{(1)}$$

for all $n \in \mathbb{N}$. If $U_C(C) \subset C$, then $\{C^{(n)}\}$ is a decreasing sequence of subsets of C. Furthermore, if $U_C(A) \subset C$, then $U_{C^{(n)}}(A) \subset C^{(n)}$ for all $n \in \mathbb{N}$ since $U_{U_a(b)}(c) = U_a U_b U_a(c)$ for all $a,b,c \in A$.

EXAMPLES 2. 1. Let A be any Banach algebra of power series with complex coefficients and B the two-sided ideal of A of those power series in A with zero constant term. It is easy to see that $B^{(1)} = 0$.

2. Let A denote the Banach space of those formal power series with complex coefficients $f = \sum_{n=1}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_n y^n$ such that $\|f\| = \sum_{n=1}^{\infty} \|a_n\| + \sum_{n=1}^{\infty} \|b_n\| < \infty$. We make A into a Banach algebra by defining the product

$$\Big(\sum_{n=1}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_n y^n\Big)\Big(\sum_{n=1}^{\infty} c_n x^n + \sum_{n=1}^{\infty} d_n y^n\Big)$$
$$= \sum_{n=2}^{\infty} \sum_{i+j=n} a_i c_j x^n + \sum_{n=1}^{\infty} \Big[\Big(\sum_{i=1}^{\infty} a_i\Big) d_n + b_n \Big(\sum_{i=1}^{\infty} c_n\Big)\Big] y^n + \sum_{n=2}^{\infty} \sum_{i+j=n} b_i d_j y^n.$$

An easy computation shows that $A^{(1)} = \{f \in A : a_n = 0 \text{ for all } n \in \mathbb{N}\}$ and $A^{(2)} = 0$. Furthermore, A is easily seen to be semiprime.

The *Baer radical* of an algebra A, from now on denoted by $\mathrm{Rad}_{\mathcal{B}}(A)$, is the intersection of all prime ideals of A. A is *semiprime* if $\mathrm{Rad}_{\mathcal{B}}(A) = 0$.

In order to study, in the fifth section, the nilpotency of the separating subspace of a partially defined Jordan derivation we require the following results about the Baer radical.

LEMMA 2. *Let I be a Jordan ideal of an algebra A satisfying $b^n \in \mathrm{Rad}_{\mathcal{B}}(A)$ for all $b \in I$ for some fixed $n \in \mathbb{N}$. Then $I \subset \mathrm{Rad}_{\mathcal{B}}(A)$.*

Proof. First, we claim that $b^2 \in \mathrm{Rad}_{\mathcal{B}}(A)$ for all $b \in I$. Indeed, let $b \in I$ and consider the quotient linear space $A/\mathrm{Rad}_{\mathcal{B}}(A)$ endowed with the product defined by

$$(a_1 + \mathrm{Rad}_{\mathcal{B}}(A)) \circ (a_2 + \mathrm{Rad}_{\mathcal{B}}(A)) = a_1 b^2 a_2 + \mathrm{Rad}_{\mathcal{B}}(A).$$

For each $a \in A$ we have $bab \in I$ and therefore $(bab)^n \in \mathrm{Rad}_{\mathcal{B}}(A)$, which gives $(a + \mathrm{Rad}_{\mathcal{B}}(A))^{n+2} = 0$. [17; Proposition 4.4.10] shows that $(A/\mathrm{Rad}_{\mathcal{B}}(A))^N = 0$ for a suitable $N \in \mathbb{N}$. From this it may be concluded that the principal two-sided ideal I_{b^2} of A generated by b^2 satisfies $I_{b^2}^{2N+1} \subset \mathrm{Rad}_{\mathcal{B}}(A)$. Hence $I_{b^2} \subset \mathrm{Rad}_{\mathcal{B}}(A)$, which proves our claim.

If $a \in A$ and $b \in I$, then $ab + ba \in I$ and consequently $(ab+ba)^2 \in \mathrm{Rad}_{\mathcal{B}}(A)$. Therefore $babab = b(ab+ba)^2 - bab^2a - b^2a^2b - b^2aba \in \mathrm{Rad}_{\mathcal{B}}(A)$.

Arguing as before with the product on $A/\mathrm{Rad}_{\mathcal{B}}(A)$ replaced by

$$(a_1 + \mathrm{Rad}_{\mathcal{B}}(A)) \circ (a_2 + \mathrm{Rad}_{\mathcal{B}}(A)) = a_1 b a_2 + \mathrm{Rad}_{\mathcal{B}}(A)$$

we deduce that a suitable power of the principal two-sided ideal I_b of A generated by b is contained in $\mathrm{Rad}_{\mathcal{B}}(A)$. Thus $b \in \mathrm{Rad}_{\mathcal{B}}(A)$. ∎

LEMMA 3. *Let B be a hereditary subalgebra of a Banach algebra A. Then $B \cap \mathrm{Rad}_{\mathcal{B}}(A) \subset \mathrm{Rad}_{\mathcal{B}}(B)$ and $B\mathrm{Rad}_{\mathcal{B}}(B)B \subset \mathrm{Rad}_{\mathcal{B}}(A)$.*

Proof. Let $b \in B \cap \mathrm{Rad}_{\mathcal{B}}(A)$. It is a simple matter to check that A becomes a Banach algebra for the product $a_1 \circ a_2 = a_1 b a_2$ and norm $|a| = (1 + \|b\|)\|a\|$. For each $a \in A$, ba lies in $\mathrm{Rad}_{\mathcal{B}}(A)$ and therefore $(ba)^n = 0$ for a suitable $n \in \mathbb{N}$. This clearly forces that the $(n+2)$th power of a with respect to the new product equals zero. Accordingly $(A, |\cdot|, \circ)$ is a nil Banach algebra and [17; Proposition 4.4.11(b)] shows that it is nilpotent. This means that there is a fixed $N \in \mathbb{N}$ such that $a_1 b \ldots b a_N = 0$ for all $a_1, \ldots, a_N \in A$. Hence the principal two-sided ideal I_b of B generated by b satisfies $I_b^{2N} = 0$. Thus [17; Theorem 4.4.7] shows that I_b is contained in $\mathrm{Rad}_{\mathcal{B}}(B)$. Consequently, $b \in \mathrm{Rad}_{\mathcal{B}}(B)$, which proves our first assertion.

Let $b \in \mathrm{Rad}_{\mathcal{B}}(B)$ and $c,d \in B$. For each $a \in A$, $bcad \in \mathrm{Rad}_{\mathcal{B}}(B)$ and therefore $(bcad)^n = 0$ for some $n \in \mathbb{N}$. We argue as before to prove that A becomes a nilpotent Banach algebra for the product $a_1 \circ a_2 = a_1(dbc)a_2$ and norm $|a| = (1 + \|dbc\|)\|a\|$. Consequently, the principal two-sided ideal of A generated by dbc is a nilpotent ideal and therefore it is contained in $\mathrm{Rad}_{\mathcal{B}}(A)$. This establishes our second assertion. ∎

2. Sliding hump sequences for Jordan ideals in hereditary subalgebras. This section is devoted to the construction of appropriate sequences to apply a classical method in automatic continuity theory, the sliding hump argument.

Throughout this section, B denotes a hereditary subalgebra of a complex Banach algebra A, X a complex irreducible Banach left A-module, and I a Jordan ideal of B such that $BIBX \neq 0$. Let $Y = BX$ and $M = \{y \in Y : By = 0\}$. To shorten notation, we let x^M stand for the equivalence class $x + M$.

LEMMA 4. *Let $x,y \in Y$ with $Bx \neq 0$. Then there is $a \in I$ such that $B(ax - y) = 0$.*

Proof. We claim that there exists $a_1 \in I$ such that $a_1 x^M \neq 0^M$. Otherwise, for all $a \in I$ and $b \in B$, we have $0^M = (ba + ab)x^M = abx^M$. Consequently, $IBX = IBx \subset M$, a contradiction.

Assume that $\dim(Y/M) > 1$ and take $a_1 \in I$ with $a_1 x^M \neq 0^M$. Then we can choose $b_1 \in B$ such that $b_1 a_1 x^M$ and x^M are linearly independent. Therefore there exists $b_2 \in B$ satisfying $b_2 b_1 a_1 x^M = y^M$ and $b_2 x^M = 0^M$.

Now $a=(a_1\cdot b_1)\cdot b_2+a_1\cdot(b_1\cdot b_2)-(a_1\cdot b_2)\cdot b_1=a_1b_1b_2+b_2b_1a_1$ satisfies our requirements.

If $\dim(Y/M)=1$, then we can take $a_1\in I$ such that $a_1x^M=x^M$. Since $y^M=\lambda x^M$ for a suitable $\lambda\in\mathbb{C}$, $a=\lambda a_1$ has the desired properties. ■

LEMMA 5. *Let $x,y,z\in BX$ with x^M and y^M linearly independent. Then there exists $a\in I$ such that $ax=0$ and $B(ay-z)=0$.*

Proof. If $z^M=0^M$, then we can take $a=0$. Otherwise we can choose $b_1\in I$ and $b_2\in B$ such that $b_1z^M=y^M$, $b_2x^M=0^M$, and $b_2y^M=z^M$. Then $U_{b_2}(b_1)\in I$ and satisfies the required conditions. ■

LEMMA 6. *If J is a Jordan ideal of B such that $BJBX\neq 0$, then $B(I\cap J)BX\neq 0$.*

Proof. Take $x\in BX$ with $Bx\neq 0$. According to Lemma 4, we can choose $a\in I$ and $b\in J$ such that $B(ax-x)=B(bx-x)=0$. We have $bab\in I\cap J$ and $B(babx-x)=0$. Consequently, $0\neq Bx=Bbabx\subset B(I\cap J)BX$. ■

LEMMA 7. *Let $x,y\in Y$ with $Bx\neq 0$ and assume that there exist $c\in B$ and a linear functional g on B such that $B(cx-y)=0$ and $(cbc-g(b)c)X=0$ for all $b\in B$. Then there exist $a\in I$ and a linear functional f on B satisfying $B(ax-y)=0$ and $(aba-f(b)a)X=0$ for all $b\in B$.*

Proof. If $By\neq 0$, then Lemma 4 gives $a_1\in I$ such that $B(a_1y-x)=0$. Take $a=U_c(a_1)\in I$ and $f(b)=g(bca_1)$ for all $b\in B$. If $By=0$, then we choose $a_1\in I$ with $B(a_1x-y)=0$. Take $a=U_{a_1}(c)$ and $f(b)=g(a_1ba_1)$ for all $b\in B$. ■

LEMMA 8. *One of the following assertions hold*:

1. *For all $x,y\in Y$ with $Bx\neq 0$, there are $a\in I$ and a linear functional f on B such that $B(ax-y)=0$ and $(aba-f(b)a)X=0$ for all $b\in B$.*
2. *There are sequences $\{a_n\}$ in I and $\{x_n\}$ in Y such that $Ba_n\dots a_1x_n\neq 0$, $a_{n+1}\dots a_1x_n=0$, and $a_1\dots a_nBX\neq 0$.*

Proof. If $\dim b(Y/M)=1$ for some $b\in B$, then [26; Lemma 4] and the preceding lemma show that our first assertion holds.

Assume that $\dim b(Y/M)\geq 2$ for every $b\in B\setminus\{0\}$. Take $x_1\in BX$ with $Bx_1\neq 0$ and apply Lemma 4 to get $a_1\in I$ such that $B(a_1x_1-x_1)=0$. Assume that $a_1,\dots,a_n\in I$ and $x_1,\dots,x_n\in BX$ have been chosen satisfying the required conditions. Since $\dim(a_n\dots a_1)(Y/M)\geq 2$, $a_n\dots a_1x_n^M$ and $a_n\dots a_1x_{n+1}^M$ are linearly independent for a suitable $x_{n+1}\in BX$. Moreover, there is $y_n\in BX$ such that $a_1\dots a_ny_n\neq 0$. By Lemma 5 there is $a_{n+1}\in I$ such that $a_{n+1}a_n\dots a_1x_n=0$ and $a_{n+1}a_n\dots a_1x_{n+1}^M=y_n^M$. Note that $Ba_{n+1}\dots a_1x_{n+1}=By_n\neq 0$ and $a_1\dots a_{n+1}(a_n\dots a_1x_{n+1})=a_1\dots a_ny_n\neq 0$. The sequences $\{a_n\}$ and $\{x_n\}$ constructed in this way satisfy the requirements of the second assertion of the lemma. ■

3. The sliding hump procedure for partially defined operators between Banach spaces. Let X and Y be Banach spaces. A linear operator F from a linear subspace X_0 of X into Y is said to be *closed* if its graph is a closed subset of $X\times Y$, and we call F *closable* if there is a closed extension of F. We can measure the closability of F by considering its *separating subspace*, which is defined as the subspace $\mathcal{S}(F)$ of those elements y in Y for which there is a sequence $\{x_n\}$ in X_0 satisfying $\lim x_n=0$ and $\lim F(x_n)=y$. $\mathcal{S}(F)$ is a closed linear subspace of Y and F is closable if, and only if, $\mathcal{S}(F)=0$. In this case there is a smallest closed extension of F called the *closure* of F, the domain of which is the set of those $x\in X$ for which there exists a sequence $\{x_n\}$ in X_0 such that $\lim x_n=x$ and $\{Fx_n\}$ converges.

LEMMA 9. *Let X and Y be Banach spaces, and let F be a closable linear operator from a linear subspace X_0 of X into Y. If T is a continuous linear operator from X into X_0, then FT is continuous.*

Proof. Let G be the closure of F and let X_1 be its domain. X_1 becomes a Banach space for the norm given by $|x|=\|x\|+\|Gx\|$.

It is clear that G is $|\cdot|$-$\|\cdot\|$-continuous. Moreover, it is immediate that T is a closed linear operator from X into $(X_1,|\cdot|)$ and the closed graph theorem shows that it is $\|\cdot\|$-$|\cdot|$-continuous. Accordingly, the composition $FT=GT$ is $\|\cdot\|$-$\|\cdot\|$-continuous. ■

The next important results illustrate the typical sliding hump procedure. The first one was essentially stated by M. P. Thomas in [22; Proposition 1.3] for everywhere defined operators between Banach spaces. We adapt the proof given there to partially defined operators between Banach spaces.

LEMMA 10. *Let X be a Banach space, Y a normed space, $\{T_n\}$ a sequence of continuous linear operators from X into a linear subspace X_0 of X, and $\{Q_n\}$ be a sequence of continuous linear operators from Y into Banach spaces Y_n. If F is a linear operator from X_0 into Y such that $Q_nFT_1\dots T_m$ is continuous for $m>n$, then $Q_nFT_1\dots T_n$ is continuous on X_0 for sufficiently large n.*

Proof. There is no loss of generality in assuming $\|Q_n\|=\|T_n\|\leq 1$ for all $n\in\mathbb{N}$. If the result fails, then we could choose a strictly increasing sequence $\{n_k\}$ of natural numbers and a sequence $\{x_k\}$ in X_0 such that

$$\|x_k\|<2^{-k}\min\{\|Q_{n_j}FT_1\dots T_{n_i}\|:j<i\leq k\},$$

$$\|Q_nFT_1\dots T_{n_k}x_k\|\geq 1+k+\Big\|F\sum_{j=1}^{k-1}T_1\dots T_{n_j}x_j\Big\|,$$

for all $k\in\mathbb{N}$. Consider the element $x\in X$ given by $x=\sum_{j=1}^{\infty}T_1\dots T_{n_j}x_j$

and, for every $k \in \mathbb{N}$, let $y_k = x_k + \sum_{j=k+1}^{\infty} T_{n_k+1} \ldots T_{n_j} x_j$ (the series can easily be shown to converge). Note that $x = T_1 \ldots T_{n_1} y_1 \in X_0$ and $y_k = x_k + T_{n_k+1} \ldots T_{n_{k+1}} y_{k+1} \in X_0$ for all $k \in \mathbb{N}$. As in the proof of [22; Proposition 1.3], for each $k \in \mathbb{N}$ we would have $\|Fx\| \geq k$, a contradiction. ■

The following result was established by N. P. Jewell and A. M. Sinclair [13] for everywhere defined operators between Banach spaces but the basic principle was stated earlier by Sinclair [21; Lemma 1.6].

LEMMA 11. *Let X and Y be Banach spaces, $\{T_n\}$ a sequence of continuous linear operators from X into a linear subspace X_0 of X, and $\{R_n\}$ be a sequence of continuous linear operators from Y into itself. If F is a linear operator from X_0 into Y such that $FT_n - R_nF$ is continuous on X_0 for all $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $\overline{R_1 \ldots R_n \mathcal{S}(F)} = \overline{R_1 \ldots R_m \mathcal{S}(F)}$ for all $n \geq m$.*

Proof. Suppose the lemma were false. Then, as in the classical proof, we could assume that $R_1 \ldots R_{n+1} \mathcal{S}(F)$ is strictly contained in $R_1 \ldots R_n \mathcal{S}(F)$ and $\|T_n\| \leq 1$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let Q_n denote the quotient map from Y onto $Y/\overline{R_1 \ldots R_n \mathcal{S}(F)}$. [17; Proposition 6.1.9(c)] shows that $Q_n R_1 \ldots R_n F$ is closable and $Q_n R_1 \ldots R_{n-1} F$ is discontinuous for each $n \in \mathbb{N}$.

Note that for $n, m \in \mathbb{N}$ with $m \geq n$,

$$Q_n F T_1 \ldots T_m = (Q_n R_1 \ldots R_n F) T_{n+1} \ldots T_m - \sum_{k=1}^{n} R_1 \ldots R_{k-1} (R_k F - F T_k) T_{k+1} \ldots T_m$$

and the operator $\sum_{k=1}^{n} R_1 \ldots R_{k-1} (R_k F - F T_k) T_{k+1} \ldots T_m$ is continuous. If $m > n$, then Lemma 9 shows that $(Q_n R_1 \ldots R_n F) T_{n+1} \ldots T_m$ and hence $Q_n F T_1 \ldots T_m$ are continuous. The preceding lemma now yields that $Q_n F T_1 \ldots T_n$ and consequently $Q_n R_1 \ldots R_n F$ are continuous for sufficiently large n. The rest of the proof runs as in [13, 21]. The details are left to the reader. ■

LEMMA 12. *Let X and Y be Banach spaces, T a continuous linear operator from X into a linear subspace X_0 of X, and let F be a linear operator from X_0 into Y such that $FT - TF$ is continuous on X_0. Then $\bigcap_{n=1}^{\infty} T^n(\mathcal{S}(F))$ is dense in $T^m(\mathcal{S}(F))$ for a suitable $m \in \mathbb{N}$.*

Proof. By the preceding lemma, there is $m \in \mathbb{N}$ such that $\overline{T^n(\mathcal{S}(F))} = \overline{T^m(\mathcal{S}(F))}$ for all $n \geq m$. Since

$$T^{m+1}(\mathcal{S}(F)) \subset T(\overline{T^m(\mathcal{S}(F))}) \subset \overline{T(T^m(\mathcal{S}(F))} = \overline{T^m(\mathcal{S}(F))},$$

T is a continuous linear operator from the Banach space $\overline{T^m(\mathcal{S}(F))}$ onto a dense linear subspace of it. The Mittag-Leffler theorem [4; Theorem 5.3] now shows that $\bigcap_{n=1}^{\infty} T^n(\overline{T^m(\mathcal{S}(F))})$ is dense in $\overline{T^m(\mathcal{S}(F))}$. $T(\mathcal{S}(F))$ is easily checked to be a subset of $\mathcal{S}(F)$. Hence $T^m(\mathcal{S}(F)) \subset \mathcal{S}(F)$ and therefore $\overline{T^m(\mathcal{S}(F))} \subset \mathcal{S}(F)$. Consequently,

$$\bigcap_{n=1}^{\infty} T^n(\overline{T^m(\mathcal{S}(F))}) \subset \bigcap_{n=1}^{\infty} T^n(\mathcal{S}(F)) \subset T^m(\mathcal{S}(F)),$$

which establishes the desired conclusion. ■

4. Quasinilpotency of the separating subspace of a Jordan derivation. From now on, D stands for a Jordan derivation on a complex Banach algebra A defined on a hereditary subalgebra B of A.

In the sequel, $\mathrm{Rad}_{\mathcal{J}}(A)$ denotes the *Jacobson radical* of A so that $\mathrm{Rad}_{\mathcal{J}}(A)$ is the intersection of all the primitive ideals of A. It is well known that $\mathrm{Rad}_{\mathcal{J}}(A)$ is the largest two-sided ideal of A, each element a of which is quasinilpotent, i.e. $\lim \|a^n\|^{1/n} = 0$.

The separating subspace of D is easily checked to satisfy

$$B \cdot \mathcal{S}(D) \subset \mathcal{S}(D).$$

Consequently, $B \cap \mathcal{S}(D)$ is a Jordan ideal of B.

LEMMA 13. *Let $\{X_n\}$ be a sequence of complex irreducible Banach left A-modules. Assume that $\{a_n\}$ is a sequence in B and, for each $n \in \mathbb{N}$, there exists $x_n \in X_n$ satisfying $Ba_n \ldots a_1 x_n \neq 0$, $a_{n+1} \ldots a_1 x_n = 0$, and $a_1 \ldots a_n B X_n \neq 0$. Then $B[B \cap \mathcal{S}(D)]BX_n = 0$ for some $n \in \mathbb{N}$.*

Proof. We apply Lemma 11 to the operators $F = D$ and $R_n = T_n = U_{a_n}$ for $n \in \mathbb{N}$ to deduce that

$$\overline{a_1 \ldots a_{m+1} \mathcal{S}(D) a_{m+1} \ldots a_1} = \overline{a_1 \ldots a_m \mathcal{S}(D) a_m \ldots a_1}$$

for a suitable $m \in \mathbb{N}$. Since $a_1 \ldots a_{m+1} \mathcal{S}(D) a_{m+1} \ldots a_1 x_m = 0$, it follows that $a_1 \ldots a_m \mathcal{S}(D) a_m \ldots a_1 x_m = 0$. We claim that $B[B \cap \mathcal{S}(D)]BX_m = 0$. Suppose, contrary to our claim, that $B[B \cap \mathcal{S}(D)]BX_m \neq 0$. We conclude from Lemma 4 that $B \cap \mathcal{S}(D) a_m \ldots a_1 x_m = BX_m$ and hence that $a_1 \ldots a_m B X_m = 0$, a contradiction. ■

LEMMA 14. *Let X be a complex irreducible Banach left A-module and I a Jordan ideal of B such that $BIBX \neq 0$. Assume that there are $a_1, \ldots, a_n \in B$ such that $a_1 \ldots a_n BX \neq 0$, $Ba_n \ldots a_1 X \neq 0$, and $\dim U_{a_1} \ldots U_{a_n}(I) < \infty$. Then $B[B \cap \mathcal{S}(D)]BX = 0$.*

Proof. Let $x \in X$ and $y \in BX$ such that $Ba_n \ldots a_1 x \neq 0$ and $a_1 \ldots a_n y \neq 0$. Lemma 4 shows that $ba_n \ldots a_1 x = y$ for a suitable $b \in I$.

If $B[B\cap\mathcal{S}(D)]BX$ were not 0, Lemma 4 would give an $a\in\mathcal{S}(D)$ such that $ay=a_n\ldots a_1x$. Since $U_b(B)\subset I$, we have $\dim U_{a_1}\ldots U_{a_n}U_b(B)<\infty$ and therefore $DU_{a_1}\ldots U_{a_n}U_b$ is continuous. On the other hand, it is immediate that $DU_{a_1}\ldots U_{a_n}U_b-U_{a_1}\ldots U_{a_n}U_bD$ is continuous. Accordingly, $U_{a_1}\ldots U_{a_n}U_bD$ is continuous and hence $U_{a_1}\ldots U_{a_n}U_b(\mathcal{S}(D))=0$. Therefore we would have $0=a_1\ldots a_nbaba_n\ldots a_1x=a_1\ldots a_ny$, which contradicts the choice of y. ∎

Since, for every $b\in B$, DU_b-U_bD is easily seen to be continuous, a straightforward application of Lemma 12 gives the following.

LEMMA 15. *If $b\in B$, then $\bigcap_{n=1}^{\infty}b^n\mathcal{S}(D)b^n$ is dense in $b^m\mathcal{S}(D)b^m$ for a suitable $m\in\mathbb{N}$.*

LEMMA 16. *Let X be a complex irreducible Banach left A-module, I a Jordan ideal of B contained in $\mathcal{S}(D)$ such that $BIBX\neq 0$, and $p\in\mathbb{N}$. Then, for all $x,y\in BX$ with $Bx\neq 0$, there exist $a\in I^{(p)}$ and a linear functional f on B such that $B(ax-y)=0$ and $(aba-f(b)a)X=0$ for all $b\in B$.*

Proof. Fix $x\in BX$ with $Bx\neq 0$. Let us prove that for every $p\in\mathbb{N}$ there exist a_p and f_p on B satisfying the desired conditions with $y=x$.

From Lemmas 8 and 13 we deduce that there are $b_1\in I$ and a linear functional g_1 on B such that $B(b_1x-x)=0$ and $(b_1bb_1-g(b)b_1)X=0$ for all $b\in B$. According to the preceding lemma, $\bigcap_{n=1}^{\infty}b_1^n\mathcal{S}(D)b_1^n$ is dense in $b_1^m\mathcal{S}(D)b_1^m$ for a suitable $m\in\mathbb{N}$. Since $b_1^{2m+1}x-x=\sum_{k=0}^{2m}b_1^k(b_1x-x)=b_1x-x$ we have $B(b_1^{2m+1}x-x)=0$, which gives $Bb_1^{2m+1}x=Bx\neq 0$. As $b_1^{2m+1}\in b_1^m\mathcal{S}(D)b_1^m$ there exists

$$b_2\in\bigcap_{n=1}^{\infty}b_1^n\mathcal{S}(D)b_1^n=\bigcap_{n=1}^{\infty}U_{b_1}^n(b_1^2\mathcal{S}(D)b_1^2)\subset\bigcap_{n=1}^{\infty}U_{b_1}^n(U_{b_1}(B))\subset I^{(1)}$$

such that $Bb_2x\neq 0$. If b_2x^M and x^M are linearly independent, then we choose $b_3\in I$ satisfying $b_3x^M=b_3b_2x^M=x^M$. Otherwise $b_2x^M=\lambda^2x^M$ for a nonzero $\lambda\in\mathbb{C}$ and we set $b_3\in I$ such that $b_3x^M=\lambda^{-1}x^M$. Anyway, we have $a_1=U_{b_3}b_2\in I^{(1)}$ and $B(a_1x-x)=0$. We can put $b_2=b_1b_1'b_1$ for a suitable $b_1'\in\mathcal{S}(D)$. Define f_1 on B by $f_1(b)=g(b_1'b_1b_3bb_3)$. It is immediate that $(a_1ba_1-f(b)a_1)X=0$ for all $b\in B$.

Assume that there exist $a_p\in I^{(p)}$ and a linear functional f_p on B such that $B(a_px-x)=0$ and $(a_pba_p-f_p(b)a_p)X=0$. We can apply the preceding argument again, with b_1 replaced by a_p. The only points to note here are that $U_{a_p}(B)\subset I^{(p)}$, which gives $\bigcap_{n=1}^{\infty}U_{a_p}^n(U_{a_p}(B))\subset(I^{(p)})^{(1)}=I^{(p+1)}$, and $a_{p+1}\in I^{(p+1)}$.

Finally, let $x,y\in BX$ with $Bx\neq 0$ and $p\in\mathbb{N}$. If x^M and y^M are linearly independent, then we choose $a_p\in I^{(p)}$, $a_p'\in B$, and a linear functional f_p on B satisfying $B(a_py-y)=B(a_p'x-y)=B(a_p'y-y)=0$, and $(a_pba_p-f_p(b)a_p)X=0$ for all $b\in B$. In this case the element $a=a_p'a_pa_p'$ and the linear functional f defined on B by $f(b)=f_p(a_p'ba_p')$ satisfy our requirements. If $y^M=\lambda x^M$ for some $\lambda\in\mathbb{C}$, then we take $a_p\in I^{(p)}$ and a linear functional f_p on B satisfying $B(a_px-x)=0$ and $(a_pba_p-f_p(b)a_p)X=0$ for all $b\in B$. Now $a=\lambda a_p$ and $f=\lambda f_p$ have the desired properties. ∎

To simplify notation, here and subsequently, we write $\langle C\rangle$ to denote the linear subspace generated by the subset C of A.

We can now improve the main result in [26] as follows.

THEOREM 1. *If $\dim\langle B^{(p)}\cap\mathrm{Rad}_{\mathcal{J}}(A)\rangle<\infty$ for some $p\in\mathbb{N}$, then $B[\mathcal{S}(D)\cap B]B\subset\mathrm{Rad}_{\mathcal{J}}(A)$.*

Proof. Suppose, contrary to our claim, that the set $\mathcal{P}$ of those primitive ideals P of A for which $B[\mathcal{S}(D)\cap B]B\not\subset P$ is nonempty.

Define $I_0=B\cap\mathcal{S}(D)$. We take $P_1\in\mathcal{P}$, which is the kernel of a continuous irreducible representation of A on a complex Banach space X_1. Since $BI_0BX_1\neq 0$ we can choose $x_1\in BX_1$ with $Bx_1\neq 0$. Lemma 16 now gives $a_1\in I_0^{(p)}$ and a linear functional f_1 on B such that $B(a_1x_1-x_1)=0$ and $a_1ba_1-f_1(b)a_1\in P_1$ for all $b\in B$. We observe that $f_1(a_1)=1$ and ${a_1}^3-a_1\in P$.

Suppose that $P_1,\ldots,P_n$, $X_1,\ldots,X_n$, $x_1,\ldots,x_n$, $a_1,\ldots,a_n$, and $f_1,\ldots,f_n$ have been chosen satisfying the following conditions for $k=1,\ldots,n$:

(i) $P_k\in\mathcal{P}$ is the kernel of a continuous irreducible representation of A on a complex Banach space X_k,

(ii) $x_k\in X_k$, $a_k\in I_{k-1}^{(p)}$, where $I_k=I_0\cap P_1\cap\ldots\cap P_k$, $a_1\ldots a_kBX_k\neq 0$ and $Ba_k\ldots a_1x_k\neq 0$,

(iii) f_k is a linear functional on B such that $a_kba_k-f_k(b)a_k\in P_k$ for all $b\in B$,

We claim that there is $P_{n+1}\in\mathcal{P}$ such that $I_n\not\subset P_{n+1}$ and $a_1\ldots a_nB$, $Ba_n\ldots a_1\not\subset P_{n+1}$. Suppose the claim were false. Then we would have $a_1\ldots a_nB\cap Ba_n\ldots a_1\subset\bigcap_{P\in\mathcal{P},\,I_n\not\subset P}P$ from which we deduce that $\dim U_{a_1}\ldots U_{a_n}(I_{n-1})<\infty$. Indeed, for every $a\in I_0$, we have

$$\begin{aligned}a_1aa_1-f_1(a)a_1^3&\in P_1\cap BI_0B\cap\Big(\bigcap_{P\in\mathcal{P},\,I_1\not\subset P}P\Big)\cap B^{(p)}\\&\subset BI_0B\cap I_1\cap\Big(\bigcap_{P\in\mathcal{P},\,I_1\not\subset P}P\Big)\cap B^{(p)}\\&\subset BI_0B\cap\Big(\bigcap_{P\in\mathcal{P}}P\Big)\cap B^{(p)}\subset\mathrm{Rad}_{\mathcal{J}}(A)\cap B^{(p)}\end{aligned}$$

and, for every $a \in I_{n-1}$, $a_1 \ldots a_n a a_n \ldots a_1 - f_n(a) a_1 \ldots a_n \ldots a_1$ lies in

$$P_n \cap BI_{n-1}B \cap \Big(\bigcap_{P \in \mathcal{P},\, I_n \not\subset P} P \Big) \cap B^{(p)} \subset \mathrm{Rad}_{\mathcal{J}}(A) \cap B^{(p)}.$$

Lemma 14 now shows that $B[B \cap \mathcal{S}(D)]B \subset P_n$, a contradiction. Choose P_{n+1} with the claimed properties and let X_{n+1} be a complex irreducible Banach left A-module such that $P_{n+1} = \{a \in A : aX_{n+1} = 0\}$. Since $a_1 \ldots a_n B X_{n+1} \neq 0$ and $Ba_n \ldots a_1 X_{n+1} \neq 0$, we can take $x_{n+1} \in X_{n+1}$ and $y_{n+1} \in BX_{n+1}$ satisfying $Ba_n \ldots a_1 x_{n+1} \neq 0$ and $a_1 \ldots a_n y_{n+1} \neq 0$. Let $b \in B$ such that $B(ba_n \ldots a_1 x_{n+1} - x_{n+1}) = 0$. Hence

$$0 \neq Ba_n \ldots a_1 x_{n+1} = Ba_n \ldots a_1 b a_n \ldots a_1 x_{n+1} \in BI_{n-1}BX_{n+1},$$

and Lemma 6 shows that $BI_nBX_{n+1} \neq 0$, since $B(B \cap P_n)BX_{n+1} \neq 0$. According to Lemma 16, there are $a_{n+1} \in I_n^{(p)}$ and a linear functional f_{n+1} on B such that

$$B(a_{n+1}a_n \ldots a_1 x_{n+1} - y_{n+1}) = 0 \quad \text{and} \quad a_{n+1} b a_{n+1} - f_{n+1}(b) a_{n+1} \in P_{n+1}$$

for all $b \in B$. Thus $Ba_{n+1} \ldots a_1 x_{n+1} = By_{n+1} \neq 0$ and

$$a_1 \ldots a_{n+1}(a_n \ldots a_1 x_{n+1}) = a_1 \ldots a_n y_{n+1} \neq 0.$$

From (ii) we see that the sequences $\{a_n\}$, $\{X_n\}$, and $\{x_n\}$ satisfy the requirements of Lemma 13 and therefore $B[B \cap \mathcal{S}(D)]B \subset P_n$ for some n, which contradicts the choice of P_n. ∎

5. Nilpotency of the separating subspace of a Jordan derivation. In this section we use Theorem 1 to study whether $B[B \cap \mathcal{S}(D)]B \subset \mathrm{Rad}_{\mathcal{B}}(A)$.

LEMMA 17. *If* $\dim\langle B^{(p)} \cap \mathrm{Rad}_{\mathcal{J}}(A)\rangle < \infty$ *for some* $p \in \mathbb{N}$, *then*

$$U_{[B \cap \mathcal{S}(D)]^{(p)}}(\mathcal{S}(D)) = 0.$$

Proof. Theorem 1 gives $B[B \cap \mathcal{S}(D)]B \subset \mathrm{Rad}_{\mathcal{J}}(A)$ and therefore $[B \cap \mathcal{S}(D)]^{(p)} \subset B^{(p)} \cap \mathrm{Rad}_{\mathcal{J}}(A)$. On the other hand, for $a \in [B \cap \mathcal{S}(D)]^{(p)}$ we have $U_a(B) \subset [B \cap \mathcal{S}(D)]^{(p)}$. This gives $\dim U_a(B) < \infty$ and hence DU_a is continuous. Consequently, $U_a(\mathcal{S}(D)) = 0$, as required. ∎

LEMMA 18. *Let I be a subset of $B \cap \mathcal{S}(D)$ satisfying $U_I(B) \subset I$ and $b^q = 0$ for all $b \in I^{(1)}$ for some fixed $q \in \mathbb{N}$. Then $b^4 \in \mathrm{Rad}_{\mathcal{B}}(A)$ for all $b \in I$.*

Proof. For each $b \in I$, Lemma 15 shows that $\bigcap_{n=1}^{\infty} b^n \mathcal{S}(D) b^n$ is dense in $b^m \mathcal{S}(D) b^m$ for a suitable $m \in \mathbb{N}$. Further, we note that

$$\bigcap_{n=1}^{\infty} b^n \mathcal{S}(D) b^n = \bigcap_{n=1}^{\infty} b^n b(b\mathcal{S}(D)b)bb^n \subset \bigcap_{n=1}^{\infty} U_b^n(I) \subset I^{(1)}.$$

Consequently, $b^{2m+1} \in b^m \mathcal{S}(D) b^m \subset \overline{I^{(1)}}$. By the preceding lemma we have $U_{I^{(1)}}(I^{(1)}) = 0$. Hence $U_{\overline{I^{(1)}}}(\overline{I^{(1)}}) = 0$ and therefore $b^{3(2m+1)} = 0$.

Fix $b \in I$. For each $a \in A$, $b^2ab^2 \in U_b(B) \subset I$ and therefore $(b^2ab^2)^n = 0$ for some $n \in \mathbb{N}$. We argue as in Lemma 3 to prove that A becomes a nilpotent Banach algebra for the product $a_1 \circ a_2 = a_1 b^4 a_2$ and norm $|a| = (1+\|b^4\|)\|a\|$. Hence the principal two-sided ideal I_b of A generated by b^4 is a nilpotent ideal of A and therefore it is contained in $\mathrm{Rad}_{\mathcal{B}}(A)$. Consequently, $b^4 \in \mathrm{Rad}_{\mathcal{B}}(A)$ whenever b lies I. ∎

THEOREM 2. *If* $\dim\langle B^{(1)} \cap \mathrm{Rad}_{\mathcal{J}}(A)\rangle < \infty$, *then* $B \cap \mathcal{S}(D) \subset \mathrm{Rad}_{\mathcal{B}}(B)$. *Accordingly*, $B[B \cap \mathcal{S}(D)]B \subset \mathrm{Rad}_{\mathcal{B}}(A)$.

Proof. Write $I = B \cap \mathcal{S}(D)$. From Lemma 17 we see that $U_{I^{(1)}}(\mathcal{S}(D_{|I}) = 0$. Since $I^{(1)} \subset \mathcal{S}(D)$, we have $U_{I^{(1)}}(I^{(1)}) = 0$. The preceding result shows that $b^4 \in \mathrm{Rad}_{\mathcal{B}}(A)$ for all $b \in I$. Lemma 2 now leads to $I \subset \mathrm{Rad}_{\mathcal{B}}(B)$. ∎

LEMMA 19. *Let I be a subset of $B \cap \mathcal{S}(D)$ satisfying $U_I(B) \subset I$ and $b^q = 0$ for all $b \in I^{(p)}$ for suitable $p, q \in \mathbb{N}$. If $B \cap \mathrm{Rad}_{\mathcal{B}}(A)$ is nilpotent, then there is $m \in \mathbb{N}$ such that $b^m = 0$ for all $b \in I$.*

Proof. If $p = 1$, then Lemma 18 shows that $b^4 \in B \cap \mathrm{Rad}_{\mathcal{B}}(A)$ and consequently $b^{4N} = 0$ for all $b \in I$, where $N \in \mathbb{N}$ satisfies $(B \cap \mathrm{Rad}_{\mathcal{B}}(A))^N = 0$. Assume the result holds for p. If there is $q \in \mathbb{N}$ such that $b^q = 0$ for all $b \in I^{(p+1)} = (I^{(p)})^{(1)}$, then $b^{4N} = 0$ for all $b \in I^{(p)}$ and therefore $b^{4N} = 0$ for all $b \in I$. ∎

THEOREM 3. *If* $\dim\langle B^{(p)} \cap \mathrm{Rad}_{\mathcal{J}}(A)\rangle < \infty$ *for some* $p \in \mathbb{N}$ *and* $B \cap \mathrm{Rad}_{\mathcal{B}}(A)$ *is nilpotent, then* $B \cap \mathcal{S}(D) \subset \mathrm{Rad}_{\mathcal{B}}(B)$. *Accordingly*, $B[B \cap \mathcal{S}(D)]B \subset \mathrm{Rad}_{\mathcal{B}}(A)$.

Proof. Let $I = B \cap \mathcal{S}(D)$. Lemma 17 shows that $U_{I^{(p)}}(I^{(p)}) = 0$. From Lemma 19 we deduce that $b^m = 0$ for all $b \in I$ for some fixed $m \in \mathbb{N}$. Lemma 2 completes the proof. ∎

COROLLARY 1. *Assume that* $\dim\langle [B \cap \mathrm{Rad}_{\mathcal{J}}(A)]^{(p)}\rangle < \infty$ *for some* $p \in \mathbb{N}$ *and that* $B \cap \mathrm{Rad}_{\mathcal{B}}(A)$ *is nilpotent. Then* $B \cap \mathrm{Rad}_{\mathcal{J}}(A) \cap \mathcal{S}(D) \subset \mathrm{Rad}_{\mathcal{B}}(B)$.

Proof. Let $I = B \cap \mathrm{Rad}_{\mathcal{J}}(A)$ and consider the restriction $D_{|I}$. According to the preceding theorem we have $I \cap \mathcal{S}(D_{|I}) \subset \mathrm{Rad}_{\mathcal{B}}(I) = \mathrm{Rad}_{\mathcal{B}}(B)$. It is immediate that $U_I(\mathcal{S}(D)) \subset \mathcal{S}(D_{|I})$ and therefore

$$U_{I \cap \mathcal{S}(D)}(I \cap \mathcal{S}(D)) \subset I \cap \mathcal{S}(D_{|I}) \subset \mathrm{Rad}_{\mathcal{B}}(B).$$

Consequently, $b^3 \in \mathrm{Rad}_{\mathcal{B}}(B)$ whenever b lies in $I \cap \mathcal{S}(D)$. Lemma 2 now gives $I \cap \mathcal{S}(D) \subset \mathrm{Rad}_{\mathcal{B}}(B)$ as required. ∎

COROLLARY 2. *If* $\dim\langle [bB \cap \mathrm{Rad}_{\mathcal{J}}(A)]^{(p)}\rangle < \infty$ *for some* $p \in \mathbb{N}$ *and some nonzero* $b \in B$ *and* A *is an integral domain, then* D *is closable.*

Proof. Consider the restriction $D_{|bB}$. The preceding corollary shows that $bB\cap \mathrm{Rad}_{\mathcal{J}}(A)\cap \mathcal{S}(D_{|bB})\subset \mathrm{Rad}_{\mathcal{B}}(bB)=0$. Since $b\mathcal{S}(D)\subset \mathcal{S}(D_{|bB})$ we have

$$b^3\mathcal{S}(D)\mathrm{Rad}_{\mathcal{J}}(A)\subset bB\cap \mathrm{Rad}_{\mathcal{J}}(A)\cap \mathcal{S}(D_{|bB})=0.$$

If $\mathcal{S}(D)=0$, then D is closable. Otherwise $\mathrm{Rad}_{\mathcal{J}}(A)=0$ and Theorem 1 shows that $B[B\cap \mathcal{S}(D)]B=0$. From this we deduce that $B^2\mathcal{S}(D)B^2=0$, which gives either $B=0$ or $\mathcal{S}(D)=0$ and hence D is closable. ∎

6. Densely defined derivations. Throughout this section, B is assumed to be a dense hereditary subalgebra of A. If in addition A is semiprime, then it is easy to check that the Cusack Theorem [2; Theorem 4] remains true and D is a derivation whose separating subspace is a two-sided ideal of A.

COROLLARY 3. *If* $\dim\langle B^{(p)}\cap \mathrm{Rad}_{\mathcal{J}}(A)\rangle<\infty$ *for some* $p\in\mathbb{N}$, *then* $\mathcal{S}(D)\subset \mathrm{Rad}_{\mathcal{J}}(A)$.

Proof. From Theorem 1 we deduce that

$$B^2\mathcal{S}(D)B^2\subset B[B\cap \mathcal{S}(D)]B\subset \mathrm{Rad}_{\mathcal{J}}(A)$$

and therefore $A^2\mathcal{S}(D)A^2\subset \mathrm{Rad}_{\mathcal{J}}(A)$, which gives the desired conclusion. ∎

COROLLARY 4. *If* $\dim\langle B^{(p)}\cap \mathrm{Rad}_{\mathcal{J}}(A)\rangle<\infty$ *for some* $p\in\mathbb{N}$ *and* A *is semiprime, then* D *is closable.*

Proof. B is easily seen to be semiprime and Theorem 3 yields $B\cap \mathcal{S}(D)=0$. On the other hand, $U_B(\mathcal{S}(D))\subset B\cap \mathcal{S}(D)=0$. By density, $U_A(\mathcal{S}(D))=0$. Since $\mathcal{S}(D)$ is a two-sided ideal of A, we conclude that $\mathcal{S}(D)\subset \mathrm{Rad}_{\mathcal{B}}(A)=0$. ∎

COROLLARY 5. *If* $\dim\langle [bB\cap \mathrm{Rad}_{\mathcal{J}}(A)]^{(p)}\rangle<\infty$ *for some* $p\in\mathbb{N}$ *and some* $b\in B$ *with* $b^2\neq 0$ *and* A *is prime, then* D *is closable.*

Proof. Consider the restriction $D_{|bB}$. From Corollary 1, it follows that

$$bB\cap \mathrm{Rad}_{\mathcal{J}}(A)\cap \mathcal{S}(D_{|bB})\subset \mathrm{Rad}_{\mathcal{B}}(bB).$$

From Lemma 3 we have $bB\mathrm{Rad}_{\mathcal{B}}(bB)bB\subset \mathrm{Rad}_{\mathcal{B}}(A)=0$ and therefore $b\overline{A\mathrm{Rad}_{\mathcal{B}}(bB)}bA=0$. Since $\overline{\mathrm{Rad}_{\mathcal{B}}(bB)}$ is a two-sided ideal of A and $b\neq 0$, the primeness of A shows that $\mathrm{Rad}_{\mathcal{B}}(bB)=0$. On the other hand, $b\mathcal{S}(D)\subset \mathcal{S}(D_{|bB})$. Consequently,

$$b^2\mathcal{S}(D)B\mathrm{Rad}_{\mathcal{J}}(A)B\subset bB\cap \mathrm{Rad}_{\mathcal{J}}(A)\cap \mathcal{S}(D_{|bB})=0.$$

Hence $b^2\mathcal{S}(D)A\mathrm{Rad}_{\mathcal{J}}(A)A=0$. Since $b^2\neq 0$, the primeness of A gives either $\mathcal{S}(D)=0$ or $\mathrm{Rad}_{\mathcal{J}}(A)=0$. In the last case, Corollary 3 shows that D is closable. ∎

COROLLARY 6. *If* A *is semiprime, then* D *is closable if, and only if,* $[\mathcal{S}(D)]^{(p)}=0$ *for some* $p\in\mathbb{N}$.

Proof. If $[\mathcal{S}(D)]^{(p)}=0$, then we apply Theorem 3 to $D_{|B\cap\mathcal{S}(D)}$ to obtain $B\cap\mathcal{S}(D)\cap\mathcal{S}(D_{|B\cap\mathcal{S}(D)})=0$. Note that

$$(B\cap\mathcal{S}(D))^2\subset (B\cap\mathcal{S}(D))\mathcal{S}(D)\subset B\cap\mathcal{S}(D)\cap\mathcal{S}(D_{|B\cap\mathcal{S}(D)})=0.$$

Lemma 2 shows that $B\cap\mathcal{S}(D)=0$ and therefore $B\mathcal{S}(D)B=0$. By density, $A\mathcal{S}(D)A=0$, which implies $\mathcal{S}(D)=0$. ∎

7. Examples. 1. *Algebras of power series.* Let $\mathcal{A}$ be a complex Banach algebra and let $\mathcal{A}[[x]]$ denote the algebra of formal power series in one indeterminate x with coefficients in $\mathcal{A}$. It follows easily that if $\mathcal{A}$ is prime, then so is any subalgebra of $\mathcal{A}[[x]]$ containing $\mathcal{A}x^n$ for some $n\in\mathbb{N}_0$. A Banach algebra of power series with coefficients in $\mathcal{A}$ is a Banach space A which is a subalgebra of $\mathcal{A}[[x]]$ containing $\mathcal{A}x$ such that the coefficient functionals are continuous. The closed graph theorem shows that multiplication is separately continuous, and so A becomes a Banach algebra for an equivalent norm.

For an example of such an algebra, let ω be an algebra weight on $\mathbb{N}_0$, that is, ω is a function from $\mathbb{N}_0$ to $\mathbb{R}^+$ satisfying $\omega(0)=1$ and $\omega(m+n)\le\omega(m)\omega(n)$ for all $m,n\in\mathbb{N}$. Let

$$\ell^1(\mathcal{A},\omega)=\Big\{f=\sum_{n=0}^{\infty}a_nx^n:\|f\|=\sum_{n=0}^{\infty}\|a_n\|\omega(n)<\infty\Big\}.$$

Then $\ell^1(\mathcal{A},\omega)$ is a Banach algebra of power series with coefficients in $\mathcal{A}$. It is well known that $\ell^1(\mathbb{C},\omega)$ is semisimple if $\lim(\omega(n))^{1/n}>0$, and $\mathrm{Rad}_{\mathcal{J}}(\ell^1(\mathbb{C},\omega))=\{f\in\ell^1(\mathbb{C},\omega):f(0)=0\}$ if $\lim(\omega(n))^{1/n}=0$. For a thorough discussion of this class of Banach algebras with complex coefficients, we refer to [4] and [11].

It is well known that any everywhere defined derivation on a Banach algebra of power series with complex coefficients is continuous. This was first shown in [15]. On account of Corollaries 2 and 5 we deduce that any derivation with a hereditary domain B on a Banach algebra A of power series with coefficients in a complex prime Banach algebra $\mathcal{A}$ is closable provided that either $\mathcal{A}$ is commutative or B is dense in A. To prove this we show that there is $b\in B$ with $b^4\neq 0$. In the first case it is obvious. If our assertion were false in the second case, then we would have $B^N=0$ for some $N\in\mathbb{N}$. By density we would deduce that $A^N=0$, a contradiction. We have $b^2x=bxb\in B$, $(b^2x)^2=b^4x^2\neq 0$, and $[(b^2x)B]^{(1)}=0$.

2. *Weighted convolution algebras on the half-line.* Let ω be a positive measurable function on $[0,\infty[$ such that $\omega(s+t)\le\omega(s)\omega(t)$ for all $s,t\in[0,\infty[$; such a function is said to be a *weight function* on $[0,\infty[$. Let $L^1(\omega)$ denote the

space of equivalence classes with respect to Lebesgue measure of complex-valued measurable functions f on $[0,\infty[$ for which $\|f\|_\omega = \int_0^\infty |f(t)|\omega(t)\,dt < \infty$. $L^1(\omega)$ becomes a complex Banach algebra for the norm $\|\cdot\|_\omega$ and the convolution $(f*g)(t) = \int_0^t f(t-s)g(s)\,ds$ as product. For a thorough treatment of these algebras we refer the reader to [4, 5, 9]. It is known [4] that $L^1(\omega)$ is an integral domain. It is semisimple if $\lim_{t\to\infty}\omega(t)^{1/t} > 0$, and radical if $\lim_{t\to\infty}\omega(t)^{1/t} = 0$. For $f \in L^1(\omega)\setminus\{0\}$, define $\alpha(f) = \sup\{\delta \ge 0 : f = 0$ almost everywhere on $[0,\delta]\}$, and set $\alpha(0) = \infty$.

For each weight function ω, everywhere defined derivations on $L^1(\omega)$ are continuous (see [4, 5, 10, 13]). From Corollary 2 we deduce that any derivation defined on a hereditary subalgebra B of $L^1(\omega)$ is automatically closable. Choose $g \in L^1(\omega)\setminus\{0\}$ with $\alpha(g) > 0$ and $h \in B\setminus\{0\}$. The function $f = h*g*h$ lies in $B\setminus\{0\}$ and Titchmarsh's convolution theorem [4; Theorem 7.4] shows that $\alpha(f) = 2\alpha(h)+\alpha(g) > 0$ and $(fB)^{(1)} = 0$.

3. *Disc algebras.* Set $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$. Given a complex Banach algebra $\mathcal{A}$, let $A(\mathbb{D},\mathcal{A})$ denote the set of all continuous $\mathcal{A}$-valued functions on $\mathbb{D}$ which are holomorphic on its interior. $A(\mathbb{D},\mathcal{A})$ becomes a Banach algebra with pointwise algebraic operations and the supremum norm. From the uniqueness theorem for holomorphic functions we deduce that $A(\mathbb{D},\mathcal{A})$ is prime if $\mathcal{A}$ is. Any derivation defined on a hereditary subalgebra B of $A(\mathbb{D},\mathcal{A})$ is closable provided that either $\mathcal{A}$ is an integral domain or $\mathcal{A}$ is prime and B is dense in $A(\mathbb{D},\mathcal{A})$. Indeed, we can argue as in the first example to find $g \in B$ with $g^4 \ne 0$. The function f defined on $\mathbb{D}$ by $f(z) = g(z)^2 z$ lies in B, $f^2 \ne 0$, and $[fB]^{(1)} = 0$.

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Received October 10, 1997
Revised version December 22, 1997 (3975)