References


Department of System Science
Graduate School of Natural Science and Technology
Okayama University
Okayama 700-8530, Japan

Received October 10, 1997 (3972)

STUDIA MATHEMATICA 130 (3) (1998)

Derivations with a hereditary domain, II

by

A. R. VILLENA (Granada)

Abstract. The nilpotency of the separating subspace of an everywhere defined derivation on a Banach algebra is an intriguing question which remains unsolved, even for commutative Banach algebras. On the other hand, closability of partially defined derivations on Banach algebras is a fundamental problem motivated by the study of time evolution of quantum systems. We show that the separating subspace $S(D)$ of a Jordan derivation defined on a subalgebra $B$ of a complex Banach algebra $A$ satisfies $B[B \cap S(D)]B \subseteq \text{Rad}_2(A)$ provided that $BAB \subseteq A$ and $\dim(\text{Rad}_2(A) \cap \bigcap_{n=1}^{\infty} B^n) < \infty$, where $\text{Rad}_2(A)$ and $\text{Rad}_2(A)$ denote the Jacobson and the Baer radicals of $A$, respectively. From this we deduce the closability of partially defined derivations on complex semiprime Banach algebras with appropriate domains. The result applies to several relevant classes of algebras.

0. Introduction. The study of partially defined derivations is motivated by the time evolution and spatial translation in quantum physics. The general theory of partially defined derivations on Banach algebras is mainly concerned with the theory of closability, generator properties and classification of closed derivations. For a thorough treatment of this topic we refer the reader to [1] and [20]. On the other hand, any everywhere defined derivation on a nonassociative complete normed algebra $A$ yields a meaningful partially defined derivation on the Banach algebra $L(A)$ of all continuous linear operators on $A$ (see [23, 24] for more details).

It is appropriate to point out that there are examples of nonclosable partially defined derivations on $C^*$-algebras (see [1; Example 1.4.4]). In contrast, the case where derivations are everywhere defined is by far more satisfactory. B. E. Johnson and A. M. Sinclair [14] showed that everywhere defined derivations on semisimple Banach algebras are automatically continuous. However, at present the answers to the following equivalent questions remain open, even for commutative Banach algebras (see [3, 7, 8, 16, 19]).

1. Is the separating subspace of any everywhere defined derivation on a Banach algebra contained in the Baer radical of the algebra?

1991 Mathematics Subject Classification: Primary 46H40.
2. Are everywhere defined derivations continuous on semiprime Banach algebras?
3. Are everywhere defined derivations continuous on prime Banach algebras?

R. V. Gamidov, [7, 8] and V. Runde [19] give some affirmative answers to the preceding questions provided that suitable additional requirements on the algebra are fulfilled. J. Cusack [3] showed that if the answer to any of the preceding question is negative, then there exists a topologically simple radical Banach algebra, no example of which is currently known.

In this paper we continue the work started in [25, 26] by investigating the questions listed above for a Jordan derivation $D$ defined on a subalgebra $B$, which is not assumed to be closed or dense, of a complex Banach algebra $A$, satisfying $BAB \subset A$. We prove that $B[B \cap S(D)]B \subset \text{Rad}_{\sigma}(A)$ if $\dim(\text{Rad}_{\sigma}(A) \cap \bigcap_{n=1}^{\infty} B^n) < \infty$, where $\text{Rad}_{\sigma}(A)$ and $\text{Rad}_{\sigma}(A)$ denote the Jacobson and the Baer radicals of $A$ respectively. From this we deduce that $D$ is automatically closable in each of the following situations.

1. $A$ is semiprime, $B$ is dense in $A$, and $\dim(\text{Rad}_{\sigma}(A) \cap \bigcap_{n=1}^{\infty} B^n) < \infty$.
2. $A$ is prime, $B$ is dense, and $\dim\bigcap_{n=1}^{\infty}[B \cap \text{Rad}_{\sigma}(A)]^n < \infty$ for some $b \in B$ with $b^2 \neq 0$.
3. $A$ is an integral domain and $\dim\bigcap_{n=1}^{\infty}[B \cap \text{Rad}_{\sigma}(A)]^n < \infty$ for some nonzero $b \in B$.

1. Algebraic preliminaries. We call a subalgebra $B$ of an algebra $A$ hereditary if it satisfies

$$BAB \subset B.$$ 

For a historical account of this concept we refer the reader to [17], where several remarkable properties of hereditary subalgebras can be found.

The important point to note here is the following result stated in [26; Lemma 1].

**Lemma 1.** Let $B$ be a hereditary subalgebra of a complex Banach algebra $A$ acting irreducibly on a complex Banach space $X$ such that $B^X \neq 0$. Then $B$ is strictly dense on the quotient complex normed space $BX/M$, where $M = \{y \in BX : By = 0\}$, with the natural action $b(y + M) = By + M$.

**Examples 1.1.** Pedersen [18, 15.1] defines the hereditary subalgebras of a C*-algebra $A$ as those *-subalgebras $B$ of $A$ for which $0 \leq a \leq b$, $b \in B$ and $a \in A$ implies $a \in B$. Assume that $B$ is a hereditary subalgebra of a C*-algebra $A$ in the sense of Pedersen. Let $b \in B$. If $a \in A$ with $0 \leq a$, then [18; Proposition 1.3.5] shows that $b^*ab \leq \|b\|b^*b \in B$ and therefore $b^*ab$ lies in $B$. For an arbitrary $a \in A$ we can write $a = (a_1 - a_2) + i(a_3 - a_4)$ for suitable $a_k \in A$ with $0 \leq a_k$, for $k = 1, 2, 3, 4$. Hence $b^*ab = b^*a_1b - b^*a_2b + i(b^*a_3b - b^*a_4b) \in B$. If $b_1, b_2 \in B$ and $a \in A$ we have $b_1ab_2 = \frac{1}{4} \sum_{k=0}^{3} (-i)^k(b_1^* + ib_2) a(b_1^* + ib_2) b_2 \in B$. Consequently, $BAB \subset B$. Actually, it is known [6] that for a closed *-subalgebra $B$ of $A$, Pedersen’s condition and the property $BAB \subset B$ are equivalent.

2. The one-sided ideals of an algebra $A$ are easily checked to be hereditary subalgebras of $A$.

3. For any elements $a, b$ in an algebra $A$, $aAb$ is a hereditary subalgebra of $A$.

4. The intersection of an arbitrary family of hereditary subalgebras of an algebra $A$ is a hereditary subalgebra of $A$. Accordingly, the intersection of a left ideal and a right ideal of $A$ is a hereditary subalgebra of $A$. However, arbitrary hereditary subalgebras of $A$ may be far from having this form. We illustrate this fact in the following example.

5. Let $A$ be the commutative Banach algebra of all bounded complex-valued functions on the interval $[0, 1]$. The set $B$ of those $f$ in $A$ which are differentiable at 0 and satisfy $f(0) = 0$ is easily seen to be a hereditary subalgebra of $A$ which is not an ideal of $A$.

Let $A$ be an algebra. A linear map $D$ from a subalgebra $B$ of $A$ to $A$ is said to be a derivation if it satisfies

$$D(ab) = D(a)b + aD(b) \quad \forall a, b \in B.$$ 

$D$ is said to be a Jordan derivation if

$$D(a^2) = D(a)a + aD(a) \quad \forall a \in B.$$ 

The Jordan derivation identity is equivalent to assuming that

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b) \quad \forall a, b \in B,$$

where

$$a \cdot b = \frac{1}{2}(ab + ba)$$

is the Jordan product of $a$ and $b$ in $A$. It should be noted that every derivation is a Jordan derivation. I. N. Herstein [12] showed that any everywhere defined Jordan derivation on a prime ring of characteristic other than 2 is a derivation, and J. M. Cusack [2] extended this result to the case of any semiprime ring in which $2a = 0$ implies $a = 0$.

A linear subspace $I$ of $A$ is said to be a Jordan ideal of $A$ if

$$I \cdot A + A \cdot I \subset I.$$ 

We also define

$$U_a(b) = 2a \cdot (a \cdot b) - a^2 \cdot b = aba.$$
for all \( a, b \in A \). For an arbitrary subset \( C \) of \( A \) we define

\[
C^{(1)} = \bigcap_{k=1}^{\infty} U_C . \cdot b_k^k U_C(C) \quad \text{and} \quad C^{(n+1)} = (C^{(n)})^{(1)}
\]

for all \( n \in \mathbb{N} \). If \( U_C(C) \subseteq C \), then \( \{C^{(n)}\} \) is a decreasing sequence of subsets of \( C \). Furthermore, if \( U_C(C) \subseteq C \), then \( U_{C^{(n)}}(A) \subseteq C^{(n)} \) for all \( n \in \mathbb{N} \) since \( U_{\alpha_i}(b) = U_\alpha U_\beta U_\gamma(c) \) for all \( a, b, c \in A \).

**Examples 2.1.** Let \( A \) be any Banach algebra of power series with complex coefficients and \( B \) the two-sided ideal of \( A \) of those power series in \( A \) with zero constant term. It is easy to see that \( B^{(1)} = 0 \).

2. Let \( A \) denote the Banach space of those formal power series with complex coefficients \( f = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n y^n \) such that \( ||f|| = \sum_{n=1}^{\infty} ||a_n|| + \sum_{n=1}^{\infty} ||b_n|| < \infty \). We make \( A \) into a Banach algebra by defining the product

\[
\left( \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n y^n \right) \left( \sum_{n=1}^{\infty} c_n z^n + \sum_{n=1}^{\infty} d_n y^n \right)
= \sum_{n=1}^{\infty} \sum_{i+j=n} a_i c_j z^n + \sum_{i+j=n} \left[ \sum_{i=1}^{\infty} a_i d_n \right] y^n + \sum_{i+j=n} b_i d_j y^n.
\]

An easy computation shows that \( A^{(1)} = \{ f \in A : a_n = 0 \text{ for all } n \in \mathbb{N} \} \) and \( A^{(2)} = 0 \). Furthermore, \( A \) is easily seen to be semiprime.

The *Baer radical* of an algebra \( A \), from now on denoted by \( \text{Rad}_B(A) \), is the intersection of all prime ideals of \( A \). \( A \) is semiprime if \( \text{Rad}_B(A) = 0 \).

In order to study, in the fifth section, the nilpotency of the separating subspace of a partially defined Jordan derivation we require the following results about the Baer radical.

**Lemma 2.2.** Let \( I \) be a Jordan ideal of an algebra \( A \) satisfying \( b^n \in \text{Rad}_B(A) \) for all \( b \in I \) for some fixed \( n \in \mathbb{N} \). Then \( I \subseteq \text{Rad}_B(A) \).

**Proof.** First, we claim that \( b^2 \in \text{Rad}_B(A) \) for all \( b \in I \). Indeed, let \( b \in I \) and consider the quotient linear space \( A/\text{Rad}_B(A) \) endowed with the product defined by

\[
(a_1 + \text{Rad}_B(A)) \circ (a_2 + \text{Rad}_B(A)) = a_1 b^2 a_2 + \text{Rad}_B(A).
\]

For each \( a \in A \) we have \( bab \in I \) and therefore \( (bab)^n \in \text{Rad}_B(A) \), which gives \( (a + \text{Rad}_B(A))^{n+2} = 0 \). [17, Proposition 4.4.10] shows that \( (A/\text{Rad}_B(A))^{n+2} = 0 \) for a suitable \( n \in \mathbb{N} \). From this it may be concluded that the principal two-sided ideal \( I_b^2 \) of \( A \) generated by \( b^2 \) satisfies \( I_b^{n+1} \subseteq \text{Rad}_B(A) \). Hence \( I_b \subseteq \text{Rad}_B(A) \), which proves our claim.

If \( a, b \in I \), then \( ab + ba \in I \) and consequently \( ab + ba \in I \) and \( ab + ba \in I \) and consequently \( ab + ba \in I \) and consequently \( b^2 \in \text{Rad}_B(A) \). Therefore \( bab = b(ab + ba) - b(ab + ba) - b(ab + ba) \in \text{Rad}_B(A) \).

Arguing as before with the product on \( A/\text{Rad}_B(A) \) replaced by

\[
(a_1 + \text{Rad}_B(A)) \circ (a_2 + \text{Rad}_B(A)) = a_1 b a_2 + \text{Rad}_B(A)
\]

we deduce that a suitable power of the principal two-sided ideal \( I_b \) of \( A \) generated by \( b \) is contained in \( \text{Rad}_B(A) \). Thus \( b \in \text{Rad}_B(A) \).

**Lemma 3.** Let \( B \) be a hereditary subalgebra of a Banach algebra \( A \). Then \( B \cap \text{Rad}_B(A) \subseteq \text{Rad}_B(B) \) and \( B \text{Rad}_B(B) \subseteq \text{Rad}_B(A) \).

**Proof.** Let \( b \in B \cap \text{Rad}_B(A) \). It is a simple matter to check that \( A \) becomes a Banach algebra for the product \( a \circ b = a b a_2 \) and norm \( ||a|| = (1 + ||b||)||a|| \). For each \( a \in A, ba \) lies in \( \text{Rad}_B(A) \) and therefore \( (ba)^n = 0 \) for a suitable \( n \in \mathbb{N} \). This clearly forces that the \((n+2)th\) power of a with respect to the new product equals zero. Accordingly \( (A, \circ, \cdot, \cdot) \) is a nil Banach algebra and [17, Proposition 4.4.11(b)] shows that it is nilpotent. This means that there is a fixed \( N \in \mathbb{N} \) such that \( a b \ldots a b N = 0 \) for all \( a_1, \ldots, a_N \in A \). Hence the principal two-sided ideal \( I_b \) of \( B \) generated by \( b \) satisfies \( I_b^N = 0 \). Thus [17, Theorem 4.4.7] shows that \( I_b \) is contained in \( \text{Rad}_B(B) \). Consequently, \( b \in \text{Rad}_B(B) \), which proves our first assertion.

Let \( b \in \text{Rad}_B(B) \) and \( c, d \in B \). For each \( a \in A, b b d c \in \text{Rad}_B(B) \) and therefore \( (b b d c)^n = 0 \) for some \( n \in \mathbb{N} \). We argue as before to prove that \( A \) becomes a nilpotent Banach algebra for the product \( a \circ b = a (b c) a_2 \) and norm \( ||a|| = (1 + ||b c||)||a|| \). Consequently, the principal two-sided ideal \( A \) generated by \( b c a \) is nilpotent and therefore it is contained in \( \text{Rad}_B(A) \). This establishes our second assertion.

**2. Sliding hump sequences for Jordan ideals in hereditary subalgebras.** This section is devoted to the construction of appropriate sequences to apply a classical method in automatic continuity theory, the sliding hump argument.

Throughout this section, \( B \) denotes a hereditary subalgebra of a complex Banach algebra \( A \), \( X \) a complex irreducible Banach left \( A \)-module, and \( I \) a Jordan ideal of \( B \) such that \( B I B X \neq 0 \). Let \( Y = B X \) and \( M = \{ y \in Y : B y = 0 \} \). To shorten notation, we let \( x + M \) stand for the equivalence class \( x + M \).

**Lemma 4.** Let \( x, y \in Y \) with \( B x \neq 0 \). Then there is \( a \in I \) such that \( B (a x - y) = 0 \).

**Proof.** We claim that there exists \( a \in I \) such that \( a x + M \neq 0 \). Otherwise, for all \( a \in I \) and \( b \in B \), we have \( 0 M = (b a + a b) x + M = a b x + M \). Consequently, \( IB X = B X \subseteq M \), a contradiction.

Assume that \( \dim(Y/M) > 1 \) and take \( a \in I \) with \( a x + M \neq 0 \). Then we can choose \( b_1 \in B \) such that \( b_1 a x + M \) and \( x + M \) are linearly independent. Therefore there exists \( b_2 \in B \) satisfying \( b_2 b_1 a x + M = y + M \) and \( b_2 x + M = 0 \).
Now \( a = (a_1 \cdot b_1) \cdot b_2 + a_1 \cdot (b_1 \cdot b_2) - (a_1 \cdot b_2) \cdot b_1 = a_1 b_2 b_2 + b_2 b_1 a_1 \) satisfies our requirements.

If \( \dim(Y/M) = 1 \), then we can take \( a_1 \in I \) such that \( a_1 x^M = x^M \). Since \( y^M = \lambda x^M \) for a suitable \( \lambda \in \mathbb{R} \), \( a = a_1 \lambda \) has the desired properties.

**Lemma 5.** Let \( x, y, \xi \in BX \) with \( x^M \) and \( y^M \) linearly independent. Then there exists \( a \in I \) such that \( ax = 0 \) and \( B(ay - \xi) = 0 \).

**Proof.** If \( x^M = 0^M \), then we can take \( a = 0 \). Otherwise we can choose \( b_1 \in I \) and \( b_2 \in B \) such that \( b_1 x^M = y^M \), \( b_2 x^M = 0^M \), and \( b_2 y^M = z^M \). Then \( B(b_1 b_2) \in I \) and satisfies the required conditions.

**Lemma 6.** If \( I \) is a Jordan ideal of \( B \) such that \( BJX \neq 0 \), then \( B(I \cap J)BX \neq 0 \).

**Proof.** Take \( x \in BX \) with \( Bx \neq 0 \). According to Lemma 4, we can choose \( a \in I \) and \( b \in J \) such that \( B(ax - x) = B(bx - x) = 0 \). We have \( bab \in I \cap J \) and \( B(babx - x) = 0 \). Consequently, \( 0 \neq Bx = B(babx) \subset B(I \cap J)BX \).

**Lemma 7.** Let \( x, y \in Y \) with \( Bx \neq 0 \) and assume that there exist \( \epsilon \in B \) and a linear functional \( f \) on \( B \) such that \( B(\epsilon x - y) = 0 \) and \( (\epsilon f)(b)X = 0 \) for all \( b \in B \). Then there exists \( a \in I \) and a linear functional \( f \) on \( B \) satisfying \( B(ax - y) = 0 \) and \( (ab - f(b)a)X = 0 \) for all \( b \in B \).

**Proof.** If \( By \neq 0 \), then Lemma 4 gives \( a_1 \in I \) such that \( B(a_1 y - x) = 0 \). Take \( a = Ua_1 \in I \) and \( f(b) = g(bu_1) \) for all \( b \in B \). If \( By = 0 \), then we choose \( a_1 \in I \) with \( B(a_1 x - y) = 0 \). Take \( a = Ua_1 \) and \( f(b) = g(a_1 b a_1) \) for all \( a_1 \in B \).

**Lemma 8.** One of the following assertions hold:

1. For all \( x, y \in Y \) with \( Bx \neq 0 \), there are \( a \in I \) and a linear functional \( f \) on \( B \) such that \( B(ax - y) = 0 \) and \( (ab - f(b)a)X = 0 \) for all \( b \in B \).
2. There are sequences \( \{a_n\} \) in \( I \) and \( \{x_n\} \) in \( Y \) such that \( B a_n \cdots a_1 x_n \neq 0, a_n \cdots a_1 x_n^M = 0, \) and \( a_n \cdots a_1 X_n \neq 0 \).

**Proof.** If \( \dim Y/M = 1 \) for some \( b \in B \), then [26; Lemma 4] and the preceding lemma show that our first assertion holds.

Assume \( \dim Y/M \geq 2 \) for every \( b \in B \setminus \{0\} \). Take \( x_1 \in BX \) with \( Bx_1 \neq 0 \) and apply Lemma 4 to get \( a_1 \in I \) such that \( B(a_1 x_1 - x_1) = 0 \). Assume that \( a_2 \ldots, a_n \in B \), \( a_2 \ldots, a_n \in BX \), and \( a_2 \ldots, a_n \in BX \) have been chosen satisfying the required conditions. Since \( \dim(a_n \cdots a_1 Y/M) \geq 2 \), \( a_n \cdots a_1 x_n^M \) and \( a_n \cdots a_1 x_n^M \) are linearly independent for a suitable \( x_n \in BX \). Moreover, there is \( y_n \in BX \) such that \( a_n \cdots a_1 y_n \neq 0 \). By Lemma 5 there is \( \alpha_{n+1} \in I \) such that \( a_n \cdots a_1 x_n \alpha_{n+1} = 0 \) and \( a_n \cdots a_1 x_n \alpha_{n+1} = y_n \). Note that \( a_n \cdots a_1 x_n + a_2 \ldots, a_n x_n + a_1 \ldots, a_n x_n + a_1 \ldots, a_n x_n = a_1 \ldots, a_n y_n \neq 0 \). The sequences \( \{a_n\} \) and \( \{x_n\} \) constructed in this way satisfy the requirements of the second assertion of the lemma.

3. The sliding hump procedure for partially defined operators between Banach spaces. Let \( X \) and \( Y \) be Banach spaces. A linear operator \( F \) from a linear subspace \( X_0 \) of \( X \) into \( Y \) is said to be closed if its graph is a closed subset of \( X \times Y \), and we call \( F \) closable if there is a closed extension of \( F \). We can measure the closability of \( F \) by considering its separating subspace, which is defined as the subspace \( S(F) \) of those elements \( y \) in \( Y \) for which there is a sequence \( \{x_n\} \) in \( X_0 \) satisfying \( \lim x_n = 0 \) and \( \lim F(x_n) = y \). \( S(F) \) is a closed linear subspace of \( Y \) and \( F \) is closable if, and only if, \( S(F) = 0 \). In this case there is a smallest closed extension of \( F \) called the closure of \( F \), the domain of which is the set of those \( x \in X \) for which there exists a sequence \( \{x_n\} \) in \( X_0 \) such that \( \lim x_n = x \) and \( \{F x_n\} \) converges.

**Lemma 9.** Let \( X \) and \( Y \) be Banach spaces, and let \( F \) be a closable linear operator from a linear subspace \( X_0 \) of \( X \) into \( Y \). If \( T \) is a continuous linear operator from \( X \) into \( X_0 \), then \( FT \) is continuous.

**Proof.** Let \( G \) be the closure of \( F \) and let \( X_1 \) be its domain. \( X_1 \) becomes a Banach space for the norm given by \( ||x|| = ||x|| + ||Gx|| \).

It is clear that \( G \) is \( ||x|| \cdot || \cdot || \)-continuous. Moreover, it is immediate that \( T \) is a closed linear operator from \( (X_1, || \cdot ||) \) and the closed graph theorem shows that it is \( || \cdot || \cdot || \)-continuous. Accordingly, the composition \( FT = GT \) is \( || \cdot || \cdot || \)-continuous.

The next important results illustrate the typical sliding hump procedure. The first one was essentially stated by M. P. Thomas in [22; Proposition 1.3] for everywhere defined operators between Banach spaces. We adapt the proof given here to partially defined operators between Banach spaces.

**Lemma 10.** Let \( X \) be a Banach space, \( Y \) a normed space, \( \{T_n\} \) a sequence of continuous linear operators from \( X \) into a linear subspace \( X_0 \) of \( X \), and \( \{Q_n\} \) be a sequence of continuous linear operators from \( Y \) into Banach spaces \( Y_n \). If \( F \) is a linear operator from \( X_0 \) into \( Y \) such that \( Q_n F T_1 \ldots T_m \) is continuous for \( m > n \), then \( Q_n F T_1 \ldots T_m \) is continuous on \( X_0 \) for sufficiently large \( n \).

**Proof.** There is no loss of generality in assuming \( ||Q_n|| = ||T_n|| \leq 1 \) for all \( n \). If the result fails, then we could choose a strictly increasing sequence \( \{n_k\} \) of natural numbers and a sequence \( \{x_n\} \) in \( X_0 \) such that

\[
||x_k|| < 2^{-k} \min\{||Q_j F T_1 \ldots T_m|| : j < k \},
\]

\[
||Q_n F T_1 \ldots T_m x_k|| \geq 1 + k + \left| \sum_{j=1}^{k-1} T_1 \ldots T_m x_j \right|
\]

for all \( k \in \mathbb{N} \). Consider the element \( x \in X \) given by \( x = \sum_{k=1}^\infty T_1 \ldots T_m x_j \).
and, for every $k \in \mathbb{N}$, let $y_k = x_k + \sum_{j=k+1}^{\infty} T_n x_j$ (the series can easily be shown to converge). Note that $x = T_1 \ldots T_n y_1 \in X_0$ and $y_k = x_k + T_{n,k+1} \ldots T_n x_j \in X_0$ for all $k \in \mathbb{N}$. As in the proof of [22; Proposition 1.3], for each $k \in \mathbb{N}$ we would have $\|F x\| \geq k$, a contradiction.

The following result was established by N. P. Jewell and A. M. Sinclair [13] for everywhere defined operators between Banach spaces but the basic principle was stated earlier by Sinclair [21; Lemma 1.6].

**Lemma 11.** Let $X$ and $Y$ be Banach spaces, $\{T_n\}$ a sequence of continuous linear operators from $X$ into a linear subspace $X_0$ of $X$, and $\{R_n\}$ be a sequence of continuous linear operators from $Y$ into itself. If $F$ is a linear operator from $X_0$ into $Y$ such that $FT_n - R_n$ is continuous on $X_0$ for all $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $R_1 \ldots R_m S(F) = R_1 \ldots R_m S(F)$ for all $n \geq m$.

**Proof.** Suppose the lemma were false. Then, as in the classical proof, we could assume that $R_1 \ldots R_m S(F)$ is strictly contained in $R_1 \ldots R_m S(F)$ and $\|T_n\| \leq 1$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $Q_n$ denote the quotient map from $Y$ onto $Y/R_1 \ldots R_m S(F)$. [17; Proposition 6.1.9(c)] shows that $Q_n R_1 \ldots R_m F$ is closable and $Q_n R_1 \ldots R_{m-1} F$ is discontinuous for each $n \in \mathbb{N}$.

Note that for $n, m \in \mathbb{N}$ with $m \geq n$,

$$Q_n F T_1 \ldots T_m = (Q_n R_1 \ldots R_m F) T_{n+1} \ldots T_m$$

$$- \sum_{k=1}^{n} R_1 \ldots R_{k-1} (R_k F - FT_k) T_{k+1} \ldots T_m$$

and the operator $\sum_{k=1}^{n} R_1 \ldots R_{k-1} (R_k F - FT_k) T_{k+1} \ldots T_m$ is continuous. If $m > n$, then Lemma 9 shows that $(Q_n R_1 \ldots R_m F) T_{n+1} \ldots T_m$ and hence $Q_n F T_1 \ldots T_m$ is continuous. The preceding lemma now yields that $Q_n F T_1 \ldots T_m$ and consequently $Q_n R_1 \ldots R_m F$ are continuous for sufficiently large $n$. The rest of the proof runs as in [13, 21]. The details are left to the reader. 

**Lemma 12.** Let $X$ and $Y$ be Banach spaces, $T$ a continuous linear operator from $X$ into a linear subspace $X_0$ of $X$, and let $F$ be a linear operator from $X_0$ into $Y$ such that $F T - T F$ is continuous on $X_0$. Then $\bigcap_{n=1}^{\infty} T^n S(F)$ is dense in $T^n S(F)$ for a suitable $m \in \mathbb{N}$.

**Proof.** By the preceding lemma, there is $m \in \mathbb{N}$ such that $T^n S(F) = T^n S(F)$ for all $n \geq m$. Since

$$T^{m+1} S(F) \subset T(T^n S(F)) \subset T(T^n S(F)) = T^n S(F),$$

$T$ is a continuous linear operator from the Banach space $T^n S(F)$ onto a dense linear subspace of it. The Mittag-Leffler theorem [4; Theorem 5.3] now shows that $\bigcap_{n=1}^{\infty} T^n S(F)$ is dense in $T^n S(F)$. $T(S(F))$ is easily checked to be a subset of $S(F)$. Hence $T^n S(F) \subset S(F)$ and therefore $T^n S(F) \subset S(F)$. Consequently,

$$\bigcap_{n=1}^{\infty} T^n (T^n S(F)) \subset \bigcap_{n=1}^{\infty} T^n S(F) \subset T^n S(F),$$

which establishes the desired conclusion.

**4. Quasinilpotency of the separating subspace of a Jordan derivation.** From now on, $D$ stands for a Jordan derivation on a complex Banach algebra $A$ defined on a hereditary subalgebra $B$ of $A$.

In the sequel, $\text{Rad}_D(A)$ denotes the Jacobson radical of $A$ so that $\text{Rad}_D(A)$ is the intersection of all the primitive ideals of $A$. It is well known that $\text{Rad}_D(A)$ is the largest two-sided ideal of $A$, each element $a$ of which is quasinilpotent, i.e. $\lim \|a^n\|^{1/n} = 0$.

The separating subspace of $D$ is easily checked to satisfy

$$B \cdot S(D) \subset S(D).$$

Consequently, $B \cap S(D)$ is a Jordan ideal of $B$.

**Lemma 13.** Let $\{X_n\}$ be a sequence of complex irreducible Banach left $A$-modules. Assume that $\{a_n\}$ is a sequence in $B$ and, for each $n \in \mathbb{N}$, there exists $x_n \in X_n$, satisfying $B a_n \ldots a_1 x_n = 0$, $a_{n-1} \ldots a_1 x_n = 0$, and $a_{n-1} \ldots a_1 x_n = 0$. Then $B [B \cap S(D)] B X_n = 0$ for some $n \in \mathbb{N}$.

**Proof.** We apply Lemma 11 to the operators $F = D$ and $R_n = T_n = U_n$ for $n \in \mathbb{N}$ to deduce that

$$a_1 \ldots a_{n+1} S(D) a_{n+1} \ldots a_1 = a_1 \ldots a_m S(D) a_m \ldots a_1$$

for a suitable $m \in \mathbb{N}$. Since $a_1 \ldots a_{n+1} S(D) a_{n+1} \ldots a_2 x_m = 0$, it follows that $a_1 \ldots a_m S(D) a_{n+1} \ldots a_2 x_m = 0$. We claim that $B [B \cap S(D)] B X_n = 0$. Suppose, contrary to our claim, that $B [B \cap S(D)] B X_n \neq 0$. We conclude from Lemma 4 that $B \cap S(D) a_{n+1} \ldots a_2 x_m = B x_m$ and hence that $a_1 \ldots a_m B x_m = 0$, a contradiction.

**Lemma 14.** Let $X$ be a complex irreducible Banach left $A$-module and $I$ a Jordan ideal of $B$ such that $B I B X \neq 0$. Assume that there are $a_1, \ldots, a_n \in B$ such that $a_1 \ldots a_n B X \neq 0$, $B a_n \ldots a_1 x \neq 0$, and $u a_1 \ldots u a_n (I) < 0$. Then $B [B \cap S(D)] B X = 0$.

**Proof.** Let $x \in X$ and $y \in B X$ such that $B a_n \ldots a_1 x = 0$ and $a_1 \ldots a_n y \neq 0$. Lemma 4 shows that $b a_n \ldots a_1 x = y$ for a suitable $b \in I$. 


If \( B[\mathcal{B} \cap \mathcal{S}(D)]B \) were not zero, Lemma 4 would give an \( a \in \mathcal{S}(D) \) such that \( ay = a_0 \ldots a_n x \). Since \( U_B(B) \subset I \), we have \( \dim U_B \ldots U_B U_B(B) < \infty \) and therefore \( DU_B \ldots U_B U_B \) is continuous. On the other hand, it is immediate that \( DU_{a_0} \ldots U_B U_B U_B \) is continuous. Accordingly, \( U_{a_0} \ldots U_B U_B \) is continuous and hence \( U_{a_0} \ldots U_B U_B (\mathcal{S}(D)) = 0 \). Therefore we would have \( 0 = a_0 \ldots a_n b_a b_m \ldots \ldots a_n x = a_0 \ldots a_n y \), which contradicts the choice of \( y \).

Since, for every \( b \in B \), \( DU_B \ldots U_B \) is easily seen to be continuous, a straightforward application of Lemma 12 gives the following.

**Lemma 15.** If \( b \in B \), then \( \sum_{n=0}^{\infty} b^n S(D)b^n \) is dense in \( b^n S(D)b^n \) for a suitable \( B \) in \( \mathbb{N} \).

**Lemma 16.** Let \( B \) be a complex irreducible Banach left \( A \)-module, \( I \) a Jordan ideal of \( B \) contained in \( S(D) \) such that \( BIBX \not= 0 \), and \( \nu \in I \).

Then, for all \( x, y \in BX \) with \( Bx \neq 0 \), there exists \( a \in I \) and a linear functional \( f_y \) on \( B \) satisfying \( B(ax - y) = 0 \) and \( abf = f(b)aBx = 0 \) for all \( b \in B \).

**Proof.** Fix \( x \in BX \) with \( Bx \neq 0 \). Let us prove that for every \( n \in \mathbb{P} \) there exist \( a_n \) and \( f_n \) on \( B \) satisfying the desired conditions with \( y = x \).

From Lemmas 8 and 13 we deduce that there are \( b_1 \in I \) and a linear functional \( g_1 \) on \( B \) such that \( B(b_1 x - z) = 0 \) and \( b_1 b_2 g(b_2 - x) \). According to the preceding lemma, \( \sum_{n=0}^{\infty} b^n S(D)b^n \) is dense in \( b^n S(D)b^n \) for a suitable \( B \) in \( \mathbb{N} \). Since \( b_1^{n+1} x - z = \sum_{k=0}^{n+1} b_1^{k+1} - x \), we have \( B(b_1^{n+1} x - z) = 0 \), which gives \( b_1^{n+1} x - Bx = 0 \). As \( b_1^{n+1} \in b^n S(D)b^n \), there exists \( b_2 \in B \) such that \( Bb_2x \neq 0 \). If \( b_2x = 0 \) and \( x = 0 \), then \( b_2x = x \). Otherwise \( b_2x = x \). Anyway, we have \( a_1 \in U_B U_B \subseteq I \) and \( B(ax - y) = 0 \). We can put \( b_2 = b_2b_1 \) for a suitable \( b_1 \in S(D) \). Define \( f_1 \) on \( B \) by \( f_1(x) = g(b_1 b_2 x) \). It is immediate that \( a_1 b_2 = f(b_2x) \). For all \( b \in B \), we have

\[
U_B U_B S(D)b^n \subseteq U_B U_B S(D)b^n \subseteq U_B U_B(B) \subseteq I(1)
\]

such that \( Bb_2x \neq 0 \). If \( b_2x = 0 \) and \( x = 0 \), then \( b_2x = x \). Otherwise \( b_2x = x \). Anyway, we have \( a_1 \in U_B U_B \subseteq I \) and \( B(ax - y) = 0 \). We can put \( b_2 = b_2b_1 \) for a suitable \( b_1 \in S(D) \). Define \( f_1 \) on \( B \) by \( f_1(x) = g(b_1 b_2 x) \). It is immediate that \( a_1 b_2 = f(b_2x) \). For all \( b \in B \), we have

\[
U_B U_B S(D)b^n \subseteq U_B U_B S(D)b^n \subseteq U_B U_B(B) \subseteq I(1)
\]

Finally, let \( x, y \in BX \) with \( Bx \neq 0 \) and \( y \in I \). If \( x = 0 \) and \( y = 0 \), then we choose \( a_0 \in I \), \( a_0 \in B \), and a linear functional \( f_0 \) on \( B \) satisfying \( B(a_0y - y) = B(a_0x - y) = B(a_0y - y) = 0 \), and \( (a_0 b_0 - f_0(b_0 a_0)X = 0 \) for all \( b \in B \). In this case the element \( a = a_0 b_0 a_0 \) and the linear functional \( f \) defined on \( B \) by \( f(b) = f_0(b_0 a_0) \) satisfy our requirements. If \( y = \lambda x \) for some \( \lambda \in \mathbb{C} \), then we take \( a_0 \in I \) and a linear functional \( f_0 \) on \( B \) satisfying \( B(a_0 x - y) = 0 \) and \( (a_0 b_0 f_0(b_0 a_0)X = 0 \) for all \( b \in B \). Now \( a = \lambda a_0 \) and \( f = \lambda f_0 \) have the desired properties.

To simplify notation, here and subsequently, we write \( (C) \) to denote the linear subspace generated by the subset \( C \) of \( A \).

We can now improve the main result in [26] as follows.

**Theorem 1.** If \( \dim (B^{(p)} \cap \mathcal{R} \mathcal{J}(A)) < \infty \) for some \( p \in \mathbb{N} \), then \( B[S(D) \cap B] \subseteq \mathcal{R} \mathcal{J}(A) \).

**Proof.** Suppose, contrary to our claim, that the set \( \mathcal{P} \) of those primitive ideals \( p \) of \( A \) for which \( B[S(D) \cap B] \subseteq \mathcal{P} \) is nonempty.

Define \( I_0 = B[S(D)]B \). We take \( P \subseteq \mathcal{P} \), which is the kernel of a continuous irreducible representation of \( A \) on a complex Banach space \( X_k \).

If \( Bx \neq 0 \) we can choose \( x \in BX \) with \( Bx \neq 0 \). Lemma 16 now gives \( a_0 \in I_0 \) and a linear functional \( f_0 \) on \( B \) such that \( B(a_0 x - x) = 0 \) and \( a_0 b_0 f_0(b_0 a_0)X = 0 \) for all \( b \in B \). We observe that \( f_0(a_0) = 1 \) and \( a_0 = 0 \).

Suppose that \( P_1, \ldots, P_n, X_1, \ldots, X_n, a_1, \ldots, a_n, a_1, \ldots, a_n \) and \( f_1, \ldots, f_n \) have been chosen satisfying the following conditions for \( k = 1, \ldots, n \):

(i) \( P_k \subseteq \mathcal{P} \) is the kernel of a continuous irreducible representation of \( A \) on a complex Banach space \( X_k \).

(ii) \( x_k \in X_k, a_0 \in I_{k-1} \), where \( I_{k-1} = I_0 \cap P_1 \cap \ldots \cap P_{k-1} \).

(iii) \( f_k \) is a linear functional on \( B \) such that \( a_k b_0 f_k(b_0 a_k)X = 0 \) for all \( b \in B \).

We claim that there is a \( P_{n+1} \in \mathcal{P} \) such that \( I_n \not\subseteq P_{n+1} \) and \( a_1 \ldots a_n B \not\subseteq P_{n+1} \). Suppose the claim were false. Then we would have \( a_1 \ldots a_n B \not\subseteq P_{n+1} \) and \( a_1 \ldots a_n B \not\subseteq P_{n+1} \) from which we deduce that \( \dim U_{a_0} \ldots U_{a_0} \subseteq I_{n-1} \subseteq I_{n+1} \). Indeed, for every \( a \in I_0 \), we have

\[
\text{dim} \left( \bigcap_{P \in \mathcal{P}, a \in P} P \right) = \infty
\]

Finally, let \( x, y, \subset BX \) with \( Bx \neq 0 \) and \( y \subseteq I \). If \( x = 0 \) and \( y = 0 \), we choose \( a_0 \in I \). Then \( f_0(\bar{a}_0) = 0 \) and the linear functional \( f_0 \) on \( B \) satisfying \( B(a_0y - y) = B(a_0x - y) = B(a_0y - y) = 0 \), and \( (a_0 b_0 f_0(b_0 a_0)X = 0 \) for all \( b \in B \). In this case the element \( a = a_0 b_0 a_0 \) and the linear functional \( f \) defined on \( B \) by \( f(b) = f_0(b_0 a_0) \) satisfy our requirements. If \( y = \lambda x \) for some \( \lambda \in \mathbb{C} \), then we take \( a_0 \in I \) and a linear functional \( f_0 \) on \( B \) satisfying \( B(a_0 x - y) = 0 \) and \( (a_0 b_0 f_0(b_0 a_0)X = 0 \) for all \( b \in B \). Now \( a = \lambda a_0 \) and \( f = \lambda f_0 \) have the desired properties.
Consequently, \( b^{2m+1} \in \mathcal{B}^n(S(D))b^{m+1} \subseteq \mathcal{I}^{(1)} \). By the preceding lemma we have \( U_{\mathcal{I}^{(1)}}(I) = 0 \). Hence \( U_{\mathcal{I}^{(1)}}(I^{(3)}) = 0 \) and therefore \( b^{2m+1} = 0 \).

Fix \( b \in I \). For each \( a \in A \), \( b^2ab^2 \in U_b(B) \subseteq I \) and therefore \( b^2ab^2 \equiv 0 \) for some \( n \in \mathbb{N} \). We argue as in Lemma 3 to prove that \( A \) becomes a nilpotent Banach algebra for the product \( a_1 \cdots a_n = a_1b^2a_2 \) and norm \( \|a\| = (1 + \|b^2\|)^n \|a\| \). Hence the principal two-sided ideal \( I_b \) of \( A \) generated by \( b^2 \) is a nilpotent ideal of \( A \) and therefore it is contained in \( \text{Rad}_G(A) \). Consequently, \( b^4 \in \text{Rad}_G(A) \) whenever \( b \) lies \( I \).

**Theorem 2.** If \( \dim(B^{(1)} \cap \text{Rad}_G(A)) < \infty \), then \( B \cap S(D) \subseteq \text{Rad}_G(B) \).

Accordingly, \( B[\mathcal{B} \cap S(D)]B \subset P_n \) for some \( n \), which contradicts the choice of \( P_n \).

**Lemma 17.** If \( \dim(B^{(p)} \cap \text{Rad}_G(A)) < \infty \) for some \( p \in \mathbb{N} \), then \( U_{B[B \cap S(D)]}^{(p)}(S(D)) = 0 \).

**Proof.** Theorem 1 gives \( B[B \cap S(D)] \subseteq \text{Rad}_G(A) \) and therefore \( \dim(B \cap S(D))^{(p)} \subseteq B^{(p)} \cap \text{Rad}_G(A) \). On the other hand, for \( a \in [B \cap S(D)]^{(p)} \) we have \( U_a(B) \subseteq [B \cap S(D)]^{(p)} \). This gives \( \dim U_a(B) < \infty \) and hence \( DU_a \) is continuous. Consequently, \( U_a(S(D)) = 0 \), as required.

**Lemma 18.** Let \( B \subseteq B \cap S(D) \) satisfying \( U_b(B) \subseteq I \) and \( b^2 = 0 \) for all \( b \in I^{(1)} \) for some fixed \( q \in \mathbb{N} \). Then \( b^4 \in \text{Rad}_G(A) \) for all \( b \in I \).

**Proof.** For each \( b \in I \), Lemma 15 shows that \( \bigcap_{n=1}^{\infty} b^nS(D)b^n \) is dense in \( b^nS(D)b^n \) for a suitable \( m \in \mathbb{N} \). Further, we note that

\[
\bigcap_{n=1}^{\infty} b^nS(D)b^n = \bigcap_{n=1}^{\infty} b^nS(D)(b)b^n \subseteq \bigcap_{n=1}^{\infty} U_b^n(I) \subseteq I^{(1)}.
\]

5. Nilpotency of the separating subspace of a Jordan derivation. In this section we use Theorem 1 to study whether \( B[B \cap S(D)] \subseteq \text{Rad}_G(A) \).

**Lemma 19.** Let \( I \) be a subset of \( B \cap S(D) \) satisfying \( U_b(B) \subseteq I \) and \( b^2 = 0 \) for all \( b \in I^{(1)} \) for some fixed \( q \in \mathbb{N} \). Then \( b^4 \in \text{Rad}_G(A) \) for all \( b \in I \).

**Proof.** Let \( I = B \cap S(D) \). From Lemma 17 we see that \( U_{I^{(1)}}(S(D)) = 0 \). Since \( I^{(1)} \subseteq S(D) \), we have \( U_{I^{(1)}}(I^{(1)}) = 0 \). The preceding result shows that \( b^4 \in \text{Rad}_G(A) \) for all \( b \in I \), Lemma 2 now leads to \( I \subseteq \text{Rad}_G(B) \).

**Theorem 3.** If \( \dim(B^{(p)} \cap \text{Rad}_G(A)) < \infty \) for some \( p \in \mathbb{N} \) and \( B \cap \text{Rad}_G(A) \) is nilpotent, then \( B \cap S(D) \subseteq \text{Rad}_G(B) \). Accordingly, \( B[B \cap S(D)]B \subseteq \text{Rad}_G(A) \).

**Proof.** Let \( I = B \cap S(D) \). From Lemma 17 we deduce that \( b^m = 0 \) for all \( b \in I \) for some fixed \( m \in \mathbb{N} \). Lemma 2 completes the proof.

**Corollary 1.** Assume that \( \dim([B \cap \text{Rad}_G(A)]^{(p)}) < \infty \) for some \( p \in \mathbb{N} \) and that \( B \cap \text{Rad}_G(A) \) is nilpotent. Then \( B \cap \text{Rad}_G(A) \subseteq S(D) \subseteq \text{Rad}_G(B) \).

**Proof.** Let \( I = B \cap \text{Rad}_G(A) \) and consider the restriction \( D \). Accordingly, the preceding theorem and we have \( I \cap S(D) \subseteq \text{Rad}_G(I) = \text{Rad}_G(B) \). It is immediate that \( U(I \cap S(D)) \subseteq S(D) \) and therefore

\[
U_{I \cap S(D)}(I \cap S(D)) \subseteq I \cap S(D) \subseteq \text{Rad}_G(B) \subseteq \text{Rad}_G(B).
\]

Consequently, \( b^4 \in \text{Rad}_G(B) \) whenever \( b \) lies in \( I \cap S(D) \). Lemma 2 now gives \( I \cap S(D) \subseteq \text{Rad}_G(B) \) as required.

**Corollary 2.** If \( \dim([B \cap \text{Rad}_G(A)]^{(p)}) < \infty \) for some \( p \in \mathbb{N} \) and some nonzero \( b \in B \) and \( A \) is an integral domain, then \( D \) is d closable.
Proof. Consider the restriction $D_{[B]}$. The preceding corollary shows that $b^2 \cap \text{Rad}_A(A) \cap S(D_{[B]}) \subset \text{Rad}_B(bB) = 0$. Since $bS(D) \subset S(D_{[B]})$ we have

$$b^2 S(D) \cap \text{Rad}_A(A) \subset bB \cap \text{Rad}_A(A) \cap S(D_{[B]}) = 0.$$ 

If $S(D) = 0$, then $D$ is closable. Otherwise $\text{Rad}_A(A) = 0$ and Theorem 1 shows that $B \cap S(D) = 0$. From this we deduce that $B^2 S(D) = 0$, which gives either $B = 0$ or $S(D) = 0$ and hence $D$ is closable. \hfill \Box

6. Densely defined derivations. Throughout this section, $B$ is assumed to be a dense hereditary subalgebra of $A$. If in addition $A$ is semiprime, then it is easy to check that the Cusack theorem [2; Theorem 4] remains true and $D$ is a derivation whose separating subspace is a two-sided ideal of $A$.

Corollary 3. If $\dim(B^{[p]} \cap \text{Rad}_A(A)) < \infty$ for some $p \in \mathbb{N}$, then $S(D) \subset \text{Rad}_A(A)$.

Proof. From Theorem 1 we deduce that

$$B^2 S(D) \subset B \cap S(D) \subset \text{Rad}_A(A)$$

and therefore $A^2 S(D) A \subset \text{Rad}_A(A)$, which gives the desired conclusion. \hfill \Box

Corollary 4. If $\dim(B^{[p]} \cap \text{Rad}_A(A)) < \infty$ for some $p \in \mathbb{N}$ and $A$ is semiprime, then $D$ is closable.

Proof. $B$ is easily seen to be semiprime and Theorem 3 yields $B \cap S(D) = 0$. On the other hand, $U_B(S(D)) \subset B \cap S(D) = 0$. By density, $U_A(S(D)) = 0$. Since $S(D)$ is a two-sided ideal of $A$, we conclude that $S(D) \subset \text{Rad}_A(A)$. \hfill \Box

Corollary 5. If $\dim([bB \cap \text{Rad}_A(A)]^{[p]}) < \infty$ for some $p \in \mathbb{N}$ and some $b \in B$ with $b^2 \neq 0$ and $A$ is prime, then $D$ is closable.

Proof. Consider the restriction $D_{[B]}$. From Corollary 1, it follows that

$$bB \cap \text{Rad}_A(A) \subset S(D_{[B]}) \subset \text{Rad}_B(bB).$$

From Lemma 3 we have $bB \cap \text{Rad}_A(A) \subset \text{Rad}_B(bB) = 0$ and therefore $bB \subset \text{Rad}_B(bB) = 0$. Since $bB \subset \text{Rad}_B(bB)$ we have $bB \subset \text{Rad}_B(\text{Rad}_B(bB)) = 0$. Consequently, $b^2 S(D) B \subset bB \cap \text{Rad}_A(A) \subset S(D_{[B]}) = 0$. Hence $b^2 S(D) A \subset \text{Rad}_A(A) A = 0$. Since $b^2 \neq 0$, the primeness of $A$ gives either $S(D) = 0$ or $\text{Rad}_A(A) = 0$. In the last case, Corollary 3 shows that $D$ is closable. \hfill \Box

Corollary 6. If $A$ is semiprime, then $D$ is closable if, and only if, $[S(D)]^{[p]} = 0$ for some $p \in \mathbb{N}$.

Proof. If $[S(D)]^{[p]} = 0$, then we apply Theorem 3 to $D_{[B]} S(D)$ to obtain $B \cap S(D) \subset S(D_{[B]} S(D)) = 0$. Note that

$$(B \cap S(D))^2 \subset (B \cap S(D)) S(D) \subset B \cap S(D) \subset S(D_{[B]} S(D)) = 0.$$ 

Lemma 2 shows that $B \cap S(D) = 0$ and therefore $BS(D) B = 0$. By density, $AS(D) A = 0$, which implies $D = 0$. \hfill \Box

7. Examples. 1. Algebras of power series. Let $A$ be a complex Banach algebra and let $A[[x]]$ denote the algebra of formal power series in one indeterminate $x$ with coefficients in $A$. It follows easily that if $A$ is prime, then so is any subalgebra of $A[[x]]$ containing $A x^m$ for some $m \in \mathbb{N}_0$. A Banach algebra of power series with coefficients in $A$ is a Banach space $A$ which is a subalgebra of $A[[x]]$ containing $A x$ such that the coefficient functionals are continuous. The closed graph theorem shows that multiplication is separately continuous, and so $A$ becomes a Banach algebra for an equivalent norm.

For an example of such an algebra, let $\omega$ be an algebra weight on $\mathbb{N}_0$, that is, $\omega$ is a function from $\mathbb{N}_0$ to $\mathbb{R}^+$ satisfying $\omega(0) = 1$ and $\omega(m + n) \leq \omega(m) \omega(n)$ for all $m, n \in \mathbb{N}$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n : ||f|| = \sum_{n=0}^{\infty} ||a_n||\omega(n) < \infty.$$ 

Then $f(A, \omega)$ is a Banach algebra of power series with coefficients in $A$. It is well known that $f(A, \omega)$ is semisimple if $\lim(\omega(n))^{1/n} > 0$, and $\text{Rad}_A(f(A, \omega)) = \{ f \in f(A, \omega) : f(0) = 0 \}$ if $\lim(\omega(n))^{1/n} = 0$. For a thorough discussion of this class of Banach algebras with complex coefficients, we refer to [4] and [11].

It is well known that any everywhere defined derivation on a Banach algebra of power series with complex coefficients is continuous. This was first shown in [15]. On account of Corollaries 2 and 5 we deduce that any derivation with a hereditary domain $B$ on a Banach algebra $A$ of power series with coefficients in a complex prime Banach algebra $A$ is closable provided that either $A$ is commutative or $B$ is dense in $A$. To prove this we show that is there is $b \in B$ with $b^4 \neq 0$. In the first case it is obvious. If our assertion were false in the second case, then we would have $B^N = 0$ for some $N \in \mathbb{N}$. By density we would deduce that $A^N = 0$, a contradiction. We have $b^2 x = b x b \in B$, $(b^2 x)^2 = b x b^2 \neq 0$, and $(b^2 x)^3 B(1) = 0$.

2. Weighted convolution algebras on the half-line. Let $\omega$ be a positive measurable function on $[0, \infty)$ such that $\omega(s + t) \leq \omega(s) \omega(t)$ for all $s, t \in [0, \infty]$, such a function is said to be a weight function on $[0, \infty)$. Let $L^1(\omega)$ denote the
space of equivalence classes with respect to Lebesgue measure of complex-valued measurable functions \( f \) on \([0, \infty)\) for which \( \|f\|_\omega = \int_0^\infty |f(t)| \omega(t) \, dt < \infty \). \( L^1(\omega) \) becomes a complex Banach algebra for the norm \( \|\cdot\|_\omega \) and the convolution \( (f * g)(t) = \int_0^t f(t-s)g(s) \, ds \) as product. For a thorough treatment of these algebras we refer the reader to \([4, 5, 9]\). It is known \([4]\) that \( L^1(\omega) \) is an integral domain. It is semisimple if \( \operatorname{lim}_{t \to \infty} \omega(t)^{1/t} > 0 \), and radical if \( \operatorname{lim}_{t \to \infty} \omega(t)^{1/t} = 0 \). For \( f \in L^1(\omega) \setminus \{0\} \), define \( \alpha(f) = \sup \{ \delta > 0 : f = \chi_{[0, \delta]} \text{ almost everywhere on } [0, \delta] \} \), and set \( \alpha(0) = \infty \).

For each weight function \( \omega \), everywhere defined derivations on \( L^1(\omega) \) are continuous (see \([4, 5, 10, 13]\)). From Corollary 2 we deduce that any derivation defined on a hereditary subalgebra \( \mathcal{B} \) of \( L^1(\omega) \) is automatically closable. Choose \( g \in L^1(\omega) \setminus \{0\} \) with \( \alpha(g) > 0 \) and \( h \in B \setminus \{0\} \). The function \( f = h \ast g \ast h \) lies in \( B \setminus \{0\} \) and Titchmarsh's convolution theorem \([4, \text{Theorem 7.4}]\) shows that \( \alpha(f) = 2\alpha(h) + \alpha(g) > 0 \) and \( (fB)^{(1)} = 0 \).

3. Disc algebra. Set \( \mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \} \). Given a complex Banach algebra \( \mathcal{A} \), let \( A(\mathbb{D}, \mathcal{A}) \) denote the set of all continuous \( \mathcal{A} \)-valued functions on \( \mathbb{D} \) which are holomorphic on its interior. \( A(\mathbb{D}, \mathcal{A}) \) becomes a Banach algebra with pointwise algebraic operations and the supremum norm. From the uniqueness theorem for holomorphic functions we deduce that \( A(\mathbb{D}, \mathcal{A}) \) is prime if \( \mathcal{A} \) is. Any derivation defined on a hereditary subalgebra \( \mathcal{B} \) of \( A(\mathbb{D}, \mathcal{A}) \) is closable provided that either \( \mathcal{A} \) is an integral domain or \( \mathcal{A} \) is prime and \( \mathcal{B} \) is dense in \( A(\mathbb{D}, \mathcal{A}) \). Indeed, we can argue as in the first example to find \( g \in B \) with \( g^2 \neq 0 \). The function \( f \) defined on \( \mathbb{D} \) by \( f(z) = g(z)z \) lies in \( B \), \( f^2 \neq 0 \), and \( (fB)^{(1)} = 0 \).

References


Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, Spain
E-mail: avillena@goliat.ugr.es

Received October 10, 1997
Revised version December 28, 1997