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Time-dependent perturbation theory for abstract evolution equations of second order

by

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Abstract. A condition on a family $\{B(t) : t \in [0,T]\}$ of linear operators is given under which the inhomogeneous Cauchy problem for

$$u''(t) = (A + B(t))u(t) + f(t) \quad \text{for } t \in [0,T]$$

has a unique solution, where A is a linear operator satisfying the conditions characterizing infinitesimal generators of cosine families except the density of their domains. The result obtained is applied to the partial differential equation

$$\begin{cases} u_{tt} = u_{xx} + b(t,x)u_x(t,x) + c(t,x)u(t,x) + f(t,x) & \text{for } (t,x) \in [0,T] \times [0,1], \\ u(t,0) = u(t,1) = 0 \quad \text{for } t \in [0,T], \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x) & \text{for } x \in [0,1] \end{cases}$$

in the space of continuous functions on $[0,1]$.

1. Introduction. In this paper we discuss the inhomogeneous Cauchy problem of second order

$$(\mathrm{CP}; ((u_0, v_0), f))_2 \qquad \begin{cases} u''(t) = (A + B(t))u(t) + f(t) & \text{for } t \in [0,T], \\ u(0) = u_0, \quad u'(0) = v_0 \end{cases}$$

in a general Banach space X with norm $\|\cdot\|$. We begin by setting up basic hypotheses on A, $\{B(t) : t \in [0,T]\}$ and f appearing in (CP; $((u_0, v_0), f))_2$.

(a) A is a closed linear operator in X. There exist $K \ge 1$ and $\omega > 0$ such that $\{\lambda^2 : \lambda > \omega\} \subset \varrho(A)$ (the resolvent set of A) and

$$\|(1/(m-1)!)(\lambda - \omega)^m (d/d\lambda)^{m-1} \lambda(\lambda^2 - A)^{-1} u\| \le K\|u\|$$

for $u \in X$, $\lambda > \omega$ and $m \ge 1$.

(b) $\{B(t) : t \in [0,T]\}$ is a family of linear operators in X satisfying the following conditions:

(b$_1$) $D(A) \subset D(B(t))$ for $t \in [0,T]$.

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(b$_2$) For $u \in X$ and $t \in [0,T]$, $B(t)(\lambda^2 - A)^{-1}u$ is infinitely differentiable in $\lambda > \omega$. There exists $M \ge 1$ such that

$$\|(1/(m-1)!)(\lambda-\omega)^m (d/d\lambda)^{m-1} B(t)(\lambda^2 - A)^{-1}u\| \le M\|u\|$$

for $u \in X$, $t \in [0,T]$, $\lambda > \omega$ and $m \ge 1$.

(b$_3$) For $u \in D(A)$, $B(t)u \in C^1([0,T];X)$.

(f) $f \in C^1([0,T];X)$.

If A is a densely defined operator satisfying condition (a), then it is the infinitesimal generator of a cosine family on X (see Sova [8]). In this case, problem (CP; $((u_0,v_0),0))_2$ was extensively studied by several authors. Among others, Lutz [3] dealt with the case where $B(t)$ is in $B(X,X)$ for $t \in [0,T]$, and Serizawa and Watanabe [7] improved his result so that $B(t)$ can be replaced by a differential operator.

Our purpose in this paper is to show that their results remain true without assuming the density of the domain of A, and with their conditions on $\{B(t) : t \in [0,T]\}$ replaced by the weaker condition (b). In the case where $B(t)$ is independent of t, condition (b) was proposed by Serizawa and Watanabe [6].

Problem (CP; $((u_0,v_0),f))_2$ is an abstract version of the following initial-boundary value problem for a partial differential equation, which will be discussed in the final part of this paper:

$$(1.1)\quad \begin{cases} u_{tt} = u_{xx} + b(t,x)u_x(t,x) + c(t,x)u(t,x) + f(t,x) \\ \qquad\qquad\qquad\qquad \text{for } (t,x) \in [0,T]\times[0,1], \\ u(t,0) = u(t,1) = 0 \quad \text{for } t \in [0,T], \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x) \quad \text{for } x \in [0,1]. \end{cases}$$

As we want to find a solution of this problem such that (1.1) holds pointwise in $(t,x) \in [0,T]\times[0,1]$, in the applications of the abstract theory to problem (1.1) we choose as X the Banach space of continuous functions on $[0,1]$ with supremum norm. This setting leads us to consider operators A whose domains are not necessarily dense in X. The study of inhomogeneous abstract Cauchy problems of first order in such settings was initiated by Da Prato and Sinestrari [1]. Their result has recently been extended to the quasi-linear case by Tanaka [9]. Our argument in this paper needs his result (Theorem A).

The author wishes to express her thanks to Professor Tanaka for suggesting the problem and for many stimulating conversations.

2. Main Theorem. In this section we state the main theorem along with some comments.

It is natural to reduce problem (CP; $((u_0,v_0),f))_2$ to the Cauchy problem of first order

$$(2.1)\quad \begin{cases} \widehat{u}'(t) = (\widehat{A}_0 + \widehat{B}_0(t))\widehat{u}(t) + \widehat{f}(t) \quad \text{for } t \in [0,T], \\ \widehat{u}(0) = (u_0,v_0), \end{cases}$$

where

$$\widehat{A}_0 = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \quad \widehat{B}_0(t) = \begin{pmatrix} 0 & 0 \\ B(t) & 0 \end{pmatrix}, \quad \widehat{f}(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

for $t \in [0,T]$. There is the "traditional" method of solving this Cauchy problem of first order in a suitable phase space $\widehat{X} \subset \widehat{Z} := X \times X$, if $B(t) = 0$ and $f(t) = 0$ for $t \in [0,T]$. This method is found in the paper by Kisyński [2], and it was shown that the phase space is given by $E \times X$ where

$$(2.2)\quad \begin{aligned} &E = \{u \in X : C(t)u \text{ is continuously differentiable in } t \in \mathbb{R}\}, \\ &|u|_E = \|u\| + \sup\{\|(d/dt)C(t)u\| : t \in [0,1]\} \quad \text{for } u \in E, \end{aligned}$$

if A is the inifinitesimal generator of a cosine family $\{C(t) : t \in \mathbb{R}\}$ on X.

We recall the following fundamental existence theorem concerning the inhomogeneous Cauchy problem of first order, which will be used in later arguments.

THEOREM A ([9, Theorem 1.10]). *Let $\widehat{X}$ be a Banach space with norm $\|\cdot\|_{\widehat{X}}$ and $\widehat{Y}$ another Banach space which is continuously imbedded in $\widehat{X}$. Suppose that a family $\{\widehat{A}(t) : t \in [0,T]\}$ of closed linear operators in $\widehat{X}$ satisfies the following conditions:*

(A$_1$) *$D(\widehat{A}(t)) = \widehat{Y}$ for $t \in [0,T]$.*

(A$_2$) *There exist $\widehat{M} \ge 1$ and $\widehat{\omega} \ge 0$ such that $(\widehat{\omega},\infty) \subset \varrho(\widehat{A}(t))$ for $t \in [0,T]$, and*

$$\Big\|\prod_{k=1}^{m}(\lambda - \widehat{A}(t_k))^{-1}\Big\|_{\widehat{X}} \le \widehat{M}(\lambda-\widehat{\omega})^{-m} \quad \text{for } \lambda > \widehat{\omega}$$

and every finite sequence $\{t_k\}_{k=1}^m$ with $0 \le t_1 \le \ldots \le t_m \le T$ and $m = 1,2,\ldots$

(A$_3$) *For $\widehat{u} \in \widehat{Y}$, $\widehat{A}(t)\widehat{u} \in C^1([0,T];\widehat{X})$.*

If $\widehat{f} \in C^1([0,T];\widehat{X})$ and $\widehat{u}_0 \in \widehat{Y}$ satisfy the compatibility condition that $\widehat{A}(0)\widehat{u}_0 + \widehat{f}(0) \in \overline{\widehat{Y}}^{\widehat{X}}$ (the closure of $\widehat{Y}$ in $\widehat{X}$), *then the inhomogeneous Cauchy problem of first order*

$$\begin{cases} (d/dt)\widehat{u}(t) = \widehat{A}(t)\widehat{u}(t) + \widehat{f}(t) \quad \text{for } t \in [0,T], \\ \widehat{u}(0) = \widehat{u}_0, \end{cases}$$

has a unique solution $\widehat{u}$ in the class $C([0,T];\widehat{Y}) \cap C^1([0,T];\widehat{X})$.

The central part of this paper is to determine the phase space $\widehat{X} = V\times X$ so that $D(A)\subset V$ and (A_1) through (A_3) of Theorem A are satisfied with the family $\{\widehat{A}+\widehat{B}(t): t\in[0,T]\}$ of linear operators in $\widehat{X}$ defined by

$$(2.3)\qquad \begin{aligned} \widehat{A}(u,v) &= (v, Au) &&\text{for } (u,v)\in D(A)\times V =: \widehat{Y},\\ \widehat{B}(t)(u,v) &= (0, B(t)u) &&\text{for } (u,v)\in D(A)\times X. \end{aligned}$$

Now, we write for simplicity

$$F(\lambda,m) = \frac{(\lambda-\omega)^m}{(m-1)!}\left(\frac{d}{d\lambda}\right)^{m-1} A(\lambda^2-A)^{-1}\quad \text{for } \lambda>\omega \text{ and } m\ge 1$$

and

$$G(\lambda,m,t) = \frac{(\lambda-\omega)^m}{(m-1)!}\left(\frac{d}{d\lambda}\right)^{m-1} \lambda B(t)(\lambda^2-A)^{-1}$$
$$\text{for } \lambda>\omega,\ m\ge 1 \text{ and } t\in[0,T],$$

and define two Banach spaces V_1 and V_2 by

$$\begin{aligned} V_1 &= \{u\in X : \sup\{\|F(\lambda,m)u\| : \lambda>\omega,\ m\ge 1\}<\infty\},\\ |u|_{V_1} &= \sup\{\|F(\lambda,m)u\| : \lambda>\omega,\ m\ge 1\} + \|u\| \quad \text{for } u\in V_1 \end{aligned}$$

and

$$\begin{aligned} V_2 &= \{u\in X : \sup\{\|G(\lambda,m,t)u\| : \lambda>\omega,\ m\ge 1,\ t\in[0,T]\}<\infty\},\\ |u|_{V_2} &= \sup\{\|G(\lambda,m,t)u\| : \lambda>\omega,\ m\ge 1,\ t\in[0,T]\} + \|u\| \quad \text{for } u\in V_2. \end{aligned}$$

Note that $D(A)\subset V_1\cap V_2$. Indeed, $B(t)\in B([D(A)],X)$ by (b_2), where $[D(A)]$ is the Banach space $D(A)$ equipped with the graph norm of A. Condition (a) and this fact together imply $D(A)\subset V_2$. By Leibniz's rule we have

$$(2.4)\qquad \|(1/(m-1)!)(\lambda-\omega)^m(d/d\lambda)^{m-1}(\lambda^2-A)^{-1}u\| \le (K/\omega)\|u\|$$

for $u\in X$, $\lambda>\omega$ and $m\ge 1$, which implies $D(A)\subset V_1$. Hence $D(A)\subset V_1\cap V_2$.

In Section 3, it will be proved that the Banach space V defined by

$$(2.5)\qquad V = V_1\cap V_2,\qquad |u|_V = \max(|u|_{V_1}, |u|_{V_2})\quad \text{for } u\in V$$

is as desired. The main result of this paper is given by the following theorem.

MAIN THEOREM. *Assume that conditions* (a), (b) *and* (f) *are satisfied. If* $u_0\in D(A)$, $v_0\in \overline{D(A)}^V$ *and* $(A+B(0))u_0+f(0)\in \overline{V}^X$, *then* (CP; $((u_0,v_0),f))_2$ *has a unique solution* u *in the class*

$$(2.6)\qquad C([0,T];D(A))\cap C^1([0,T];V)\cap C^2([0,T];X).$$

COROLLARY ([7]). *Assume that* A *is the infinitesimal generator of a cosine family on* X, $B(t)\in B(E,X)$ *for* $t\in[0,T]$, $B(t)u\in C^1([0,T];X)$ *for* $u\in E$, *where* E *is the Banach space defined by* (2.2), *and* $f\in C^1([0,T];X)$. *If* $u_0\in D(A)$ *and* $v_0\in E$, *then* (CP; $((u_0,v_0),f))_2$ *has a unique solution* u *in the class*

$$C([0,T];D(A))\cap C^1([0,T];E)\cap C^2([0,T];X).$$

REMARK. (i) The main theorem is a generalization of [6, Theorem].

(ii) The condition that $u_0\in D(A)$, $v_0\in\overline{D(A)}^V$ and $(A+B(0))u_0+f(0)\in\overline{V}^X$ in the Main Theorem is necessary for (CP; $((u_0,v_0),f))_2$ to have a solution in the class (2.6).

Indeed, let u be a solution of (CP; $((u_0,v_0),f))_2$ in the class (2.6); hence $u(t)\in D(A)$ and $u'(t)\in V$ for $t\ge 0$. Since $\lim_{h\downarrow 0}(u(h)-u(0))/h = v_0$ in V and $\lim_{h\downarrow 0}(u'(h)-u'(0))/h = (A+B(0))u_0+f(0)$ in X we have $v_0\in\overline{D(A)}^V$ and $(A+B(0))u_0+f(0)\in\overline{V}^X$.

3. Proofs of the Main Theorem and Corollary.

We set $\widehat{Z} = X\times X$ and define linear operators $\widehat{A}_0$ and $\widehat{B}_0(t)$ in $\widehat{Z}$ by

$$\widehat{A}_0(u,v) = (v, Au)\quad \text{for } (u,v)\in D(A)\times X$$

and

$$\widehat{B}_0(t)(u,v) = (0, B(t)u)\quad \text{for } (u,v)\in D(B(t))\times X.$$

Condition (a) implies $(\omega,\infty)\subset\varrho(\widehat{A}_0)$ and

$$(3.1)\qquad \begin{aligned} &(\lambda-\widehat{A}_0)^{-m}(u,v)\\ &\quad = \frac{(-1)^{m-1}}{(m-1)!}\left(\frac{d}{d\lambda}\right)^{m-1}(\lambda(\lambda^2-A)^{-1}u + (\lambda^2-A)^{-1}v,\\ &\qquad\qquad A(\lambda^2-A)^{-1}u + \lambda(\lambda^2-A)^{-1}v) \end{aligned}$$

for $m\ge 1$, $\lambda>\omega$ and $(u,v)\in\widehat{Z}$.

First introduce the Banach space $\widehat{X}_1$ defined by

$$\begin{aligned} \widehat{X}_1 &= \{\widehat{u}\in\widehat{Z} : \sup\{(\lambda-\omega)^m\|(\lambda-\widehat{A}_0)^{-m}\widehat{u}\|_{\widehat{Z}} : \lambda>\omega,\ m\ge 0\}<\infty\},\\ |\widehat{u}|_{\widehat{X}_1} &= \sup\{(\lambda-\omega)^m\|(\lambda-\widehat{A}_0)^{-m}\widehat{u}\|_{\widehat{Z}} : \lambda>\omega,\ m\ge 0\}\quad \text{for } \widehat{u}\in\widehat{X}_1. \end{aligned}$$

Clearly, $\|(u,v)\|_{\widehat{Z}} := \|u\|+\|v\| \le |(u,v)|_{\widehat{X}_1}$ for $(u,v)\in\widehat{X}_1$. It is known [4, Section 5] that $(\lambda-\widehat{A}_0)^{-1}(\widehat{X}_1)\subset\widehat{X}_1$ for $\lambda>\omega$, and

$$(3.2)\qquad |(\lambda-\widehat{A}_0)^{-1}\widehat{u}|_{\widehat{X}_1} \le (\lambda-\omega)^{-1}|\widehat{u}|_{\widehat{X}_1}\quad \text{for } \widehat{u}\in\widehat{X}_1 \text{ and } \lambda>\omega;$$

that is, $\widehat{X}_1$ is a Banach space on which $(\lambda-\omega)(\lambda-\widehat{A}_0)^{-1}$ becomes a contraction for $\lambda>\omega$. This is the reason why we introduce the Banach space $\widehat{X}_1$. If $\widehat{B}_0(t)(\lambda-\widehat{A}_0)^{-1}(\widehat{X}_1)\subset\widehat{X}_1$ for $t\in[0,T]$, and

$$|\widehat{B}_0(t)(\lambda-\widehat{A}_0)^{-1}\widehat{u}|_{\widehat{X}_1} \le L(\lambda-\omega)^{-1}|\widehat{u}|_{\widehat{X}_1}$$

for $\hat{u} \in \hat{X}_1$, $t \in [0,T]$ and $\lambda > \omega$, then it is not difficult to check condition (A_2) of Theorem A with $\hat{X} = \hat{X}_1$. However, we do not know whether the fact above is true or not.

Now, it is necessary for us to use a Banach space $\hat{X}_2$ defined by

$$\hat{X}_2 = \{\hat{u} \in \hat{Z} : \sup\{(\lambda-\omega)^m \|\hat{B}_0(t)(\lambda-\hat{A}_0)^{-m}\hat{u}\|_{\hat{Z}} : \lambda > \omega,\ m \ge 1,\ t \in [0,T]\} < \infty\},$$

$$|\hat{u}|_{\hat{X}_2} = \max(\|\hat{u}\|_{\hat{Z}}, \sup\{(\lambda-\omega)^m \|\hat{B}_0(t)(\lambda-\hat{A}_0)^{-m}\hat{u}\|_{\hat{Z}} : \lambda > \omega,\ m \ge 1,\ t \in [0,T]\}) \quad \text{for } \hat{u} \in \hat{X}_2.$$

Note that $\hat{B}_0(t) \in B([D(\hat{A}_0)], \hat{Z})$ for $t \in [0,T]$ and that

$$\begin{aligned}(3.3)\quad \hat{B}_0(t)(\lambda-\hat{A}_0)^{-m}(u,v) &= \biggl(0, \frac{(-1)^{m-1}}{(m-1)!}\Bigl(\frac{d}{d\lambda}\Bigr)^{m-1} B(t)\lambda(\lambda^2-A)^{-1}u \\ &\quad + \frac{(-1)^{m-1}}{(m-1)!}\Bigl(\frac{d}{d\lambda}\Bigr)^{m-1} B(t)(\lambda^2-A)^{-1}v\biggr)\end{aligned}$$

for $m \ge 1$, $\lambda > \omega$, $t \in [0,T]$ and $(u,v) \in \hat{Z}$. Indeed, the first assertion follows readily from the fact that $B(t) \in B([D(A)], X)$. Since $\lambda(\lambda^2-A)^{-1}u$ and $(\lambda^2-A)^{-1}u$ are differentiable in $[D(A)]$ with respect to $\lambda > \omega$, the desired equality (3.3) is proved by the definition of $\hat{B}_0(t)$ and (3.1).

The following lemma is useful for the proof of condition (A_2) of Theorem A.

LEMMA 1. *Let $\hat{X}_0$ be the Banach space defined by*

$$\hat{X}_0 = \hat{X}_1 \cap \hat{X}_2, \qquad |\hat{u}|_{\hat{X}_0} = \max(|\hat{u}|_{\hat{X}_1}, |\hat{u}|_{\hat{X}_2}) \quad \textit{for } \hat{u} \in \hat{X}_0.$$

Then:

(i) *For $\lambda > \omega$, $(\lambda-\hat{A}_0)^{-1}(\hat{X}_0) \subset \hat{X}_0$ and*

$$|(\lambda-\hat{A}_0)^{-1}\hat{u}|_{\hat{X}_0} \le (\lambda-\omega)^{-1}|\hat{u}|_{\hat{X}_0} \quad \textit{for } \hat{u} \in \hat{X}_0.$$

(ii) *For $t \in [0,T]$ and $\lambda > \omega$, $\hat{B}_0(t)(\lambda-\hat{A}_0)^{-1}(\hat{X}_0) \subset \hat{X}_0$ and*

$$|\hat{B}_0(t)(\lambda-\hat{A}_0)^{-1}\hat{u}|_{\hat{X}_0} \le L(\lambda-\omega)^{-1}|\hat{u}|_{\hat{X}_0} \quad \textit{for } \hat{u} \in \hat{X}_0,$$

where $L = (1+1/\omega)K + M$.

Proof. To prove (i), we first note that for $\hat{u} \in \hat{X}_1$, $\mu > \lambda > \omega$ and $l \ge 1$, the series $\sum_{k=l-1}^{\infty} {}_kC_{l-1}(\mu-\lambda)^{k-l+1}(\mu-\hat{A}_0)^{-(k+1)}\hat{u}$ is absolutely convergent in $\hat{Z}$, and

$$(3.4)\qquad (\lambda-\hat{A}_0)^{-l}\hat{u} = \sum_{k=l-1}^{\infty} {}_kC_{l-1}(\mu-\lambda)^{k-l+1}(\mu-\hat{A}_0)^{-(k+1)}\hat{u}.$$

Let $\hat{u} \in \hat{X}_0$ and $m \ge 1$. Since $\hat{B}_0(t) \in B([D(\hat{A}_0)], \hat{Z})$ we have by (3.4) with $l = 1$,

$$(\mu-\omega)^m \hat{B}_0(t)(\mu-\hat{A}_0)^{-m}(\lambda-\hat{A}_0)^{-1}\hat{u} = (\mu-\omega)^m \sum_{k=0}^{\infty}(\mu-\lambda)^k \hat{B}_0(t)(\mu-\hat{A}_0)^{-(m+k+1)}\hat{u},$$

and the right-hand side is estimated by $(\lambda-\omega)^{-1}|\hat{u}|_{\hat{X}_2}$ for $\mu > \lambda$. For μ with $\lambda > \mu > \omega$, the same estimate is obtained by using (3.4) with $l = m$. Moreover, it is clear that $\|(\lambda-\hat{A}_0)^{-1}\hat{u}\|_{\hat{Z}} \le (\lambda-\omega)^{-1}|\hat{u}|_{\hat{X}_1}$ by (3.2). It follows that

$$|(\lambda-\hat{A}_0)^{-1}\hat{u}|_{\hat{X}_2} \le (\lambda-\omega)^{-1}|\hat{u}|_{\hat{X}_0}$$

for $\lambda > \omega$. Assertion (i) is proved by combining this estimate and (3.2).

To prove (ii), let $(u,v) \in \hat{X}_0$, $t \in [0,T]$ and $\lambda > \omega$. By (3.1) and (3.3) we represent each component of $(\mu-\hat{A}_0)^{-m}\hat{B}_0(t)(\lambda-\hat{A}_0)^{-1}(u,v)$ in terms of A and $B(t)$ for $\mu > \omega$ and $m \ge 0$, and then use condition (a). This yields

$$\begin{aligned}(\mu-\omega)^m \|(\mu-\hat{A}_0)^{-m}\hat{B}_0(t)(\lambda-\hat{A}_0)^{-1}(u,v)\|_{\hat{Z}} \\ \le (1+1/\omega)K\|B(t)\lambda(\lambda^2-A)^{-1}u + B(t)(\lambda^2-A)^{-1}v\|,\end{aligned}$$

and the right-hand side is equal to $(1+1/\omega)K\|\hat{B}_0(t)(\lambda-\hat{A}_0)^{-1}(u,v)\|_{\hat{Z}}$ by (3.3) again. Hence

$$(3.5)\qquad |\hat{B}_0(t)(\lambda-\hat{A}_0)^{-1}(u,v)|_{\hat{X}_1} \le (1+1/\omega)K(\lambda-\omega)^{-1}|(u,v)|_{\hat{X}_2}.$$

By (3.3) we have

$$\begin{aligned}\hat{B}_0(s)(\mu-\hat{A}_0)^{-m}\hat{B}_0(t)(\lambda-\hat{A}_0)^{-1}(u,v) &= \biggl(0, \frac{(-1)^{m-1}}{(m-1)!}\Bigl(\frac{d}{d\mu}\Bigr)^{m-1} B(s)(\mu^2-A)^{-1}(B(t)\lambda(\lambda^2-A)^{-1}u \\ &\quad + B(t)(\lambda^2-A)^{-1}v)\biggr)\end{aligned}$$

for $s \in [0,T]$, $\mu > \omega$ and $m \ge 1$. Similarly to the argument above, condition (b_2) implies

$$|\hat{B}_0(t)(\lambda-\hat{A}_0)^{-1}(u,v)|_{\hat{X}_2} \le M(\lambda-\omega)^{-1}|(u,v)|_{\hat{X}_2}.$$

Assertion (ii) is obtained by combining this and (3.5). ■

LEMMA 2. *$\hat{X}_0 = V \times X$, and the two norms $|(u,v)|_{\hat{X}_0}$ and $|u|_V + \|v\|$ are equivalent.*

Proof. Let $\lambda > \omega$ and $m \ge 1$. Making use of (3.1) we have by condition (a) and (2.4),

$$\left|(\lambda-\omega)^m\left\{\|(\lambda-\widehat{A}_0)^{-m}(u,v)\|_{\widehat{Z}}-\frac{1}{(m-1)!}\left\|\left(\frac{d}{d\lambda}\right)^{m-1}A(\lambda^2-A)^{-1}u\right\|\right\}\right| \le K\|u\|+K(1+1/\omega)\|v\|$$

for $(u,v)\in\widehat{Z}$. We use (3.3) and condition (b_2) to find

$$\left|(\lambda-\omega)^m\left\{\|\widehat{B}_0(t)(\lambda-\widehat{A}_0)^{-m}(u,v)\|_{\widehat{Z}} - \frac{1}{(m-1)!}\left\|\left(\frac{d}{d\lambda}\right)^{m-1}B(t)\lambda(\lambda^2-A)^{-1}u\right\|\right\}\right| \le M\|v\|$$

for $(u,v)\in\widehat{Z}$. These estimates imply that for each $i=1,2$, $V_i\times X=\widehat{X}_i$, and the two norms $|(u,v)|_{\widehat{X}_i}$ and $|u|_{V_i}+\|v\|$ are equivalent, from which the desired claim follows readily. ■

Proof of Main Theorem. Let V be the Banach space defined by (2.5), and set $\widehat{X}=V\times X$ and $\widehat{Y}=D(A)\times V$. Clearly, the family $\{\widehat{A}+\widehat{B}(t):t\in[0,T]\}$ of linear operators defined by (2.3) satisfies conditions (A_1) and (A_3) of Theorem A.

To check condition (A_2), we note that $(\lambda-\widehat{A})^{-1}=(\lambda-\widehat{A}_0)^{-1}|_{\widehat{X}_0}$ for $\lambda>\omega$ (by the fact that $\widehat{X}_0=V\times X\supset D(A)\times X=D(\widehat{A}_0)$) and $\widehat{B}(t)\widehat{u}=\widehat{B}_0(t)\widehat{u}$ for $\widehat{u}\in D(\widehat{A})$ and $t\in[0,T]$. By Lemma 1 we have

$$(\lambda-(\widehat{A}+\widehat{B}(t)))^{-1}\widehat{u}=(\lambda-\widehat{A})^{-1}\sum_{k=0}^{\infty}(\widehat{B}(t)(\lambda-\widehat{A})^{-1})^k\widehat{u}$$

for $\widehat{u}\in\widehat{X}_0$ and $\lambda>\beta:=\omega+L$, and

$$|(\lambda-(\widehat{A}+\widehat{B}(t)))^{-1}\widehat{u}|_{\widehat{X}_0}\le(\lambda-\beta)^{-1}|\widehat{u}|_{\widehat{X}_0}$$

for $\widehat{u}\in\widehat{X}_0$ and $\lambda>\beta$, which implies that condition (A_2) is satisfied, since $\widehat{X}_0=\widehat{X}$ and their norms are equivalent by Lemma 2.

If we set $\widehat{u}_0=(u_0,v_0)$ and $\widehat{f}(t)=(0,f(t))$ for $t\in[0,T]$, then $\widehat{u}_0\in D(A)\times\overline{D(A)}^V\subset\widehat{Y}$ and

$$(\widehat{A}+\widehat{B}(0))\widehat{u}_0+\widehat{f}(0)=(v_0,(A+B(0))u_0+f(0))\in\overline{D(A)}^V\times\overline{V}^X=\overline{\widehat{Y}}^{\widehat{X}}.$$

From Theorem A we deduce that the problem

$$(d/dt)(u(t),v(t))=(\widehat{A}+\widehat{B}(t))(u(t),v(t))+\widehat{f}(t)=(v(t),(A+B(t))u(t)+f(t))$$

has a unique solution $(u,v)\in C([0,T];\widehat{Y})\cap C^1([0,T];\widehat{X})$. It follows that the desired solution of (CP; $((u_0,v_0),f))_2$ is given by u. ■

Proof of Corollary. Since $D(A)$ is dense in X and $D(A)\subset V$, V is dense in X. The condition that $(A+B(0))u_0+f(0)\in\overline{V}^X$ is thus automatically satisfied.

Now, set $\widetilde{X}=E\times X$ and define linear operators $\widetilde{A}$ and $\widetilde{B}(t)$ in $\widetilde{X}$ by

$$\widetilde{A}(u,v)=(v,Au)\quad\text{ for }(u,v)\in D(A)\times E$$

and

$$\widetilde{B}(t)(u,v)=(0,B(t)u)\quad\text{ for }(u,v)\in\widetilde{X}.$$

It is known [2] that $\widetilde{A}$ is the infinitesimal generator of $\{\widetilde{T}(t):t\ge0\}$ of class (C_0) on $\widetilde{X}$ satisfying $\|\widetilde{T}(t)\|_{\widetilde{X}}\le K_0e^{\omega_0t}$ for $t\ge0$. Note that $D(A)\subset E$. Since $(\lambda-\widetilde{A})^{-1}=(\lambda-\widehat{A}_0)^{-1}|_{\widetilde{X}}$ for $\lambda>\omega_0$ we have

$$(3.6)\quad (\lambda-\widetilde{A})^{-1}(u,v) =(\lambda(\lambda^2-A)^{-1}u+(\lambda^2-A)^{-1}v,A(\lambda^2-A)^{-1}u+\lambda(\lambda^2-A)^{-1}v)$$

for $(u,v)\in\widetilde{X}$ and $\lambda>\omega_0$. It follows that

$$(3.7)\quad \left(\frac{d}{d\lambda}\right)^{m-1}(\lambda(\lambda^2-A)^{-1}u+(\lambda^2-A)^{-1}v,A(\lambda^2-A)^{-1}u+\lambda(\lambda^2-A)^{-1}v) =\int_0^\infty(-t)^{m-1}e^{-\lambda t}\widetilde{T}(t)(u,v)\,dt$$

for $(u,v)\in E\times X$, $m\ge1$ and $\lambda>\omega_0$. To check condition (a), let $v\in X$. Setting $u=0$ in (3.7) we have

$$\|(1/(m-1)!)(d/d\lambda)^{m-1}((\lambda^2-A)^{-1}v,\lambda(\lambda^2-A)^{-1}v)\|_{\widetilde{X}} \le K_0\int_0^\infty t^{m-1}e^{-(\lambda-\omega_0)t}\,dt\,\|v\|/(m-1)!$$

for $m\ge1$ and $\lambda>\omega_0$, which implies that condition (a) is satisfied with $\omega=\omega_0$ and $K=K_0$.

By the assumption of B we have $\widetilde{B}(t)\in B(\widetilde{X})$ for $t\in[0,T]$, and $\widetilde{B}(t)\widetilde{u}\in C^1([0,T];\widetilde{X})$ for $\widetilde{u}\in\widetilde{X}$; hence there exists $M_0\ge1$ such that $\|\widetilde{B}(t)\|_{\widetilde{X}}\le M_0$ for $t\in[0,T]$, by the principle of uniform boundedness. By (3.6) we have

$$(3.8)\quad \widetilde{B}(t)(\lambda-\widetilde{A})^{-m}(u,v) =\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{d}{d\lambda}\right)^{m-1}\widetilde{B}(t)(\lambda-\widetilde{A})^{-1}(u,v) =\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{d}{d\lambda}\right)^{m-1}(0,B(t)(\lambda(\lambda^2-A)^{-1}u+(\lambda^2-A)^{-1}v))$$

for $(u,v)\in\widetilde{X}$, $\lambda>\omega_0$, $t\in[0,T]$ and $m\ge1$.

Now, we check condition (b_2). To this end, let $v \in X$. Setting $u = 0$ in (3.8) we have

$$\|(1/(m-1)!)(d/d\lambda)^{m-1}B(t)(\lambda^2 - A)^{-1}v\| = \|\widetilde{B}(t)(\lambda - \widetilde{A})^{-m}(0,v)\|_{\widetilde{X}} \le M_0(\lambda - \omega_0)^{-m}\|v\|$$

for $m \ge 1$ and $\lambda > \omega_0$. This means that condition (b_2) holds with $\omega = \omega_0$ and $M = M_0$.

It remains to show that $\overline{D(A)}^V = E$ and the norm in E is equivalent to that in V. Note that V is determined by ω_0 instead of ω. The fact that $E \subset V$ and $|u|_V \le \max(M_0, K_0)|u|_E$ for $u \in E$ is easily obtained by estimating the equalities (3.7) and (3.8) with $v = 0$. Hence $\overline{D(A)}^V \supset \overline{D(A)}^E = E$, since $D(\widetilde{A})$ ($= D(A) \times E$) is dense in $\widetilde{X}$ ($= E \times X$).

We show the converse inclusion. From Lemmas 1(i) and 2 we deduce that there exists a semigroup $\{\widehat{T}(t) : t \ge 0\}$ of class (C_0) on $\overline{D(A)}^V \times \overline{V}^X$ ($= \overline{D(\widehat{A})}^{\widehat{X}}$) given by the exponential formula

$$\widehat{T}(t)(u,v) = \lim_{m\to\infty}(1 - t\widehat{A}/m)^{-m}(u,v)$$

for $(u,v) \in \overline{D(A)}^V \times \overline{V}^X$ and $t \ge 0$. Moreover, $\|\widehat{T}(t)\|_{\widehat{X}} \le L_0 e^{\omega_0 t}$ for $t \ge 0$. By the fact above (in particular $v = 0$) we see that the limit

$$U(t)u := \lim_{m\to\infty} \frac{(-1)^{m-1}\lambda^m}{(m-1)!}\left(\frac{d}{d\lambda}\right)^{m-1} A(\lambda^2 - A)^{-1}u\Big|_{\lambda = m/t} \tag{3.9}$$

exists in X for $u \in \overline{D(A)}^V$ and $t > 0$. If we define $U(0) = 0$ (the zero operator on X) then the one-parameter family $\{U(t) : t \ge 0\}$ has the following properties:

(i) For $u \in \overline{D(A)}^V$, $U(t)u \in C([0,\infty); X)$.

(ii) $\|U(t)u\| \le L_0 e^{\omega_0 t}|u|_V$ for $u \in \overline{D(A)}^V$.

Now, let $u \in \overline{D(A)}^V$. If $\{S(t) : t \ge 0\}$ is the associated sine family on X then

$$S(t)v = \lim_{m\to\infty} \frac{(-1)^{m-1}\lambda^m}{(m-1)!}\left(\frac{d}{d\lambda}\right)^{m-1}(\lambda^2 - A)^{-1}v\Big|_{\lambda = m/t}$$

for $v \in X$ and $t > 0$. Since A is closed we have by this representation and (3.9), $S(t)u \in D(A)$ and $AS(t)u = U(t)u$ for $t \ge 0$. It is well known that $C(t)v = v + A\int_0^t S(s)v\,dt$ for $v \in X$ and $t \ge 0$. Combining the last two equalities we have $C(t)u = u + \int_0^t U(s)u\,ds$ for $t \ge 0$. Hence $C(t)u \in C^1([0,\infty); X)$ and $(d/dt)C(t)u = U(t)u$ for $t \ge 0$, which implies $u \in E$ and $|u|_E \le (L_0 e^{\omega_0} + 1)|u|_V$. ■

4. Application. In this section we give an application of the main theorem to the initial-boundary value problem

$$\begin{cases} u_{tt} = u_{xx} + b(t,x)u_x(t,x) + c(t,x)u(t,x) + f(t,x) \\ \qquad\qquad\qquad\qquad \text{for } (t,x) \in [0,T] \times [0,1], \\ u(t,0) = u(t,1) = 0 \quad \text{for } t \in [0,T], \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x) \quad \text{for } x \in [0,1]. \end{cases} \tag{4.1}$$

Here it is assumed that $b, c, f \in C^1([0,T] \times [0,1])$.

Let X be the Banach space $C[0,1]$ with the usual supremum norm $|u|_\infty$ and define a closed linear operator A in X by $(Au)(x) = u''(x)$ for $u \in D(A)$, where $D(A) = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$. It is known [5, (4.13)] that A satisfies condition (a).

We consider a family $\{B(t) : t \in [0,T]\}$ of linear operators in X defined by $(B(t)u)(x) = b(t,x)u'(x) + c(t,x)u(x)$ for $u \in D(B(t)) = C^1[0,1]$. Condition (b_2) follows readily by noting that the differential operators d/dx and $(d/d\lambda)^m$ commute [5, (4.16)]. Clearly, conditions (b_1) and (b_3) are satisfied.

Let $W = \{u \in C^1[0,1] : u(0) = u(1) = 0\}$ and $|u|_W = |u|_\infty + |u'|_\infty$ for $u \in W$. By [5, (4.17)] we have $W \subset V_1$ and $|w|_{V_1} \le |w|_W$ for $w \in W$. Since $(d/dx)((\lambda^2 - A)^{-1}w)(x) = ((\lambda^2 - A)^{-1}w')(x)$ for $w \in W$ and $x \in [0,1]$, we have by [5, (4.13)], $W \subset V_2$ and $|w|_{V_2} \le (\|b\|_\infty + \|c\|_\infty + 1)|w|_W$ for $w \in W$, where $\|b\|_\infty = \sup\{|b(t,x)| : (t,x) \in [0,T] \times [0,1]\}$. Hence $W \subset V$ and $|w|_V \le (\|b\|_\infty + \|c\|_\infty + 1)|w|_W$ for $w \in W$. By this fact we have $\{u \in C^1[0,1] : u(0) = u(1) = 0\} \subset \overline{D(A)}^V$ and $\{u \in C[0,1] : u(0) = u(1) = 0\} \subset \overline{V}^X$.

From the Main Theorem we deduce that if $u_0 \in C^2[0,1]$ and $v_0 \in C^1[0,1]$ satisfy the compatibility condition $u_0(0) = u_0(1) = 0$, $v_0(0) = v_0(1) = 0$ and

$$\begin{aligned} u_0''(0) + b(0,0)u_0'(0) + c(0,0)u_0(0) + f(0,0) \\ = u_0''(1) + b(0,1)u_0'(1) + c(0,1)u_0(1) + f(0,1) = 0, \end{aligned} \tag{4.2}$$

then problem (4.1) has a unique solution u in the class

$$C([0,T]; C^2[0,1]) \cap C^1([0,T]; C^1[0,1]) \cap C^2([0,T]; C[0,1]).$$

REMARK. If one considers solving problem (4.1) by applying the corollary, then the following technical conditions are required instead of (4.2):

$$\begin{aligned} &u_0''(0) = u_0''(1) = 0, \\ &b(t,0) = b(t,1) = 0, \quad f(t,0) = f(t,1) = 0 \quad \text{for } t \in [0,T]. \end{aligned}$$

Indeed, it is necessary to set $X = \{u \in C[0,1] : u(0) = u(1) = 0\}$ and $(Au)(x) = u''(x)$ for $u \in D(A)$, where $D(A) = \{u \in C^2[0,1] : u(0) = u(1) = 0, u''(0) = u''(1) = 0\}$.

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Derivations with a hereditary domain, II

by

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Abstract. The nilpotency of the separating subspace of an everywhere defined derivation on a Banach algebra is an intriguing question which remains still unsolved, even for commutative Banach algebras. On the other hand, closability of partially defined derivations on Banach algebras is a fundamental problem motivated by the study of time evolution of quantum systems. We show that the separating subspace $\mathcal{S}(D)$ of a Jordan derivation defined on a subalgebra B of a complex Banach algebra A satisfies $B[B\cap\mathcal{S}(D)]B\subset \mathrm{Rad}_{\mathcal{B}}(A)$ provided that $BAB\subset A$ and $\dim(\mathrm{Rad}_{\mathcal{J}}(A)\cap\bigcap_{n=1}^{\infty}B^n)<\infty$, where $\mathrm{Rad}_{\mathcal{J}}(A)$ and $\mathrm{Rad}_{\mathcal{B}}(A)$ denote the Jacobson and the Baer radicals of A respectively. From this we deduce the closability of partially defined derivations on complex semiprime Banach algebras with appropriate domains. The result applies to several relevant classes of algebras.

0. Introduction. The study of partially defined derivations is motivated by the time evolution and spatial translation in quantum physics. The general theory of partially defined derivations on Banach algebras is mainly concerned with the theory of closability, generator properties and classification of closed derivations. For a thorough treatment of this topic we refer the reader to [1] and [20]. On the other hand, any everywhere defined derivation on a nonassociative complete normed algebra A yields a meaningful partially defined derivation on the Banach algebra $L(A)$ of all continuous linear operator on A (see [23, 24] for more details).

It is appropriate to point out that there are examples of nonclosable partially defined derivations on C*-algebras (see [1; Example 1.4.4]). In contrast, the case where derivations are everywhere defined is by far more satisfactory. B. E. Johnson and A. M. Sinclair [14] showed that everywhere defined derivations on semisimple Banach algebras are automatically continuous. However, at present the answers to the following equivalent questions remain open, even for commutative Banach algebras (see [3, 7, 8, 16, 19]).

1. Is the separating subspace of any everywhere defined derivation on a Banach algebra contained in the Baer radical of the algebra?

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