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Reverse-Hölder classes in the Orlicz spaces setting

by

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Abstract. In connection with the A_ϕ classes of weights (see [K-T] and [B–K]), we study, in the context of Orlicz spaces, the corresponding reverse-Hölder classes RH_ϕ. We prove that when ϕ is Δ_2 and has lower index greater than one, the class RH_ϕ coincides with some reverse-Hölder class RH_q, $q > 1$. For more general ϕ we still get $RH_\phi \subset A_\infty = \bigcup_{q>1} RH_q$ although the intersection of all these RH_ϕ gives a proper subset of $\bigcap_{q>1} RH_q$.

1. Introduction. By a *weight* w we mean a non-negative and locally integrable function on $\mathbb{R}^n$. As is well known, a weight w is said to belong to the *reverse-Hölder class* with exponent q, RH_q, if it satisfies the inequality

$$\left(\frac{1}{|Q|}\int_Q w(x)^q\,dx\right)^{1/q} \le C\frac{1}{|Q|}\int_Q w(x)\,dx$$

for any cube $Q \subset \mathbb{R}^n$ with sides parallel to the axes; here $|Q|$ denotes the Lebesgue measure of Q. These classes appeared in connection with the A_p classes of Muckenhoupt which characterize the weights such that the Hardy–Littlewood maximal operator is bounded on $L^p(w)$, $1 < p < \infty$. To be precise, the A_p weights are defined as those weights such that

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C$$

for any cube $Q \subset \mathbb{R}^n$. The limiting case $p = 1$, A_1, is defined as the weights satisfying

$$m_Q(w) \le C \inf_{x\in Q} w(x)$$

for any cube $Q \subset \mathbb{R}^n$, where $m_Q(w)$ denotes the average of w over Q. For $p = \infty$, A_∞ consists of the weights w such that for any Q and any measurable

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subset $E \subset Q$, there exists $\delta > 0$ satisfying

$$w(E)/w(Q) \le C(|E|/|Q|)^{\delta}.$$

Note that these definitions can be extended to more general measures μ instead of Lebesgue measure, with the obvious changes. So we can, and will, talk about $A_\infty(d\mu)$, $A_1(d\mu)$ or $A_p(d\mu)$ classes and we drop the notation $d\mu$ in the case of Lebesgue measure.

The precise connection between the two classes is given by the following identities:

$$\bigcup_{q>1} RH_q = \bigcup_{p>1} A_p = A_\infty.$$

Kerman and Torchinsky [K-T] and later Bloom and Kerman [B-K] studied the boundedness of the Hardy–Littlewood maximal operator on weighted Orlicz spaces. In this context they introduced, for ϕ a Young function, the classes A_ϕ and more generally W_ϕ, both giving extensions of A_p, $p > 1$.

The aim of this paper is to study the corresponding reverse-Hölder classes of weights in the Orlicz spaces setting.

For a non-negative, increasing, continuous and convex function ϕ defined on $[0,\infty)$ we say that a weight w belongs to RH_ϕ if there exists a positive constant C such that

$$\int_Q \phi\left(\frac{\phi^{-1}(t/|Q|)}{Cm_Q(w)} w(x)\right)\frac{dx}{t} \le 1 \tag{1.1}$$

for any cube $Q \subset \mathbb{R}^n$ and $t > 0$.

It is easy to check that when $\phi(s) = s^q$, $q > 1$, the above inequality coincides with the reverse-Hölder condition with exponent q, $q > 1$. Also, for ϕ as above, the reverse inequality to (1.1) holds with $C = 1$ as a consequence of the Jensen inequality for convex functions. Finally, we remark that the parameter t is necessary to make the class RH_ϕ invariant under dilations in the sense that if $w \in RH_\phi$ then $w(\lambda x) \in RH_\phi$ with the constant C independent of $\lambda > 0$.

Relating to these classes we prove a result similar to that of Kerman and Torchinsky for the class A_ϕ, that is, when ϕ is "between power functions with exponents greater than one", in a sense that will be made precise later, the class RH_ϕ coincides with some RH_q. For more general ϕ, including functions "near the identity", although the above result is not necessarily true, we still get $RH_\phi \subset A_\infty = \bigcup_{q>1} RH_q$.

On the other hand, we also characterize the intersection of RH_ϕ for those general ϕ as the class RH_∞ introduced by Franchi ([F]), which is a proper subset of $\bigcap_{q>1} RH_q$, as shown in [CU-N]. These results correspond to those obtained in [B-K] for W_ϕ.

2. Statement of main theorems. In this paper we say that a non-negative function ϕ defined on $[0,\infty)$ is an *N-function* (or a *Young function*) if it is convex and satisfies

$$\lim_{t\to 0+} \frac{\phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\phi(t)} = 0.$$

Clearly, under these conditions ϕ has a derivative φ which is non-decreasing and non-negative with $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. For such ϕ, the complementary function defined by

$$\widetilde{\phi}(s) = \sup_{t>0}(st - \phi(t))$$

is also an N-function. Moreover, it can be proved that there exist constants C_1 and C_2 such that

$$C_1 t \le \phi^{-1}(t)\widetilde{\phi}^{-1}(t) \le C_2 t$$

for every $t > 0$.

Given an N-function ϕ and a finite Borel measure μ on $\mathbb{R}^n$, the *Orlicz space* $L_\phi(d\mu)$ consists of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ for which there exists a constant C such that

$$\int_{\mathbb{R}^n} \phi(|f(x)|C)\, d\mu(x) < \infty.$$

Furthermore, the space $L_\phi(d\mu)$ equipped with the Luxemburg norm

$$\|f\|_{L_\phi(d\mu)} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \phi(|f(x)|/\lambda)\, d\mu(x) \le 1\Big\}$$

is a Banach space. Also, if ψ denotes the complementary function of ϕ, then the Hölder inequality

$$\int_{\mathbb{R}^n} f(x)g(x)\, d\mu(x) \le C\|f\|_{L_\phi(d\mu)}\|g\|_{L_{\widetilde{\phi}}(d\mu)}$$

holds.

Sometimes, we will impose further conditions on the function ϕ. To this end, we introduce the notions of lower and upper types.

We say that ϕ is *of lower type p* if there exists a constant C such that

$$\phi(st) \le Cs^p\phi(t) \quad \text{for } s \le 1 \text{ and } t \ge 0, \tag{2.1}$$

and that ϕ is *of upper type q* if there exists a constant C such that

$$\phi(st) \le Cs^q\phi(t) \quad \text{for } s \ge 1 \text{ and } t \ge 0. \tag{2.2}$$

Observe that for our kind of functions ϕ, if p and q satisfy the above inequalities then we have

$$1 \le p \le q < \infty.$$

Whenever ϕ has a finite upper type, we say that ϕ satisfies the Δ_2-*condition*, which is equivalent to asking that $\phi(2t) \le C\phi(t)$ for all $t \ge 0$.

Given ϕ satisfying the Δ_2-condition, the *lower* and *upper indices* are defined by

$$i(\phi) = \lim_{s\to 0^+} \frac{\log h(s)}{\log s} = \sup_{0<s<1} \frac{\log h(s)}{\log s}$$

and

$$I(\phi) = \lim_{s\to\infty} \frac{\log h(s)}{\log s} = \inf_{s>1} \frac{\log h(s)}{\log s}$$

respectively, where

$$h(s) = \sup_{t>0} \frac{\phi(st)}{\phi(t)}.$$

For the existence of the above limits we refer to the book of Kokilashvili and Krbec [K-K].

This notion of index is related to that of type in the following sense: for any $\varepsilon > 0$, ϕ is of lower type $i(\phi) - \varepsilon$ and of upper type $I(\phi) + \varepsilon$. This statement may fail for $\varepsilon = 0$. Finally, we point out that when $1 < i(\phi)$ and $I(\phi) < \infty$, we have the following relationship between the indices of ϕ and those of the complementary function $\widetilde{\phi}$:

$$i(\widetilde{\phi}) = (I(\phi))' \quad \text{and} \quad I(\widetilde{\phi}) = (i(\phi))',$$

where $r' = r/(r-1)$ is the conjugate exponent of r. From this, we easily deduce that $I(\widetilde{\phi})$ is finite if and only if $i(\phi) > 1$. Whenever this happens we say that ϕ satisfies the Δ_2-*complementary condition*, or simply the Δ_2^c.

Finally, from the above definitions, we observe that a function ϕ satisfies both Δ_2 and Δ_2^c if and only if $1 < i(\phi) \le I(\phi) < \infty$ if and only if ϕ is of lower type $p > 1$ and of upper type $q < \infty$. In this work, we use freely any of these equivalent statements.

We are now in a position to state our main results.

THEOREM I. *Let ϕ be an N-function satisfying both the Δ_2 and Δ_2^c conditions. For a weight w and $q = I(\phi)$ the following assertions are equivalent*:

(a) $w \in RH_\phi$.
(b) *For any $b \ge 0$, $\phi(bw) \in A_\infty$ with a uniform constant.*
(c) $w^q \in A_\infty$.
(d) $w \in RH_q$.

THEOREM II. *If $\mathcal{N}$ denotes the class of all N-functions, we have*:

(a) $\bigcup_{\phi\in\mathcal{N}} RH_\phi = A_\infty = \{w : 1/w \in A_\infty(w\,dx)\} = \bigcup_{q>1} RH_q$.
(b) $\bigcap_{\phi\in\mathcal{N}} RH_\phi = RH_\infty = \{w : 1/w \in A_1(w\,dx)\}$, *strictly contained in* $\bigcap_{q>1} RH_q$.

3. Lemmas and preliminary results. In this section we give some equivalent definitions of RH_ϕ and we study some properties of the weights belonging to these classes.

First of all, we note that inequality (1.1) defining RH_ϕ can be written by using Orlicz norms as

$$\phi^{-1}(t/|Q|)\|\chi_Q w\|_{L_\phi(dx/t)} \le Cm_Q(w) \quad \text{for any cube } Q \subset \mathbb{R}^n \text{ and } t > 0.$$

We now give two more ways of describing the classes RH_ϕ.

(3.1) PROPOSITION. *The following statements are equivalent*:

(3.2) $w \in RH_\phi$.
(3.3) $\|\chi_Q w/\varepsilon\|_{L_\phi(dx/(|Q|\phi(1/\varepsilon)))} \le Cm_Q(w)$ *for any cube* $Q \subset \mathbb{R}^n$ *and* $\varepsilon > 0$.
(3.4) $\phi^{-1}(1/|Q|)\|\chi_Q \delta_\lambda w\|_{L_\phi(dx)} \le Cm_Q(\delta_\lambda w)$ *for any cube* $Q \subset \mathbb{R}^n$ *and* $\lambda > 0$, *where* $\delta_\lambda w(x) = w(\lambda x)$.

The proof is quite straightforward. In fact, that (3.3) is equivalent to (3.2) follows by the change of parameter $1/\varepsilon = \phi^{-1}(t/|Q|)$, and that (3.4) is equivalent to (3.2) is immediate after a change of variable in the integrals involved. A similar calculation shows that these classes remain invariant under dilations, that is, $w \in RH_\phi$ implies $\delta_\lambda w \in RH_\phi$ for all $\lambda > 0$, with the constant C appearing in (1.1) independent of λ. We finally observe that the reverse of any of the inequalities stated in the proposition holds true with $C = 1$, as a consequence of the Hölder inequality in Orlicz spaces.

As is easy to check, a weight $w \in RH_\phi$ with $i(\phi) > 1$ also belongs to RH_{ϕ^r} for $1/i(\phi) < r \le 1$. We shall show that if, moreover, ϕ has finite upper type, then the above statement can be extended somewhat to the right of $r = 1$. In order to prove this result, we need some technical lemmas.

(3.5) LEMMA. *Let η be a non-negative, non-decreasing function of positive lower type a and finite upper type b. Then, for the function*

$$\overline{\eta}(t) = \int_0^t \frac{\eta(s)}{s}\,ds, \quad t > 0,$$

we have

$$\frac{1}{bC_b}\eta(t) \le \overline{\eta}(t) \le \frac{C_a}{a}\eta(t),$$

where C_a and C_b are the constants of type appearing in (2.1) *and* (2.2) *respectively.*

Proof. From the definitions of lower type a and upper type b, it follows that, for $0 < s \le 1$,

$$\frac{s^b}{C_b}\eta(t) \le \eta(ts) \le C_a s^a \eta(t).$$

Dividing by s and integrating over $\{0 \le s \le t\}$, we arrive at the desired conclusion after a change of variable. ∎

(3.6) LEMMA. *Let η and h be non-negative functions with h non-increasing. Assume also that η is non-decreasing, has a lower type $p > 1$, finite upper type q and that*

$$-\int_t^\infty \frac{\eta(v)}{v}\,dh(v) \le C\frac{\eta(t)}{t}h(t) \tag{3.7}$$

for any $t > 1$. Then there exist $r > 1$ and a constant C, both depending only on p, q, on the constants C_p and C_q associated with the types of η, and on the constant C from (3.7), *such that*

$$-\int_1^\infty \frac{\eta(v)^r}{v}\,dh(v) \le C\eta(1)^{r-1}\Big(-\int_1^\infty \frac{\eta(v)}{v}\,dh(v)\Big).$$

Proof. Without loss of generality we may assume that h has compact support. Let $r > 1$ be a number to be fixed later.

Let

$$\overline{\eta}(t) = \int_0^t \frac{\eta(s)}{s}\,ds.$$

Using Lemma (3.5), integrating by parts and defining $\gamma(t) = \overline{\eta}(t)^{r-1}$, we get

$$\begin{aligned}
-\int_1^\infty &\frac{\eta(v)^r}{v}\,dh(v)\\
&\le (qC_q)^{r-1}\Big(-\int_1^\infty \overline{\eta}(v)^{r-1}\frac{\eta(v)}{v}\,dh(v)\Big)\\
&= q^{r-1}C_q^{r-1}\Big(-\gamma(1)\int_1^\infty \frac{\eta(u)}{u}\,dh(u) - \int_1^\infty \frac{d\gamma}{dv}\Big(\int_v^\infty \frac{\eta(u)}{u}\,dh(u)\Big)dv\Big).
\end{aligned}$$

By applying the hypothesis and integrating by parts again, the second term in the sum above can be bounded by

$$Cq^{r-1}C_q^{r-1}\Big(-h(1)\int_0^1 \frac{d\gamma}{du}\frac{\eta(u)}{u}\,du - \int_1^\infty\Big(\int_0^v \frac{d\gamma}{du}\frac{\eta(u)}{u}\,du\Big)dh(v)\Big).$$

The first term in brackets is non-positive while for the second we have

$$\begin{aligned}
\int_0^v \frac{d\gamma}{du}\frac{\eta(u)}{u}\,du &= (r-1)\int_0^v \overline{\eta}(u)^{r-2}\frac{\eta(u)^2}{u^2}\,du\\
&\le (r-1)\frac{qC_qC_p^{r-1}}{p^{r-1}}\int_0^v \frac{\eta(u)^r u^{-1}}{u}\,du
\end{aligned}$$

$$\le \frac{(r-1)qC_qC_p^{2r-1}}{p^{r-1}(pr-1)}\cdot\frac{\eta(v)^r}{v},$$

where we have used Lemma (3.5) for the functions η, η^{r-1} and $\eta(u)^r u^{-1}$.

With these estimates we get

$$\begin{aligned}
-\int_1^\infty \frac{\eta(v)^r}{v}\,dh(v) \le &-q^{r-1}C_q^{r-1}\gamma(1)\int_1^\infty \frac{\eta(v)}{v}\,dh(v)\\
&-(r-1)\frac{q^rCC_q^rC_p^{2r-1}}{p^{r-1}(pr-1)}\int_1^\infty \frac{\eta(v)^r}{v}\,dh(v).
\end{aligned}$$

By Lemma (3.5),

$$\gamma(1) = \overline{\eta}(1)^{r-1} \le \frac{C_p^{r-1}}{p^{r-1}}\eta^{r-1}(1),$$

so the last inequality can be written as

$$\begin{aligned}
\Big(1-\frac{q^rCC_q^rC_p^{2r-1}}{p^{r-1}(pr-1)}(r-1)\Big)&\Big(-\int_1^\infty \frac{\eta(v)^r}{v}\,dh(v)\Big)\\
&\le \Big(\frac{C_pC_qq}{p}\Big)^{r-1}\eta(1)^{r-1}\Big(-\int_1^\infty \frac{\eta(v)}{v}\,dh(v)\Big).
\end{aligned}$$

Since $p > 1$, the constant on the left hand side can be made positive by choosing r sufficiently close to 1. This gives the desired inequality. ∎

(3.8) LEMMA. *Let η be an N-function and w a weight. For a fixed cube Q set $h(t) = w(E(t))$ with $E(t) = \{x \in Q : w(x) > t\}$. Then*

$$\int_{E(t)} \eta(w(x))\,dx = -\int_t^\infty \frac{\eta(s)}{s}\,dh(s) \quad \textit{for any } t \ge 1. \tag{3.9}$$

Proof. Since η is a convex function it is absolutely continuous on any bounded interval and therefore $\eta(s)/s$ is absolutely continuous on each bounded interval which is away from zero. Then for $t \ge 1$ we may write

$$\int_{E(t)} \eta(w(x))\,dx = \frac{\eta(t)}{t}h(t) + \int_{E(t)} w(x)\Big(\int_t^{w(x)} \frac{d}{ds}\Big(\frac{\eta(s)}{s}\Big)ds\Big)dx.$$

Since $w(x)$, being locally integrable, is finite almost everywhere, changing the order of integration and integrating by parts we get

$$\int_{E(t)} \eta(w(x))\,dx = \frac{\eta(t)}{t}h(t) + \int_t^\infty \frac{d}{ds}\left(\frac{\eta(s)}{s}\right)\Big(\int_{E(s)} w(x)\,dx\Big)ds$$
$$= \frac{\eta(t)}{t}h(t) + \int_t^\infty \frac{d}{ds}\left(\frac{\eta(s)}{s}\right)h(s)\,ds$$
$$= \lim_{s\to\infty} \frac{\eta(s)}{s}h(s) - \int_t^\infty \frac{\eta(s)}{s}\,dh(s).$$

The proof will be complete if we are able to show that under our assumptions the above limit at ∞ is 0.

First suppose that the left hand side of (3.9) is finite and observe that because of the convexity of η, $\eta(s)/s$ is a non-decreasing function. Then for $s \ge t$,

$$h(s) = \int_{\{x\in Q:w(x)>s\}} w(x)\,dx \le \frac{s}{\eta(s)}\int_{E(s)} w(x)\frac{\eta(w(x))}{w(x)}\,dx$$
$$= \frac{s}{\eta(s)}\int_{E(s)} \eta(w(x))\,dx$$

and the last integral goes to zero as $s\to\infty$ since $\eta(w(x))$ is integrable on $E(t)$ and $|E(s)|\to 0$ as $s\to\infty$.

Finally, if the right hand side of (3.9) is finite we have

$$\lim_{b\to\infty} -\int_b^\infty \frac{\eta(s)}{s}\,dh(s) = 0.$$

Using again the fact that $\eta(s)/s$ is non-decreasing gives

$$-\int_b^\infty \frac{\eta(s)}{s}\,dh(s) \ge \frac{\eta(b)}{b}\Big(-\int_b^\infty dh(s)\Big) = \frac{\eta(b)}{b}(h(b) - \lim_{s\to\infty} h(s)).$$

But $h(s) = \int_{E(s)} w(x)\,dx \to 0$ as $s\to\infty$, since w is locally integrable. ∎

With these lemmas we can now prove the key property of the classes RH_ϕ mentioned above.

(3.10) PROPOSITION. *Let ϕ be an N-function satisfying the Δ_2 and Δ_2^c conditions. Let w be a weight belonging to RH_ϕ. Then there exists $r>1$ such that w belongs to RH_{ϕ^r}.*

Proof. Let Q be a cube and assume that

$$N_Q(w) = \sup_\varepsilon \|\chi_Q w/\varepsilon\|_{L_\phi(dx/(|Q|\phi(1/\varepsilon)))} = 1.$$

Therefore

$$\frac{1}{|Q|}\int_Q \phi\Big(\frac{w(x)}{\varepsilon}\Big)\,dx \le \phi(1/\varepsilon).$$

Given $t>1$ we set $s = 2Ct$ with C the constant appearing in the reverse-Hölder-ϕ inequality. By our assumptions $N_Q(w) < s$, and hence for any fixed $\varepsilon>0$,

$$\frac{1}{|Q|}\int_Q \phi\Big(\frac{w(x)}{\varepsilon s}\Big)\,dx \le \phi(1/\varepsilon).$$

So we may apply the Calderón–Zygmund decomposition to the function $\phi(w(x)/(\varepsilon s))$ on the cube Q with $\lambda = \phi(1/\varepsilon)$ to obtain a family $\{Q_j\}$ of disjoint cubes satisfying

$$\phi(1/\varepsilon) \le \frac{1}{|Q_j|}\int_{Q_j} \phi\Big(\frac{w(x)}{\varepsilon s}\Big)\,dx \le 2^n\phi(1/\varepsilon) \tag{3.11}$$

and

$$\phi\Big(\frac{w(x)}{\varepsilon s}\Big) \le \phi(1/\varepsilon) \quad \text{for almost any } x\in Q - \bigcup_j Q_j. \tag{3.12}$$

Let $E(s) = \{x\in Q : w(x) > s\}$. The last assertion implies that up to a set of measure zero, $E(s) \subset \bigcup Q_j = G$. Then from (3.11) we have

$$\int_{E(s)} \phi\Big(w(x)\varepsilon s\Big)\,dx \le \sum_j \int_{Q_j} \phi\Big(\frac{w(x)}{\varepsilon s}\Big)\,dx \le 2^n\phi(1/\varepsilon)|G|. \tag{3.13}$$

In order to estimate $|G|$ we observe that the first inequality in (3.11) implies

$$\|\chi_{Q_j} w/\varepsilon\|_{L_\phi(dx/(|Q_j|\phi(1/\varepsilon)))} > s.$$

Since $w\in RH_\phi$ we get

$$\frac{C}{|Q_j|}\int_{Q_j} w(x)\,dx > s$$

and then

$$s|G| = s\sum_j |Q_j| \le C\sum_j \int_{Q_j} w(x)\,dx$$
$$\le C\sum_j \int_{Q_j\cap E(t)} w(x)\,dx + Ct\sum_j |Q_j|$$
$$\le C\int_{E(t)} w(x)\,dx + Ct|G|.$$

So, we get

$$|G| \le \frac{1}{t}\int_{E(t)} w(x)\,dx.$$

Inserting this estimate in (3.13), we obtain

$$\int_{E(s)} \phi\left(\frac{w(x)}{\varepsilon s}\right) \frac{dx}{\phi(1/\varepsilon)} \le \frac{C}{t} \int_{E(t)} w(x)\,dx,$$

to conclude that

$$\int_{E(t)} \phi\left(\frac{w(x)}{\varepsilon s}\right) \frac{dx}{\phi(1/\varepsilon)} \le \frac{C}{t} \int_{E(t)} w(x)\,dx + \int_{E(t)-E(s)} \phi\left(\frac{w(x)}{\varepsilon s}\right) \frac{dx}{\phi(1/\varepsilon)}.$$

Since ϕ is increasing, the last term can be bounded by $|E(t)|$, which in turn, by the Chebyshev inequality, is bounded by the first term of the sum above.

Since $s = 2Ct$ and ϕ is of finite upper type we have proved that

$$\int_{E(t)} \phi\left(\frac{w(x)}{\varepsilon t}\right) dx \le C\frac{\phi(1/\varepsilon)}{t} \int_{E(t)} w(x)\,dx.$$

Setting $\sigma = \varepsilon t$ we may apply Lemma (3.8) to the left hand side with $\eta(s) = \phi(s/\sigma)$ and the weight w, since the integrability on Q of w implies the finiteness of both integrals. Then we have

$$-\int_t^\infty \frac{\phi(s/\sigma)}{s}\,dh(s) \le C\frac{\phi(t/\sigma)}{t}h(t)$$

for any $t \ge 1$ and $\sigma > 0$, where $h(t)$ is defined by $w(E(t)) = w(\{x \in Q : w(x) > t\})$.

Now we are in a position to apply Lemma (3.6) to the functions $\eta(s) = \phi(s/\sigma)$ and $h(t)$. Since all these functions η have the same types of ϕ, with the same constants, there are $r > 1$ and a constant C such that

$$-\int_1^\infty \frac{\phi(s/\sigma)^r}{s}\,dh(s) \le C\phi(1/\sigma)^{r-1}\left(-\int_1^\infty \frac{\phi(s/\sigma)}{s}\,dh(s)\right).$$

But the integral on the right is the same as $\int_{E(1)} \phi(w(x)/\sigma)\,dx$, which is finite because of the local integrability of $\phi(w(x)/\sigma)$ implied by the reverse-Hölder-ϕ condition on w. Thus, the left hand side of the above inequality is also finite, and we may also apply Lemma (3.8) to get for any $\sigma > 0$,

$$\int_{E(1)} \phi(w(x)/\sigma)^r\,dx \le C\phi(1/\sigma)^{r-1} \int_{E(1)} \phi(w(x)/\sigma)\,dx.$$

On the other hand, for $x \in Q - E(1)$ we have

$$\phi(w(x)/\sigma)^r = \phi(w(x)/\sigma)^{r-1}\phi(w(x)/\sigma) \le \phi(1/\sigma)^{r-1}\phi(w(x)/\sigma).$$

Integrating over $Q - E(1)$ and combining with the estimate over $E(1)$ we get

$$\int_Q \phi(w(x)/\sigma)^r\,dx \le C\phi(1/\sigma)^{r-1} \int_Q \phi(w(x)/\sigma)\,dx.$$

Now, since we assume $N_Q(w) = 1$, we have

$$\int_Q \phi(w(x)/\sigma)^r \frac{dx}{|Q|\phi(1/\sigma)^r} \le C \int_Q \phi(w(x)/\sigma) \frac{dx}{|Q|\phi(1/\sigma)} \le C$$

and therefore for any $\sigma > 0$,

$$\|\chi_Q w/\sigma\|_{L_{\phi^r}(dx/(|Q|\phi^r(1/\sigma)))} \le C$$

and the reverse-Hölder-ϕ inequality for the cube Q implies

$$\sup_{\sigma>0} \|\chi_Q w/\sigma\|_{L_{\phi^r}(dx/(|Q|\phi^r(1/\sigma)))} \le C\frac{1}{|Q|} \int_Q w(x)\,dx.$$

Finally, if $N_Q(w) \ne 1$ we take $W(x) = w(x)/N_Q(w)$. Since W satisfies the reverse-Hölder-ϕ condition with the same constant and $N_Q(W) = 1$ we may apply the last inequality to W, which gives the same result for w with a constant independent of the cube Q. This finishes the proof of the theorem. ■

4. Proof of the theorems

Proof of Theorem I. We first check that (a)⇒(d). To this end we show that if $w \in RH_\phi$ then $w \in RH_{q-\varepsilon}$ for ε small enough. Then, using Proposition (3.10) we know that w also belongs to RH_{ϕ^r} for some $r > 1$. Since $I(\phi^r) = qr$ we may conclude $w \in RH_{qr-\varepsilon}$, and upon taking $\varepsilon = q(r-1) > 0$ the result follows. Now, by the definition of $q = I(\phi)$, we see that for any $\gamma \ge 1$,

$$\sup_{s>0} \frac{\phi(\gamma s)}{\phi(s)} \ge \gamma^q.$$

Therefore, for any $\gamma \ge 1$, there exists $s = s(\gamma)$ such that

$$2\phi(\gamma s) \ge \gamma^q \phi(s). \tag{4.1}$$

In particular, we can choose s_k such that the above inequality holds for $\gamma = 2^k$, $k \ge 0$.

On the other hand, our assumption $w \in RH_\phi$ can be written as

$$\int_Q \phi\left(\frac{w(x)\phi^{-1}(t/|Q|)}{Cm_Q(w)}\right)\frac{dx}{t} \le 1 \tag{4.2}$$

for any cube Q, $t > 0$ and a fixed constant C. Now, in order to check that

$w \in RH_{q-\varepsilon}$, we take a cube Q and consider the disjoint sets

$$E_k = \left\{ x \in Q : 2^k \le \frac{w(x)}{C m_Q(w)} < 2^{k+1} \right\}, \quad k \ge 0.$$

For each k we use (4.2) for $t_k = \phi(s_k)|Q|$. This together with (4.1) gives

$$1 \ge \int_{E_k} \phi\left(\frac{w(x)\phi^{-1}(t_k/|Q|)}{C m_Q(w)} \right) \frac{dx}{t_k} \ge \int_{E_k} \phi(2^k s_k) \frac{dx}{t_k}$$
$$\ge \frac{1}{2} \int_{E_k} 2^{kq} \phi(s_k) \frac{dx}{t_k} \ge C_\varepsilon \frac{2^{k\varepsilon}}{|Q| m_Q(w)^{q-\varepsilon}} \int_{E_k} w(x)^{q-\varepsilon}\, dx.$$

So we get

$$\frac{1}{|Q|} \int_{E_k} w(x)^{q-\varepsilon}\, dx \le C_\varepsilon m_Q(w)^{q-\varepsilon} 2^{-k\varepsilon}.$$

But if we set $E = \{x \in Q : w(x) \le C m_Q(w)\}$ we also have for small $\varepsilon > 0$,

$$\frac{1}{|Q|} \int_E w(x)^{q-\varepsilon}\, dx \le C_\varepsilon m_Q(w)^{q-\varepsilon}.$$

Adding up these inequalities and raising to the $1/(q-\varepsilon)$ power we get

$$\left(\frac{1}{|Q|} \int_Q w(x)^{q-\varepsilon}\, dx \right)^{1/(q-\varepsilon)} \le C_\varepsilon \frac{1}{|Q|} \int_Q w(x)\, dx,$$

so $w \in RH_{q-\varepsilon}$.

That (c) is equivalent to (d) is an already known result. See for example [S-T]. So we turn to the proof of (c)⇒(b). For Q a cube and S a measurable subset of Q we want to prove that there exist $\delta > 0$ and a constant C independent of b, Q and S such that

$$\text{(4.3)} \qquad \frac{\phi(bw)(S)}{\phi(bw)(Q)} \le C \left(\frac{|S|}{|Q|} \right)^\delta.$$

Let $S_1 = \{x \in S : \phi(bw(x)) \le \phi(bw)(Q)/|Q|\}$ and $S_2 = S - S_1$. Then we have the estimates

$$I_1 = \int_{S_1} \phi(bw(x))\, dx \le \frac{|S|}{|Q|} \int_Q \phi(bw(x))\, dx \le \left(\frac{|S|}{|Q|} \right)^\delta \phi(bw)(Q)$$

for any $\delta \le 1$. Also since ϕ is of upper type $r = q + \varepsilon$ for $\varepsilon > 0$, we have, for all $t_1 \ge t_2$,

$$\phi(t_1) \le C_r (t_1/t_2)^r \phi(t_2).$$

Since for $x \in S_2$, $bw(x) > \phi^{-1}(m_Q(\phi(bw)))$, we may apply the latter

inequality to get

$$\text{(4.4)} \qquad I_2 = \int_{S_2} \phi(bw(x))\, dx \le \frac{m_Q(\phi(bw))}{(\phi^{-1}(m_Q(\phi(bw))))^r} \int_S (bw(x))^r\, dx.$$

Now since $w^q \in A_\infty$ it is also true that $w^r \in A_\infty$ for some $r > q$; for such an r, the last integral can be bounded by

$$C_r \left(\frac{|S|}{|Q|} \right)^\delta \int_Q (bw(x))^r\, dx \le C_r |Q| \left(\frac{|S|}{|Q|} \right)^\delta \left(\frac{1}{|Q|} \int_Q bw(x)\, dx \right)^r$$

for some $\delta > 0$, where in the last inequality we made use of the fact that $w^r \in A_\infty$ implies $w \in RH_r$.

Next we observe that the convexity of ϕ gives, for any locally integrable function g,

$$\phi(m_Q(g)) \le m_Q(\phi(g)),$$

which for $g = bw$ leads to

$$m_Q(bw) \le \phi^{-1}[m_Q(\phi(bw))].$$

Inserting these estimates in (4.4) we get

$$I_2 \le C_r \left(\frac{|S|}{|Q|} \right)^\delta \left(\int_Q \phi(bw(x))\, dx \right).$$

Combining the I_1 and I_2 estimates we get (4.3) for a constant C independent of b.

Finally, we prove that (b)⇒(a). We begin by observing that, by an appropriate change of the parameter b, we may assume $m_Q(w) = 1$. Our goal is to show that for some choice of C_0,

$$\text{(4.5)} \qquad \int_Q \phi\left(\frac{w(x)\phi^{-1}(t/|Q|)}{C_0} \right) \frac{dx}{t} \le 1$$

for every $t > 0$. We use the following condition equivalent to $v \in A_\infty$: there exist $\alpha, \beta > 0$ such that

$$\text{(4.6)} \qquad |\{x \in Q : v(x) > \beta m_Q(v)\}| > \alpha |Q|.$$

Our hypothesis (b) implies that the latter inequality holds for $v = \phi(bw)$ for some α and β independent of b. We claim that then, for all $b > 0$,

$$\frac{1}{|Q|} \int_Q \phi(bw(x)) \frac{dx}{\phi(b)} \le \frac{C}{\beta},$$

where C can be taken as $2C_r/\alpha^r$, with r the upper type of ϕ and C_r the r-type constant.

Otherwise, we would have for some b the opposite inequality, which together with (4.6), applied to $\phi(bw)$, and the Chebyshev inequality would lead to

$$\alpha|Q| < \left|\left\{x \in Q : \phi(bw(x)) > \frac{\beta}{|Q|}\int_Q \phi(bw(x))\,dx\right\}\right|$$
$$\le |\{x \in Q : \phi(bw(x)) > C\phi(b)\}| = |\{x \in Q : w(x) > \phi^{-1}(C\phi(b))/b\}|$$
$$\le \frac{b}{\phi^{-1}(C\phi(b))}\int_Q w(x)\,dx = \frac{b|Q|}{\phi^{-1}(C\phi(b))}$$

and hence we would have $\phi^{-1}(C\phi(b)) < b/\alpha$ or, in other words, $C\phi(b) < \phi(b/\alpha)$. Since that ϕ is of upper type r, this would imply $C < C_r/\alpha^r$, which is a contradiction.

Therefore, the claim is true and setting $b = \phi^{-1}(t/|Q|)$ we have

$$\int_Q \phi(w(x)\phi^{-1}(t/|Q|))\,\frac{dx}{t} \le \frac{C}{\beta}.$$

Since $C/\beta > 1$ the fact that ϕ has positive lower type allows us to replace the constant outside ϕ by a perhaps different constant inside ϕ leading then to the inequality (4.5).

Proof of Theorem II. (a) That the second inequality is true follows from the fact that $w \in A_\infty$ if and only if there exist $\alpha, \beta > 0$ such that for any cube Q and any measurable subset $E \subset Q$,

$$|E|/|Q| \le \alpha \Rightarrow w(E)/w(Q) \le \beta \tag{4.7}$$

(see for example [C-F]). But, upon taking $Q - E$ in place of E the last assertion is equivalent to the existence of γ and δ such that

$$w(E)/w(Q) \le \gamma \Rightarrow |E|/|Q| \le \delta. \tag{4.8}$$

Now if $w \in A_\infty$, then $d\mu = w(x)dx$ is a doubling measure and therefore $A_\infty(d\mu)$ also coincides with the weights v for which there exist γ and δ such that

$$\mu(E)/\mu(Q) \le \gamma \Rightarrow \int_E v\,d\mu \Big/ \int_Q v\,d\mu \le \delta \tag{4.9}$$

(see [C-F]); but (4.8) gives exactly this statement for $v = 1/w$. Conversely, if a weight belongs to $A_\infty(w)$, it satisfies the weaker condition (4.9), therefore $1/w \in A_\infty(w)$ implies (4.8), which is equivalent to $w \in A_\infty$.

For the first equality, using the fact that $A_\infty = \bigcup_{p>1} RH_p$ we only need to show that $RH_\phi \subset A_\infty$ for any $\phi \in \mathcal{N}$. In fact we will check that if $w \in RH_\phi$, then (4.7) holds for w.

Let Q be a cube and E a measurable subset. Then the Hölder inequality for $\phi_t(s) = \phi(s)/t$ gives

$$w(E) = \int_E w(x)\,dx \le \|\chi_Q w\|_{L_{\phi_t}}\|\chi_E\|_{L_{\widetilde{\phi}_t}}.$$

Now since $\|\chi_E\|_{L_{\widetilde{\phi}_t}} = 1/\widetilde{\phi}_t^{-1}(1/|E|)$, using the fact that $s \simeq \phi_t^{-1}(s)\widetilde{\phi}_t^{-1}(s)$, we easily get

$$\|\chi_E\|_{L_{\widetilde{\phi}_t}} \le C|E|\phi^{-1}(t/|E|).$$

From this estimate and since $w \in RH_\phi$, we obtain

$$w(E) \le C|E|\frac{w(Q)}{|Q|}\cdot\frac{\phi^{-1}(t/|E|)}{\phi^{-1}(t/|Q|)},$$

which for $t = |Q|$ can be written as

$$\frac{w(E)}{w(Q)} \le C\frac{|E|}{|Q|}\phi^{-1}(|Q|/|E|).$$

Therefore, in order to get (4.7), we only need to show that if $|E|/|Q|$ is small enough then $(|E|/|Q|)\phi^{-1}(|Q|/|E|)$ is also small. In other words, we must show that the function $s\phi^{-1}(1/s)$ goes to zero as $s \to 0$. After setting $1/s = \phi(\sigma)$ this is equivalent to

$$\lim_{\sigma\to\infty}\frac{\sigma}{\phi(\sigma)} = 0 \quad \text{or} \quad \lim_{\sigma\to\infty}\frac{\phi(\sigma)}{\sigma} = \infty,$$

which is true for any $\phi \in \mathcal{N}$.

(b) The easy part is to show that $RH_\infty \subset \bigcap_{\phi\in\mathcal{N}} RH_\phi$. In fact, $w \in RH_\infty$ is equivalent to saying that for any cube Q we have

$$\sup_{x\in Q} w(x) \le C\frac{w(Q)}{|Q|}.$$

Therefore for $\phi \in \mathcal{N}$ and C as above, we have

$$\int_Q \phi\left(\frac{w(x)\phi^{-1}(t/|Q|)}{Cm_Q(w)}\right)\frac{dx}{t} \le \int_Q \phi(\phi^{-1}(t/|Q|))\frac{dx}{t} = 1,$$

so $w \in RH_\phi$.

Let us see now that for $\phi \in \mathcal{N}$, $RH_\phi \subset RH_\infty$.

We are going to show that if $w \notin RH_\infty$, that is, $1/w \notin A_1(w)$, then it is possible to construct $\psi \in \mathcal{N}$ such that $w \notin RH_\psi$. In fact, if $1/w \notin A_1(w)$, then for each $k \in \mathbb{N}$, there exists a cube Q_k such that

$$\sup_{Q_k} w \ge 2k^2 w(Q_k)/|Q_k|.$$

Consider now the sets

$$E_k = \{x \in Q_k : w(x) \ge k^2 w(Q_k)/|Q_k|\}.$$

Setting $t_k = w(E_k)/w(Q_k)$, we have $0 < t_k < 1$ and also the sequence $a_k = \max_{1\le i\le k} i/t_i$ is non-decreasing and, since $a_k \ge k$, it goes to infinity. Then we may choose an increasing subsequence a_{k_j} and define a continuous increasing function g such that

$$g(0)=0, \qquad g(k_j)=a_{k_j},$$

and it is linear in-between. We claim that the function ψ such that $\psi(0)=0$ and $\psi'=g$ gives the desired conclusion.

Clearly, by construction, ψ is a non-negative, increasing and convex function on $[0,\infty)$, and being quadratic near zero, it also satisfies $\psi(t)/t \to 0$ as $t\to 0$. Also by the convexity of ψ, we have

$$\frac{1}{2}\psi'\left(\frac{t}{2}\right) \le \frac{\psi(t)}{t} \le \psi'(t). \tag{4.10}$$

Since both bounds have limit infinity for t tending to infinity, we conclude that $\psi \in \mathcal{N}$.

On the other hand, for the cubes Q_{k_j} we have

$$\frac{1}{w(Q_{k_j})}\int_{Q_{k_j}} \psi'\left(\frac{|Q_{k_j}|w(x)}{k_j w(Q_{k_j})}\right) w(x)\,dx \ge \frac{1}{w(Q_{k_j})}\int_{E_{k_j}} \psi'(k_j)w(x)\,dx$$
$$= \frac{w(E_{k_j})}{w(Q_{k_j})} a_{k_j} \ge 1.$$

Together with (4.10) this gives

$$\frac{k_j}{|Q_{k_j}|}\int_{Q_{k_j}} \psi\left(\frac{2|Q_{k_j}|w(x)}{k_j w(Q_{k_j})}\right) dx \ge 1.$$

Setting $t_j = |Q_{k_j}|/k_j$ we can rewrite the above inequality as

$$\frac{1}{t_j}\int_{Q_{k_j}} \psi\left(\frac{\psi^{-1}(t_j/|Q_{k_j}|)w(x)}{\frac{w(Q_{k_j})}{|Q_{k_j}|}\cdot \frac{k_j}{2}\psi^{-1}\left(\frac{1}{k_j}\right)}\right) dx \ge 1.$$

Now set $C_j = (k_j/2)\psi^{-1}(1/k_j)$. We claim $C_j \to \infty$, which will contradict the fact that $w \in RH_\psi$. Since $k_j \to \infty$ as $j\to\infty$, it is enough to check that $\psi^{-1}(s)/s \to \infty$ as $s \to 0$. But

$$\lim_{s\to 0}\frac{\psi^{-1}(s)}{s} = \lim_{t\to 0}\frac{t}{\psi(t)} = \infty,$$

by hypothesis.

Finally, that RH_∞ is a proper subset of $\bigcap_{q>1} RH_q$ follows as in [CU-N] by taking $w(x) = \max(\log(1/|x|),1)$. ■

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