Reverse-Hölder classes in the Orlics spaces setting

by

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Abstract. In connection with the $A_p$ classes of weights (see [K-T] and [B-K]), we study, in the context of Orlics spaces, the corresponding reverse-Hölder classes $RH_q$. We prove that when $\phi$ is $A_2$ and has lower index greater than one, the class $RH_q$ coincides with some reverse-Hölder class $RH_{q'}$, $q > 1$. For more general $\phi$ we still get $RH_q \subset A_\infty = \bigcup_{q>1} RH_q$ although the intersection of all these $RH_q$ gives a proper subset of $\bigcap_{q>1} RH_q$.

1. Introduction. By a weight $w$ we mean a non-negative and locally integrable function on $\mathbb{R}^n$. As is well known, a weight $w$ is said to belong to the reverse-Hölder class with exponent $q$, $RH_q$, if it satisfies the inequality

$$\left( \frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{1/q} \leq C \frac{1}{|Q|} \int_Q w(x) \, dx$$

for any cube $Q \subset \mathbb{R}^n$ with sides parallel to the axes; here $|Q|$ denotes the Lebesgue measure of $Q$. These classes appeared in connection with the $A_p$ classes of Muckenhoupt which characterize the weights such that the Hardy–Littlewood maximal operator is bounded on $L^p(w)$, $1 < p < \infty$. To be precise, the $A_p$ weights are defined as those weights such that

$$\left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q w(x)^{q(p-1)} \, dx \right)^{1/(p-1)} \leq C$$

for any cube $Q \subset \mathbb{R}^n$. The limiting case $p = 1$, $A_{\infty}$, is defined as the weights satisfying

$$m_Q(w) \leq C \inf_{a \in \partial Q} w(a)$$

for any cube $Q \subset \mathbb{R}^n$, where $m_Q(w)$ denotes the average of $w$ over $Q$. For $p = \infty$, $A_{\infty}$ consists of the weights $w$ such that for any $Q$ and any measurable

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subset $E \subset Q$, there exists $\delta > 0$ satisfying
\[
\frac{w(E)}{w(Q)} \leq C(|E|/|Q|)^\delta.
\]

Note that these definitions can be extended to more general measures $\mu$ instead of Lebesgue measure, with the obvious changes. So we can, and will, talk about $A_\infty(d\mu)$, $A_1(d\mu)$ or $A_p(d\mu)$ classes and we drop the notation $d\mu$ in the case of Lebesgue measure.

The precise connection between the two classes is given by the following identities:
\[
\bigcup_{q > 1} RH_q = \bigcup_{p > 1} A_p = A_\infty.
\]

Kerman and Torchinsky [K-T] and later Bloom and Kerman [B-K] studied the boundedness of the Hardy-Littlewood maximal operator on weighted Orlicz spaces. In this context they introduced, for $\phi$ a Young function, the classes $A_\phi$ and more generally $W_\phi$, both giving extensions of $A_\phi$, $p > 1$.

The aim of this paper is to study the corresponding reverse-Hölder classes of weights in the Orlicz spaces setting.

For a non-negative, increasing, continuous and convex function $\phi$ defined on $[0, \infty)$ we say that a weight $w$ belongs to $RH_\phi$ if there exists a positive constant $C$ such that
\[
\int_Q \frac{\phi^{-1}(t/|Q|) w(x)}{C m_Q(w)} \frac{dx}{t} \leq 1
\]
for any cube $Q \subset \mathbb{R}^n$ and $t > 0$.

It is easy to check that when $\phi(s) = s^q$, $q > 1$, the above inequality coincides with the reverse Hölder condition with exponent $q$, $q > 1$. Also, for $\phi$ as above, the reverse inequality to (1.1) holds with $C = 1$ as a consequence of the Jensen inequality for convex functions. Finally, we remark that the parameter $t$ is necessary to make the class $RH_\phi$ invariant under dilations in the sense that if $w \in RH_\phi$ then $w(\lambda x) \in RH_\phi$ with the constant $C$ independent of $\lambda > 0$.

Relating to these classes we prove a result similar to that of Kerman and Torchinsky for the class $A_\phi$, that is, when $\phi$ is "between power functions with exponents greater than one", in a sense that will be made precise later, the class $RH_\phi$ coincides with some $RH_q$. For more general $\phi$, including functions "near the identity", although the above result is not necessarily true, we still get $RH_\phi \subset A_\infty = \bigcup_{q > 1} RH_q$.

On the other hand, we also characterize the intersection of $RH_\phi$ for those general $\phi$ as the class $RH_\infty$ introduced by Franchi ([F]), which is a proper subset of $\bigcup_{q > 1} RH_q$, as shown in [CU-N]. These results correspond to those obtained in [B-K] for $W_\phi$.

2. Statement of main theorems. In this paper we say that a nonnegative function $\phi$ defined on $[0, \infty)$ is an $N$-function (or a Young function) if it is convex and satisfies
\[
\lim_{t \to 0^+} \frac{\phi(t)}{t} = \lim_{t \to \infty} \frac{\phi(t)}{t} = 0.
\]

Clearly, under these conditions $\phi$ has a derivative $\varphi$ which is non-decreasing and non-negative with $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. For such $\phi$, the complementary function defined by
\[
\overline{\phi}(s) = \sup_{t > 0} (st - \phi(t))
\]
is also an $N$-function. Moreover, it can be proved that there exist constants $C_1$ and $C_2$ such that
\[
C_1 t \leq \phi^{-1}(t)\overline{\phi}^{-1}(t) \leq C_2 t
\]
for every $t > 0$.

Given an $N$-function $\phi$ and a finite Borel measure $\mu$ on $\mathbb{R}^n$, the Orlicz space $L_\phi(d\mu)$ consists of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ for which there exists a constant $C$ such that
\[
\int_{\mathbb{R}^n} \phi(|f(x)|) \, d\mu(x) < \infty.
\]

Furthermore, the space $L_\phi(d\mu)$ equipped with the Luxemburg norm
\[
\|f\|_{L_\phi(d\mu)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \phi(|f(x)|/\lambda) \, d\mu(x) \leq 1 \right\}
\]
is a Banach space. Also, if $\psi$ denotes the complementary function of $\phi$, then the Hölder inequality
\[
\int_{\mathbb{R}^n} f(x)g(x) \, d\mu(x) \leq C\|f\|_{L_\psi(d\mu)}\|g\|_{L_\phi(d\mu)}
\]
holds.

Sometimes, we will impose further conditions on the function $\phi$. To this end, we introduce the notions of lower and upper types.

We say that $\phi$ is of lower type $p$ if there exists a constant $C$ such that
\[
\phi(st) \leq C s^p \phi(t) \quad \text{for } s \leq 1 \text{ and } t \geq 0,
\]
and that $\phi$ is of upper type $q$ if there exists a constant $C$ such that
\[
\phi(st) \leq C s^q \phi(t) \quad \text{for } s \geq 1 \text{ and } t \geq 0.
\]

Observe that for our kind of functions $\phi$, if $p$ and $q$ satisfy the above inequalities then we have
\[
1 \leq p \leq q < \infty.
\]
Whenever $\phi$ has a finite upper type, we say that $\phi$ satisfies the $\Delta_2$-condition, which is equivalent to asking that $\phi(2t) \leq C\phi(t)$ for all $t \geq 0$.

Given $\phi$ satisfying the $\Delta_2$-condition, the lower and upper indices are defined by

$$i(\phi) = \lim_{s \to 0^+} \frac{\log h(s)}{\log s} = \sup_{0 < s < 1} \frac{\log h(s)}{\log s}$$

and

$$I(\phi) = \lim_{s \to \infty} \frac{\log h(s)}{\log s} = \inf_{s > 1} \frac{\log h(s)}{\log s}$$

respectively, where

$$h(s) = \sup_{t > 0} \frac{\phi(st)}{\phi(t)}.$$

For the existence of the above limits we refer to the book of Kokilashvili and Krbc [K-K].

This notion of index is related to that of type in the following sense: for any $\varepsilon > 0$, $\phi$ is of lower type $i(\phi) - \varepsilon$ and of upper type $I(\phi) + \varepsilon$. This statement may fail for $\varepsilon = 0$. Finally, we point out that when $1 < i(\phi)$ and $I(\phi) < \infty$, we have the following relationship between the indices of $\phi$ and those of the complementary function $\bar{\phi}$:

$$i(\bar{\phi}) = (I(\phi))'$$

where $r' = r/(r - 1)$ is the conjugate exponent of $r$. From this, we easily deduce that $I(\bar{\phi})$ is finite if and only if $i(\bar{\phi}) > 1$. Whenever this happens we say that $\phi$ satisfies the $\Delta_2$-complementary condition, or simply the $\Delta_2$.

Finally, from the above definitions, we observe that a function $\phi$ satisfies both $\Delta_2$ and $\Delta_2^p$ if and only if $1 < i(\phi) \leq I(\phi) < \infty$ if and only if $\phi$ is of lower type $p > 1$ and of upper type $q < \infty$. In this work, we use freely any of these equivalent statements.

We are now in a position to state our main results.

**Theorem I.** Let $\phi$ be an $N$-function satisfying both the $\Delta_2$ and $\Delta_2^p$ conditions. For a weight $w$ and $q = I(\phi)$ the following assertions are equivalent:

(a) $w \in RH_{\phi}$.
(b) For any $b \geq 0$, $\phi(bw) \in A_\infty$ with a uniform constant.
(c) $w^q \in A_\infty$.
(d) $w \in RH_q$.

**Theorem II.** If $N$ denotes the class of all $N$-functions, we have:

(a) $\bigcup_{\phi \in N} RH_{\phi} = A_{\infty} = \{w : 1/w \in A_\infty(wdx)\} = \bigcup_{q \geq 1} RH_q$.
(b) $\bigcap_{\phi \in N} RH_{\phi} = RH_\infty = \{w : 1/w \in A_1(wdx)\}$, strictly contained in $\bigcap_{q \geq 1} RH_q$.

**3. Lemmas and preliminary results.** In this section we give some equivalent definitions of $RH_{\phi}$ and we study some properties of the weights belonging to these classes.

First of all, we note that inequality (1.1) defining $RH_{\phi}$ can be written by using Orlicz norms as

$$\phi^{-1}(t/(|Q|))||Q\phi(w)||_{L_q(dx/t)} \leq C_{Q}(w)$$

for any cube $Q \subset \mathbb{R}^n$ and $t > 0$.

We now give two more ways of describing the classes $RH_{\phi}$.

(3.1) **Proposition.** The following statements are equivalent:

(3.2) $w \in RH_{\phi}$.
(3.3) $||Q\phi(w)||_{L_q(dx/(|Q|\phi(1/|Q|)))} \leq C_{Q}(w)$ for any cube $Q \subset \mathbb{R}^n$ and $\varepsilon > 0$.
(3.4) $\phi^{-1}(1/(|Q|))||Q\phi\delta_{\lambda}w||_{L_q(dx)} \leq C_{Q}(\delta_{\lambda}w)$ for any cube $Q \subset \mathbb{R}^n$ and $\lambda > 0$, where $\delta_{\lambda}w(x) = w(\lambda x)$.

The proof is quite straightforward. In fact, that (3.3) is equivalent to (3.2) follows by the change of parameter $1/\varepsilon = \phi^{-1}(t/(|Q|))$, and that (3.4) is equivalent to (3.2) is immediate after a change of variable in the integrals involved. A similar calculation shows that these classes remain invariant under dilations, that is, $w \in RH_{\phi}$ implies $\delta_{\lambda}w \in RH_{\phi}$ for all $\lambda > 0$, with the constant $C$ appearing in (1.1) independent of $\lambda$. We finally observe that the reverse of any of the inequalities stated in the proposition holds true with $C = 1$, as a consequence of the Hölder inequality in Orlicz spaces.

As is easy to check, a weight $w \in RH_{\phi}$ with $i(\phi) > 1$ also belongs to $RH_{\phi'}$ for $1/i(\phi) < r \leq 1$. We shall show that if, moreover, $\phi$ has finite upper type, then the above statement can be extended somewhat to the right of $r = 1$. In order to prove this result, we need some technical lemmas.

(3.5) **Lemma.** Let $\eta$ be a non-negative, non-decreasing function of positive lower type $a$ and finite upper type $b$. Then, for the function

$$\eta(t) = \frac{1}{t} \int_{0}^{s} \frac{\eta(s)}{s} ds, \quad t > 0,$$

we have

$$\frac{1}{bC_b} \eta(t) \leq \eta(t) \leq \frac{C_a}{a} \eta(t),$$

where $C_a$ and $C_b$ are the constants of type appearing in (2.1) and (2.2) respectively.

**Proof.** From the definitions of lower type $a$ and upper type $b$, it follows that, for $0 \leq s \leq 1$,

$$\frac{s^a}{C_b} \eta(t) \leq \eta(ts) \leq C_a s^a \eta(t).$$
Dividing by \( s \) and integrating over \( \{0 \leq s \leq t\} \), we arrive at the desired conclusion after a change of variable.

(3.6) Lemma. Let \( \eta \) and \( h \) be non-negative functions with \( h \) non-increasing. Assume also that \( \eta \) is non-decreasing, has a lower type \( p > 1 \), finite upper type \( q \) and that

\[
\int_0^t \frac{\eta(v)}{v} dh(v) \leq C \frac{\eta(t)}{t} h(t)
\]

for any \( t > 1 \). Then there exist \( r > 1 \) and a constant \( C \), both depending only on \( p, q \), on the constants \( C_p \) and \( C_q \) associated with the types of \( \eta \), and on the constant \( C \) from (3.7), such that

\[
\int_0^1 \frac{\eta(v)^r}{v} dh(v) \leq C \eta(1)^{r-1} \left( -\int_0^1 \frac{\eta(v)}{v} dh(v) \right).
\]

Proof. Without loss of generality we may assume that \( h \) has compact support. Let \( r > 1 \) be a number to be fixed later.

Let

\[
\hat{\eta}(t) = \int_0^t \frac{\eta(s)}{s} ds.
\]

Using Lemma (3.5), integrating by parts and defining \( \gamma(t) = \hat{\eta}(t)^{r-1} \), we get

\[
-\int_0^1 \frac{\eta(v)^r}{v} dh(v) \leq (qC_q)^{r-1} \left( -\int_0^1 \hat{\eta}(v)^{r-1} \frac{\eta(v)}{v} dh(v) \right)
\]

and

\[
= q^{-1}C_q^{r-1} \left( -\gamma(1) \int_1^\infty \frac{\eta(u)}{u} dh(u) - \int_0^1 \frac{d\gamma}{dv} \left( \int_1^\infty \frac{\eta(u)}{u} dh(u) \right) dv \right).
\]

By applying the hypothesis and integrating by parts again, the second term in the sum above can be bounded by

\[
C q^{-1} C_q^{r-1} \left( -h(1) \int_0^1 \frac{d\gamma}{du} \frac{\eta(u)}{u} du + \int_0^1 \left( \int_0^u \frac{d\gamma}{du} \frac{\eta(u)}{u} du \right) dh(u) \right).
\]

The first term in brackets is non-positive while for the second we have

\[
\int_0^v \frac{d\gamma}{du} \frac{\eta(u)}{u} du = (r-1) \int_0^v \hat{\eta}(u)^{r-2} \frac{\eta(u)^2}{u^2} du
\]

\[
\leq (r-1) \frac{qC_p C_q^{r-1}}{p^{r-1}} \int_0^v \hat{\eta}(u)^{r-2} \frac{\eta(u)^2 u^{-2}}{u} du
\]

where we have used Lemma (3.5) for the functions \( \eta, \eta^{r-1} \) and \( \eta(u)^{r-1} \).

With these estimates we get

\[
\int_0^1 \frac{\eta(v)^r}{v} dh(v) \leq -q^{-1} C_q^{r-1} \gamma(1) \int_1^\infty \frac{\eta(v)}{v} dh(v)
\]

\[
= (r-1) \frac{q C_p C_q^{r-1}}{p^{r-1} (pr-1)} \int_0^1 \frac{\eta(v)^r}{v} dh(v).
\]

By Lemma (3.5),

\[
\gamma(1) = \hat{\eta}(1)^{r-1} \leq \frac{C_p}{p^{r-1}} \eta^{r-1}(1),
\]

so the last inequality can be written as

\[
\left( 1 - \frac{q C_p C_q^{r-1}}{p^{r-1} (pr-1)} (r-1) \right) \left( -\int_1^\infty \frac{\eta(v)^r}{v} dh(v) \right)
\]

\[
\leq \left( \frac{C_p C_q}{p} \right)^{r-1} \eta(1)^{r-1} \left( -\int_1^\infty \frac{\eta(v)}{v} dh(v) \right).
\]

Since \( p > 1 \), the constant on the left hand side can be made positive by choosing \( r \) sufficiently close to 1. This gives the desired inequality.

(3.8) Lemma. Let \( \eta \) be an \( N \)-function and \( w \) a weight. For a fixed cube \( Q \) set \( h(t) = w(E(t)) \) with \( E(t) = \{ x \in Q : w(x) > t \} \). Then

\[
\int_{E(t)} \frac{\eta(w(x))}{s} dx = -\int_0^\infty \frac{\eta(s)}{s} dh(s) \quad \text{for any } t \geq 1.
\]

Proof. Since \( \eta \) is a convex function it is absolutely continuous on any bounded interval and therefore \( \eta(s)/s \) is absolutely continuous on each bounded interval which is away from zero. Then for \( t \geq 1 \) we may write

\[
\int_{E(t)} \frac{\eta(w(x))}{s} dx = \frac{\eta(t)}{t} h(t) + \int_{E(t)} w(x) \left( \int_t^w \frac{d\eta(s)}{s} ds \right) dx.
\]

Since \( w(x) \), being locally integrable, is finite almost everywhere, changing the order of integration and integrating by parts we get
\[ \int_{E(t)} \eta(w(x)) \, dx = \frac{\eta(t)}{t} h(t) + \int_{t}^{\infty} \frac{d}{ds} \left( \frac{\eta(s)}{s} \right) \left( \int_{E(s)} w(x) \, dx \right) ds \]
\[ = \frac{\eta(t)}{t} h(t) + \int_{t}^{\infty} \frac{d}{ds} \left( \frac{\eta(s)}{s} \right) h(s) \, ds \]
\[ = \lim_{s \to \infty} \frac{\eta(s)}{s} h(s) - \int_{t}^{\infty} \frac{\eta(s)}{s} \, dh(s). \]

The proof will be complete if we are able to show that under our assumptions the above limit at infinity is 0.

First suppose that the left hand side of (3.9) is finite and observe that because of the convexity of \( \eta \), \( \eta(s)/s \) is a non-decreasing function. Then for \( s \geq t \),
\[ h(s) = \int_{\{x \in Q : w(x) > s\}} w(x) \, dx \leq \frac{s}{\eta(e)} \int_{E(s)} w(x) \eta(w(x)) \frac{1}{w(x)} \, dx \]
\[ = \frac{s}{\eta(s)} \int_{E(s)} \eta(w(x)) \, dx \]
and the last integral goes to zero as \( s \to \infty \) since \( \eta(w(x)) \) is integrable on \( E(t) \) and \( |E(s)| \to 0 \) as \( s \to \infty \).

Finally, if the right hand side of (3.9) is finite we have
\[ \lim_{s \to \infty} - \int_{b}^{\infty} \frac{\eta(s)}{s} \, dh(s) = 0. \]

Using again the fact that \( \eta(s)/s \) is non-decreasing gives
\[ - \int_{b}^{\infty} \frac{\eta(s)}{s} \, dh(s) \geq \frac{\eta(b)}{b} \left( - \int_{b}^{\infty} dh(s) \right) = \frac{\eta(b)}{b} (h(b) - \lim_{s \to \infty} h(s)). \]
But \( h(s) = \int_{E(s)} w(x) \, dx \to 0 \) as \( s \to \infty \), since \( w \) is locally integrable. \( \Box \)

With these lemmas we can now prove the key property of the classes \( RH_{\phi} \) mentioned above.

(3.10) PROPOSITION. Let \( \phi \) be an N-function satisfying the \( \Delta_2 \) and \( \Delta_2^\infty \) conditions. Let \( w \) be a weight belonging to \( RH_{\phi}. \) Then there exists \( r > 1 \) such that \( w \) belongs to \( RH_{\phi^r}. \)

Proof. Let \( Q \) be a cube and assume that
\[ N_Q (w) = \sup_{e} \| \chi_Q w/\varepsilon \|_{L^\infty (dx/|Q|^{\phi(1/e)})} = 1. \]

Therefore
\[ \frac{1}{|Q|} \int_Q \varphi \left( \frac{w(x)}{\varepsilon} \right) \, dx \leq \phi(1/\varepsilon). \]

Given \( t > 1 \) we set \( s = 2Ct \) with \( C \) the constant appearing in the reverse-Hölder-\( \varphi \) inequality. By our assumptions \( N_Q(w) < s \), and hence for any fixed \( \varepsilon > 0, \)
\[ \frac{1}{|Q|} \int_Q \varphi \left( \frac{w(x)}{\varepsilon s} \right) \, dx \leq \phi(1/\varepsilon). \]

So we may apply the Calderón-Zygmund decomposition to the function \( \varphi(w(x)/(\varepsilon s)) \) on the cube \( Q \) with \( \lambda = \phi(1/\varepsilon) \) to obtain a family \( \{Q_j\} \) of disjoint cubes satisfying
\[ \phi(1/\varepsilon) \leq \frac{1}{|Q_j|} \int_{Q_j} \varphi \left( \frac{w(x)}{\varepsilon s} \right) \, dx \leq 2^n \phi(1/\varepsilon) \]
and
\[ \phi \left( \frac{w(x)}{\varepsilon s} \right) \leq \phi(1/\varepsilon) \quad \text{for almost any } x \in Q - \bigcup_j Q_j. \]

Let \( E(s) = \{x \in Q : w(x) > s\} \). The last assertion implies that up to a set of measure zero, \( E(s) \subset \bigcup Q_j = G. \) Then from (3.11) we have
\[ \int_{E(s)} \varphi \left( \frac{w(x)}{\varepsilon s} \right) \, dx \leq \sum_j \int_{Q_j} \varphi \left( \frac{w(x)}{\varepsilon s} \right) \, dx \leq 2^n \phi(1/\varepsilon)|G|. \]

In order to estimate \( |G| \) we observe that the first inequality in (3.11) implies
\[ \| \chi_{Q_j} w/\varepsilon \|_{L^\infty (dx/|Q|^{\phi(1/e)})} > s. \]

Since \( w \in RH_{\phi} \), we get
\[ \frac{C}{|Q_j|} \int_{Q_j} w(x) \, dx > s \]
and then
\[ s|G| = s \sum_j |Q_j| \leq C \sum_j \int_{Q_j} w(x) \, dx \]
\[ \leq C \sum_j \int_{Q_j \cap E(t)} w(x) \, dx + Ct \sum_j |Q_j| \]
\[ \leq C \int_{E(t)} w(x) \, dx + Ct|G|. \]

So, we get
\[ |G| \leq \frac{1}{t} \int_{E(t)} w(x) \, dx. \]
Inserting this estimate in (3.13), we obtain

\[ \int_{E(t)} \phi\left(\frac{w(x)}{es}\right) \frac{dx}{\phi(1/\epsilon)} \leq \frac{C}{t} \int_{E(1)} w(x) dx, \]

to conclude that

\[ \int_{E(t)} \phi\left(\frac{w(x)}{es}\right) \frac{dx}{\phi(1/\epsilon)} \leq \frac{C}{t} \int_{E(t)} w(x) dx + \int_{E(t) - E(s)} \phi\left(\frac{w(x)}{es}\right) \frac{dx}{\phi(1/\epsilon)}. \]

Since \( \phi \) is increasing, the last term can be bounded by \( |E(t)| \), which in turn, by the Chebyshev inequality, is bounded by the first term of the sum above.

Since \( s = 2Ct \) and \( \phi \) is of finite upper type we have proved that

\[ \int_{E(t)} \phi\left(\frac{w(x)}{es}\right) \frac{dx}{\phi(1/\epsilon)} \leq \frac{C \phi(t/\epsilon)}{t} \int_{E(1)} w(x) dx. \]

Setting \( \sigma = ct \) we may apply Lemma (3.8) to the left hand side with \( \eta(s) = \phi(s/\sigma) \) and the weight \( w \), since the integrability on \( Q \) of \( w \) implies the finiteness of both integrals. Then we have

\[ \int_{E(1)} \frac{\phi(s/\sigma)}{s} dh(s) \leq C \frac{\phi(t/\sigma)}{t} h(t) \]

for any \( t \geq 1 \) and \( \sigma > 0 \), where \( h(t) \) is defined by \( w(E(t)) = w(\{ x \in Q : w(x) > t \}) \).

Now we are in a position to apply Lemma (3.6) to the functions \( \eta(s) = \phi(s/\sigma) \) and \( h(t) \). Since all these functions \( \eta \) have the same types of \( \phi \), with the same constants, there are \( r > 1 \) and a constant \( C \) such that

\[ \int_{1}^{\infty} \frac{\phi(s/\sigma)}{s} dh(s) \leq C \phi(1/\sigma) \left( \int_{1}^{\infty} \frac{\phi(s/\sigma)}{s} dh(s) \right). \]

But the integral on the right is the same as \( \int_{E(1)} \phi(w(x)/\sigma) dx \), which is finite because of the local integrability of \( \phi(w(x)/\sigma) \) implied by the reverse Hölder-\( \phi \) condition on \( w \). Thus, the left hand side of the above inequality is also finite, and we may also apply Lemma (3.8) to get for any \( \gamma > 0 \),

\[ \int_{E(1)} \phi(w(x)/\sigma)^{\gamma} dx \leq C \phi(1/\sigma) \left( \int_{E(1)} \phi(w(x)/\sigma) dx \right). \]

On the other hand, for \( x \in Q - E(1) \) we have

\[ \phi(w(x)/\sigma)^{\gamma} = \phi(w(x)/\epsilon)^{\gamma - 1} \phi(w(x)/\sigma) \leq \phi(1/\epsilon)^{\gamma - 1} \phi(w(x)/\sigma). \]

Integrating over \( Q - E(1) \) and combining with the estimate over \( E(1) \) we get

\[ \int_{Q} \phi(w(x)/\sigma)^{\gamma} dx \leq C \phi(1/\sigma)^{\gamma - 1} \int_{Q} \phi(w(x)/\sigma) dx. \]

Now, since we assume \( N_{Q}(w) = 1 \), we have

\[ \int_{Q} \phi(w(x)/\sigma)^{\gamma} dx \leq C \int_{Q} \phi(w(x)/\sigma) dx \frac{dx}{|Q| \phi(1/\sigma)} \leq C \]

and therefore for any \( \sigma > 0 \),

\[ \| x \phi(w(x)/\sigma \|_{L^{\infty}(dx/|Q| \phi(1/\sigma))} \leq C \]

and the reverse Hölder-\( \phi \) inequality for the cube \( Q \) implies

\[ \sup_{\sigma > 0} \| x \phi(w(x)/\sigma |_{L^{\infty}(dx/|Q| \phi(1/\sigma))} \| \leq C \frac{1}{|Q|} \int_{Q} w(x) dx. \]

Finally, if \( N_{Q}(w) \neq 1 \) we take \( W(x) = w(x)/N_{Q}(w) \). Since \( W \) satisfies the reverse Hölder-\( \phi \) condition with the same constant and \( N_{Q}(W) = 1 \) we may apply the last inequality to \( W \), which gives the same result for \( w \) with a constant independent of the cube \( Q \). This finishes the proof of the theorem.
$w \in RH_{q-\epsilon}$, we take a cube $Q$ and consider the disjoint sets

$$E_k = \{ x \in Q : 2^k \leq \frac{w(x)}{Cm_Q(w)} < 2^{k+1} \}, \quad k \geq 0.$$ 

For each $k$ we use (4.2) for $t_k = \phi(s_k)/Q$. This together with (4.1) gives

$$1 \geq \frac{1}{|E_k|} \int_{E_k} \phi\left(\frac{w(x)\phi^{-1}(t_k/Q)}{Cm_Q(w)}\right) \frac{dx}{t_k} \geq \frac{1}{|E_k|} \int_{E_k} \phi(2^{k+1}s_k) \frac{dx}{t_k} \geq \frac{1}{2} \frac{2^{2k} \phi(s_k)}{t_k} \geq C \frac{2^{2k} \phi(s_k)}{|Q|^{1/q-\epsilon}} \int_{E_k} w(x)^{q-\epsilon} \frac{dx}{t_k}.$$ 

So we get

$$\frac{1}{|Q|^\epsilon} \int_{E_k} w(x)^{q-\epsilon} \frac{dx}{t_k} \leq C \frac{m_Q(\phi(bw))}{|Q|^\epsilon} \frac{1}{|Q|^\epsilon} \int_{E_k} w(x)^{q-\epsilon} \frac{dx}{t_k}.$$ 

But if we set $E = \{ x \in Q : w(x) \leq Cm_Q(w) \}$ we also have for small $\epsilon > 0$,

$$\frac{1}{|Q|^\epsilon} \int_E w(x)^{q-\epsilon} \frac{dx}{t_k} \leq C \frac{m_Q(\phi(bw))}{|Q|^\epsilon}.$$ 

Adding up these inequalities and raising to the $1/(q-\epsilon)$ power we get

$$\left(\frac{1}{|Q|^\epsilon} \int_E w(x)^{q-\epsilon} \frac{dx}{t_k}\right)^{1/(q-\epsilon)} \leq C \frac{1}{|Q|^\epsilon} \int E w(x)^{q-\epsilon} \frac{dx}{t_k},$$

so $w \in RH_{q-\epsilon}$.

That (c) is equivalent to (d) is an already known result. See for example [S-T!]. So we turn to the proof of (c)$\Rightarrow$(b). For $Q$ a cube and $S$ a measurable subset of $Q$ we want to prove that there exist $\delta > 0$ and a constant $C$ independent of $b$, $Q$ and $S$ such that

$$\frac{\phi(bw)(S)}{\phi(bw)(Q)} \leq C \left(\frac{|S|}{|Q|}\right)^\delta.$$ 

Let $S_1 = \{ x \in S : \phi(bw(x)) \leq \phi(bw(Q)/|Q|) \}$ and $S_2 = S - S_1$. Then we have the estimates

$$I_1 = \int_{S_1} \phi(bw(x)) \frac{dx}{t_1} \leq \frac{|S|}{|Q|} \int \phi(bw(x)) \frac{dx}{t_1} \leq \left(\frac{|S|}{|Q|}\right)^\delta \phi(bw)(Q)$$

for any $\delta \leq 1$. Also since $\phi$ is of upper type $r = q + \epsilon$ for $\epsilon > 0$, we have, for all $t_1 \geq t_2$,

$$\phi(t_1) \leq C_r (t_1/t_2)^r \phi(t_2).$$

Since for $x \in S_2$, $bw(x) > \phi^{-1}(m_Q(\phi(bw)))$, we may apply the latter inequality to get

$$I_2 = \int_{S_2} \phi(bw(x)) \frac{dx}{t_1} \leq \frac{m_Q(\phi(bw))}{\phi^{-1}(m_Q(\phi(bw)))} \frac{1}{|Q|^\epsilon} \int bw(x)^{q-\epsilon} \frac{dx}{t_1}.$$ 

Now since $w^\epsilon \in A_{\infty}$ it is also true that $w^\epsilon \in A_{\infty}$ for some $r > q$; for such an $r$, the last integral can be bounded by

$$C_r \left(\frac{|S|}{|Q|}\right)^\delta \int bw(x)^r \frac{dx}{t_1} \leq C_r |Q|^\epsilon \left(\frac{|S|}{|Q|}\right)^\delta \frac{1}{|Q|^\epsilon} \int bw(x)^r \frac{dx}{t_1}.$$ 

for some $\delta > 0$, where in the last inequality we made use of the fact that $w^\epsilon \in A_{\infty}$ implies $w \in RH_r$.

Next we observe that the convexity of $\phi$ gives, for any locally integrable function $g$,

$$\phi m_Q(g) \leq m_Q(\phi(g)),$$

which for $g = bw$ leads to

$$m_Q(bw) \leq \phi^{-1}(m_Q(\phi(bw)))$$

Inserting these estimates in (4.4) we get

$$I_2 \leq C_r \left(\frac{|S|}{|Q|}\right)^\delta \left(\int \phi(bw(x)) \frac{dx}{t_1}\right)^r.$$ 

Combining the $I_1$ and $I_2$ estimates we get (4.3) for a constant $C$ independent of $b$.

Finally, we prove that (b)$\Rightarrow$(a). We begin by observing that, by an appropriate choice of the parameter $\beta$, we may assume $m_Q(w) = 1$. Our goal is to show that for some choice of $C_0$,

$$\int_Q \phi \left(\frac{w(x)\phi^{-1}(t/Q)}{C_0}\right) \frac{dx}{t} \leq 1$$

for every $t > 0$. We use the following condition equivalent to $v \in A_{\infty}$: there exist $\alpha, \beta > 0$ such that

$$\int \frac{dx}{\phi(bw(x))} \frac{dx}{t_1} \leq \frac{C}{|Q|^\epsilon} \int \phi(bw(x)) \frac{dx}{t_1} \leq C \frac{1}{|Q|^\epsilon} \int \phi(bw(x)) \frac{dx}{t_1} \leq C \frac{1}{|Q|^\epsilon} \int \phi(bw(x)) \frac{dx}{t_1} \leq C.$$ 

where $C$ can be taken as $2C_0/\alpha^r$, with $r$ the upper type of $\phi$ and $C_r$ the $r$-type constant.
Otherwise, we would have for some $b$ the opposite inequality, which together with (4.6), applied to $\phi(bw)$, and the Chebyshev inequality would lead to

$$
\alpha(Q) < \left\{ x \in Q : \phi(bw(x)) > \frac{b}{|Q|} \int_Q \phi(bw(x)) dx \right\} \\
\leq \left\{ x \in Q : \phi(bw(x)) > C\phi(b) \right\} = \left\{ x \in Q : w(x) > \frac{1}{\phi^{-1}(C\phi(b))} \right\} \\
\leq \frac{b}{\phi^{-1}(C\phi(b))} \int_Q w(x) dx = \frac{b|Q|}{\phi^{-1}(C\phi(b))}
$$

and hence we would have $\phi^{-1}(C\phi(b)) < b/\alpha$ or, in other words, $C\phi(b) < \phi(b/\alpha)$. Since that $\phi$ is of upper type $r$, this would imply $C < C_r/\alpha^r$, which is a contradiction.

Therefore, the claim is true and setting $b = \phi^{-1}(t/|Q|)$ we have

$$
\int_Q \phi(w(x)) \phi^{-1}(t/|Q|) \frac{dx}{t} \leq \frac{C}{\beta}.
$$

Since $C/\beta > 1$ the fact that $\phi$ has positive lower type allows us to replace the constant outside $\phi$ by a perhaps different constant inside $\phi$ leading then to the inequality (4.5).

**Proof of Theorem II.** (a) That the second inequality is true follows from the fact that $w \in A_{\phi}$ if and only if there exist $\alpha, \beta > 0$ such that for any cube $Q$ and any measurable subset $E \subset Q$,

$$
|E|/|Q| \leq \alpha \Rightarrow w(E)/w(Q) \leq \beta
$$

(see for example [C-F]). But, upon taking $Q - E$ in place of $E$ the last assertion is equivalent to the existence of $\gamma$ and $\delta$ such that

$$
w(E)/w(Q) \leq \gamma \Rightarrow |E|/|Q| \leq \delta.
$$

Now if $w \in A_{\phi}$, then $dm = w(x)dx$ is a doubling measure and therefore $A_{\phi}(dm)$ also coincides with the weights $\nu$ for which there exist $\gamma$ and $\delta$ such that

$$
\mu(E)/\mu(Q) \leq \gamma \Rightarrow \int_E \nu dm/\int_Q \nu dm \leq \delta
$$

(see [C-F]); but (4.8) gives exactly this statement for $\nu = 1/w$. Conversely, if a weight belongs to $A_{\phi}(w)$, it satisfies the weaker condition (4.9), therefore $1/w \in A_{\phi}(w)$ implies (4.8) which is equivalent to $w \in A_{\phi}$.

For the first equality, using the fact that $A_{\phi} = \bigcup_{\phi > 1} RH_{\phi}$, we only need to show that $RH_{\phi} \subset A_{\phi}$ for any $\phi \in N$. In fact we will check that if $w \in RH_{\phi}$, then (4.7) holds for $w$.

Let $Q$ be a cube and $E$ a measurable subset. Then the Hölder inequality for $\phi^{-1}(s) / t$ gives

$$
\omega(E) = \int_E \omega(x) dx \leq \|\phi^{-1}(s) \omega\| L_{\phi} \|\phi^{-1}(t) \omega\| L_{\phi}.
$$

Now since $\|\phi^{-1}(s) \omega\| L_{\phi} \leq 1/\phi^{-1}(1/|E|)$, using the fact that $s \simeq \phi^{-1}(s) \phi^{-1}(s)$, we easily get

$$
\|\phi^{-1}(s) \omega\| L_{\phi} \leq C|E| \phi^{-1}(t/|E|).
$$

From this estimate and since $w \in RH_{\phi}$, we obtain

$$
\omega(E) \leq C|E| \frac{w(Q)}{|Q|} \frac{\phi^{-1}(t/|Q|)}{\phi^{-1}(t/|Q|)},
$$

which for $t = |Q|$ can be written as

$$
\omega(E) \leq \frac{w(Q)}{|Q|} \frac{\phi^{-1}(t/|Q|)}{\phi^{-1}(t/|Q|)}.
$$

Therefore, in order to get (4.7), we only need to show that if $|E|/|Q|$ is small enough then $(|E|/|Q|) \phi^{-1}(t/|Q|) / |E|$ is also small. In other words, we must show that the function $\phi^{-1}(1/s)$ goes to zero as $s \to 0$. After setting $1/s = \phi(\sigma)$ this is equivalent to

$$
\lim_{\sigma \to 0} \frac{\sigma}{\phi(\sigma)} = 0 \quad \text{or} \quad \lim_{\sigma \to \infty} \frac{\phi(\sigma)}{\phi(\sigma)} = \infty,
$$

which is true for any $\phi \in N$.

(2) The easy part is to show that $RH_{\phi} \subset \bigcap_{\phi \in N} RH_{\phi}$. In fact, $w \in RH_{\phi}$ is equivalent to saying that for any cube $Q$ we have

$$
sup_{x \in Q} w(x) \leq C \frac{w(Q)}{|Q|}.
$$

Therefore for $\phi \in N$ and $C$ as above, we have

$$
\int_Q \frac{\phi^{-1}(t/|Q|)}{C \omega(Q)} \frac{dx}{t} \leq \int_Q \phi^{-1}(t/|Q|) \frac{dx}{t} = 1,
$$

so $w \in RH_{\phi}$.

Let us see now that for $\phi \in N$, $RH_{\phi} \subset RH_{\phi}$.

We are going to show that if $w \notin RH_{\phi}$, that is, $1/w \notin A_{\phi}$, then it is possible to construct $\psi \in N$ such that $w \notin RH_{\phi}$. In fact, if $1/w \notin A_{\phi}$, then for each $k \in N$, there exists a cube $Q_k$ such that

$$
sup_{Q_k} w(x) \geq \frac{2k^2}{2k^2} w(Q_k)/|Q_k|.
$$

Consider now the sets

$$
E_k = \{ x \in Q_k : w(x) \geq k^2 w(Q_k)/|Q_k| \}.
$$
Setting \( t_k = w(E_{n_k})/w(Q_{n_k}) \), we have \( 0 < t_k < 1 \) and also the sequence 
\( a_k = \max_{1 \leq i \leq k} i/t_i \) is non-decreasing and, since \( a_k \geq k \), it goes to infinity. Then we may choose an increasing subsequence \( a_{k_j} \) and define a continuous increasing function \( g \) such that 
\[
g(0) = 0, \quad g(k_j) = a_{k_j},
\]
and it is linear in-between. We claim that the function \( \psi \) such that \( \psi(0) = 0 \) and \( \psi' = g \) gives the desired conclusion.

Clearly, by construction, \( \psi \) is a non-negative, increasing and convex function on \( [0, \infty) \), and being quadratic near zero, it also satisfies \( \psi(t)/t \to 0 \) as \( t \to 0 \). Also by the convexity of \( \psi \), we have
\[
\frac{1}{2} \psi \left( \frac{t}{2} \right) \leq \frac{\psi(t)}{t} \leq \psi'(t).
\]
Since both bounds have limit infinity for \( t \) tending to infinity, we conclude that \( \psi \in \mathcal{N} \).

On the other hand, for the cubes \( Q_{k_j} \) we have
\[
\frac{1}{w(Q_{k_j})} \int_{Q_{k_j}} \psi \left( \left| Q_{k_j} \right| w(x) \right) \frac{w(x)}{w(Q_{k_j})} \, dx \geq \frac{1}{w(E_{k_j})} \int_{E_{k_j}} \psi(k_j) w(x) \, dx
\]
\[
= \frac{w(E_{k_j})}{w(Q_{k_j})} a_{k_j} \geq 1.
\]
Together with (4.10) this gives
\[
\frac{k_j}{|Q_{k_j}|} \int_{Q_{k_j}} \psi \left( \frac{2|Q_{k_j}| w(x)}{k_j w(Q_{k_j})} \right) \, dx \geq 1.
\]
Setting \( t_j = |Q_{k_j}|/k_j \) we can rewrite the above inequality as
\[
\frac{1}{t_j} \int_{Q_{k_j}} \psi \left( \frac{\psi^{-1}(t_j/|Q_{k_j}|) w(x)}{w(Q_{k_j})} \right) \, dx \geq 1.
\]
Now set \( C_j = (k_j/2) \psi^{-1}(1/k_j) \). We claim \( C_j \to \infty \), which will contradict the fact that \( w \in RH_q \). Since \( k_j \to \infty \) as \( j \to \infty \), it is enough to check that 
\[
\psi^{-1}(s)/s \to \infty \quad \text{as} \quad s \to 0.
\]
But
\[
\lim_{s \to 0} \psi^{-1}(s)/s = \lim_{t \to 0} \frac{t}{\psi(t)} = \infty,
\]
by hypothesis.

Finally, that \( RH_{q, \infty} \) is a proper subset of \( \cap_{q=1} \cap_{q=1} RH_q \) follows as in [CU-N] by taking \( w(x) = \max(\log(1/|x|), 1) \).