STUDIA MATHEMATICA 130 (3) (1998)

# Averages of holomorphic mappings and holomorphic retractions on convex hyperbolic domains

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**Abstract.** Let $D$ be a hyperbolic convex domain in a complex Banach space. Let the mapping $F \in \mathrm{Hol}(D,D)$ be bounded on each subset strictly inside $D$, and have a nonempty fixed point set $\mathcal{F}$ in $D$. We consider several methods for constructing retractions onto $\mathcal{F}$ under local assumptions of ergodic type. Furthermore, we study the asymptotic behavior of the Cesàro averages of one-parameter semigroups generated by holomorphic mappings.

Let $X$ be a complex Banach space and let $D$ be a domain (open connected subset) in $X$. We denote by $\mathrm{Hol}(D,\widetilde{D})$ the family of all holomorphic mappings from $D$ into the set $\widetilde{D} \subset X$. Thus the set $\mathrm{Hol}(D,D)$ consists of all the holomorphic mappings $F$ on $D$ such that $F(D) \subset D$. This set is a semigroup with respect to composition.

A mapping $\varphi \in \mathrm{Hol}\,(D,D)$ is called a *holomorphic retraction* of $D$ if it is an idempotent of the semigroup $\mathrm{Hol}(D,D)$, i.e., $\varphi^2 = \varphi$. In other words, if we let $V = \varphi(D)$, then $\varphi|_V = I|_V$. In this case the set $V \subset D$ is called a *holomorphic retract* of $D$.

Let $F \in \mathrm{Hol}(D,D)$. We denote by $\mathcal{F} = \mathrm{Fix}(F)$ the *fixed point set* of $F$ in $D$, i.e., $\mathcal{F} = \{x \in D : x = F(x)\}$.

One can ask several questions about $\mathrm{Fix}(F)$. The first one is: *if* $\mathcal{F} = \mathrm{Fix}(F) \neq \emptyset$, *when is it a holomorphic retract of* $D$? This question in its turn leads to the second one: *if* $\mathcal{F} = \mathrm{Fix}(F)$ *is a holomorphic retract of* $D$, *how can one determine a retraction of* $D$ *onto* $\mathcal{F}$*?*

Note that once the existence of a holomorphic retraction onto $\mathcal{F}$ is established, it follows that $\mathcal{F} = \mathrm{Fix}(F)$ is a complex-analytic submanifold of $D$ [4] (see also a result of Rossi in [8, p. 102]). This explains the importance of the first question mentioned above. The structure of fixed point sets and holomorphic retracts has been studied by many mathematicians in different situations, e.g., in the finite-dimensional case (see, for example, [12, 37, 24, 21, 26, 5, 41, 42, 22, 43, 27], in Hilbert space, its powers and the hyperball

1991 *Mathematics Subject Classification*: 32A10, 46G20, 47H10, 47H20.

(see, for example, [35, 32, 40, 10, 2, 3]), and in general Banach spaces (see, for example, [38, 39, 13, 28, 29, 6, 34, 18, 15, 16]).

The second question mentioned above is also of great interest. This is because the construction of a retraction onto $\mathcal{F} = \mathrm{Fix}(F)$ is equivalent to finding a method (explicit, implicit or approximate) for solving the equation $x = F(x)$. A holomorphic retraction may not exist in general (see [28]). Even if it exists, it need not be the limit of the iterates of the operator, even in the case of a linear operator $A$ such that $\{0\} \neq \mathrm{Ker}(I-A) \neq \mathrm{Ker}(I-A)^2$.

We will now mention several results concerning both these questions. To this end we need the following definitions. The first one is motivated, inter alia, by the above-mentioned linear example.

DEFINITION 1. Let $F \in \mathrm{Hol}(D,D)$. A point $a \in \mathrm{Fix}(F)$ is said to be *quasi-regular* if the following condition holds:

$$(*) \qquad \mathrm{Ker}(I - F'(a)) \oplus \mathrm{Im}(I - F'(a)) = X.$$

If, in addition, $\mathrm{Ker}(I - F'(a)) = \{0\}$, i.e., the linear operator $I - F'(a)$ is invertible, then we say that $a$ is a *regular fixed point* of $F$.

By the implicit function theorem (see, for example, [20]), it is clear that a regular fixed point is an isolated point of the set $\mathcal{F}$, and that in the case of a finite-dimensional $X$, each fixed point is quasi-regular (or in particular, regular).

DEFINITION 2. Let $D$ be a domain in a complex Banach space $X$. A net $\{F_j\}_{j\in\mathcal{A}} \subset \mathrm{Hol}(D,X)$ is said to converge to a mapping $F \in \mathrm{Hol}(D,X)$ in the topology of local uniform convergence over $D$ (or briefly *$T$-converge*) if for every ball $B \Subset D$,

$$\lim_{j\in\mathcal{A}} \sup_{x\in B} \|F_j(x) - F(x)\| = 0.$$

In this case we write

$$F = \operatorname*{T-lim}_{j\in\mathcal{A}} F_j.$$

If $D$ is a bounded domain in a finite-dimensional $X$, then its T-topology is equivalent to the compact open topology on $D$, i.e., the topology of uniform convergence on compact subsets of $D$.

DEFINITION 3. A mapping $F \in \mathrm{Hol}(D,D)$ is said to be *power convergent* on $D$ (with respect to the T-topology) if the sequence of iterates $\{F^n : F^0 = I,\ F^{n+1} = F \circ F^n,\ n = 0,1,2,\ldots\}$ T-converges to a mapping $\varphi \in \mathrm{Hol}(D,D)$.

It is obvious that if $F \in \mathrm{Hol}(D,D)$ is power convergent, then $\mathcal{F} = \mathrm{Fix}(F) \neq \emptyset$ and $\varphi = \text{T-lim}_{n\to\infty} F^n$ is a retraction onto $\mathcal{F}$.



Moreover, we will see below that if the domain $D$ is bounded, then the set $\mathcal{F}$ consists of only quasi-regular (or regular) points.

The converse, of course, is not true. To see this, consider, for example, a rotation $F(x) = e^{i\theta}x$, $0 < \theta < 2\pi$, of the open unit disk $\Delta$ in $\mathbb{C}$. This mapping has a unique and regular fixed point (the origin) in $\Delta$, but $F$ is not power convergent.

A criterion for $F \in \mathrm{Hol}(D,D)$ to be power convergent was given by E. Vesentini [38], [39]. The linear case is due to J. J. Koliha [19].

Let $\sigma(A)$ denote the spectrum of the linear operator $A : X \to X$ and let $\Delta$ be the open unit disk in $\mathbb{C}$. As above, assume that $\mathcal{F} = \mathrm{Fix}(F) \neq \emptyset$ for some $F \in \mathrm{Hol}(D,D)$ and that $a \in D$. Setting $A = F'(a)$ and using the Cauchy integral formula [9], the chain rule, and the boundedness of $D$, we see that the powers of $A$ are uniformly bounded, i.e.,

$$(1) \qquad \|A^n\| \le M < \infty.$$

This implies that $\sigma(A) \subset \bar{\Delta}$. Now in our context Vesentini's theorem takes the following form.

THEOREM A. *Let $D$ be a bounded convex domain in $X$ and let $F \in \mathrm{Hol}(D,D)$ with $\mathcal{F} = \mathrm{Fix}(F) \neq \emptyset$. Then $F$ is power convergent if and only if there exists $a \in \mathcal{F}$ such that for the linear operator $A = F'(a)$ the following conditions hold*:

$$(2) \qquad \sigma(A) \subset \Delta \cup \{1\}$$

*and*

(**) 1 *is a pole of at most the first order for the resolvent of $A$.*

REMARK 1. In this form the theorem is also true for bounded domains which are not necessarily convex, but satisfy the following maximum modulus principle: For each $f \in \mathrm{Hol}(D, \overline{D})$ such that $f(D) \cap \partial D \neq \emptyset$, we have $f(D) \subset \partial D$.

However, we will mainly consider convex domains in $X$ because this is all that is needed in order to describe $\mathrm{Fix}(F)$ locally in each bounded $D$. Indeed, it has recently been shown by P. Mazet [27] that if $D$ is bounded, then for each $F \in \mathrm{Hol}(D,D)$ and each $a \in \mathrm{Fix}(F)$, there is a convex neighborhood $U \subset D$ such that $a \in U$ and $F(U) \subset U$, i.e., $F \in \mathrm{Hol}(U,U)$.

In addition, we need the convexity to consider different types of mean ergodic procedures.

REMARK 2. It is clear that if, in the setting of Theorem A, $\sigma(A) \subset \Delta$, then $a \in \mathcal{F}$ is a regular point.

In addition, it can be shown that such a point is the unique fixed point of $F$ in $D$ (see [17]). As a matter of fact, as we will see in the sequel, the regularity of a point $a \in \text{Fix}(F)$ is already sufficient for its uniqueness, i.e., if $a$ is regular, then $\text{Fix}(F) = \{a\}$.

As a matter of fact, if the conditions of Theorem A hold for at least one point of $\mathcal{F}$, then they hold for all the points of $\mathcal{F}$. Moreover, if $1 \in \sigma(A)$, then $\mathcal{F}$ contains infinitely many points because it is a retract of $D$, hence a connected submanifold of $D$, tangent to $\text{Ker}(I - A)$. We will see below that the latter fact is true whenever condition (**) holds, even if the spectrum $\sigma(A)$ contains other points on the boundary $\partial\Delta$ of the open unit disk $\Delta$ different from 1. But in this case, of course, by Vesentini's theorem $F$ is not power convergent and therefore the question of approximating its fixed points is still open.

Nevertheless, if $D$ is a bounded convex domain in $X$, and $F \in \text{Hol}(D, D)$ has at least one quasi-regular fixed point in $D$, then there is another mapping $\varphi \in \text{Hol}(D, D)$ with $\text{Fix}(F) = \text{Fix}(\varphi)$ which is power convergent. Hence, in this case $\mathcal{F}$ is a holomorphic retract of $D$. For the finite-dimensional case this was established by J.-P. Vigué [41, 42, 43], and in the general case by P. Mazet and J.-P. Vigué [28]. They used nonlinear analogues of mean ergodic constructions. More precisely, they considered the Cesàro averages

$$C_n = \frac{1}{n}\sum_{k=0}^{n-1} F^k \tag{3}$$

and proved the following results.

THEOREM B. *Let $D$ be a bounded convex domain in $X = \mathbb{C}^n$, and let $F \in \text{Hol}(D, D)$ with $\mathcal{F} = \text{Fix}(F) \neq \emptyset$. Then there is a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ which T-converges to a power convergent holomorphic mapping $\varphi : D \to D$ with $\text{Fix}(\varphi) = \mathcal{F}$.*

THEOREM C. *Let $D$ be a bounded convex domain in an arbitrary complex Banach space $X$, and let $F \in \text{Hol}(D, D)$ with $\mathcal{F} = \text{Fix}(F) \neq \emptyset$. If $\mathcal{F}$ contains at least one quasi-regular point, then there exists an integer $p$ such that the mapping $C_p$ defined by (3) is power convergent to a holomorphic retraction onto $\mathcal{F}$.*

REMARK 3. Both Theorems B and C state that $\mathcal{F} = \text{Fix}(F)$ is a holomorphic retract of $D$. But a deficiency of Theorem B is that we do not know how to choose a convergent subsequence of $\{C_n\}$ in order to approximate fixed points.

Theorem C (which also covers the finite-dimensional case) would improve this situation, if we could determine the minimal number $p$ for which $C_p$ defined by (3) is power convergent.



We give below, inter alia, the simplest possible answer to this question, namely, $p = 2$.

The following partial generalization of Theorem C to unbounded domains was obtained by Do Duc Thai [6]. We first recall the following definition.

DEFINITION 4. A domain $D$ in a complex Banach space $X$ is said to be *hyperbolic* if the Kobayashi pseudo-distance $K_D$ generates the relative topology of $D$ in $X$ (see, for example, [9]).

THEOREM D. *Let $D$ be a convex hyperbolic domain in $X$, and let $F \in \text{Hol}(D, D)$ be such that $F(D)$ is contained in some compact convex subset of $X$. If $\mathcal{F} = \text{Fix}(F) \neq \emptyset$, then it is a holomorphic retract of $D$.*

Theorem D is only a partial generalization of Theorem C because of its compactness hypothesis. Indeed, according to a theorem of Krasnosel'skiĭ's [20], $F'(x)$ is compact for all $x \in D$ and hence each point in $\text{Fix}(F)$ is quasi-regular. In addition, Theorem D does not provide a construction of a holomorphic retraction onto $\text{Fix}(F)$. However, we will show below that this compactness hypothesis is unnecessary. Moreover, we will present a simple method for constructing such a retraction.

An additional question which arises in connection with Theorem C is the following: *What happens when 1 is the unique point of the spectrum $\sigma(F'(a))$ on the boundary of the open unit disk $\Delta$ in $\mathbb{C}$?*

The key to the answers to both questions is combining Theorem A with the following old result due to I. Gohberg and A. Markus [11] (see also [36] and [23]).

THEOREM E. *Let $A$ be a closed linear operator in $X$. A point $\lambda_0 \in \mathbb{C}$ is a pole of the resolvent $(\lambda I - A)^{-1}$ of order at most $\nu$ if and only if the following decomposition holds:*

$$\text{Ker}\,(\lambda_0 I - A)^\nu \oplus E = X,$$

*where $E \subset X$ is an invariant subspace of $A$ such that the restriction of $\lambda_0 I - A$ to $E$ is continuously invertible.*

Since conditions (*) and (**) are equivalent, Theorem A can be used to show that one can take $p = 2$ in Theorem C. More precisely, using Theorems A and E, we will now present a simple method for finding a retraction onto $\text{Fix}(F)$. This will also provide a proof of Theorem C.

THEOREM 1. *Let $D$ be a hyperbolic convex domain in $X$, and let $F \in \text{Hol}(D, D)$ be bounded on each subset which is strictly inside $D$.*

(i) *If $\mathcal{F} = \text{Fix}(F)$ contains at least one quasi-regular point $a \in D$, then for each $\lambda \in (0, 1)$ the averaged mapping*

(4) $$F_\lambda = \lambda I + (1-\lambda)F$$

*is power convergent.*

(ii) *If for at least one* $\lambda \in (0,1)$ *the mapping* $F_\lambda$ *defined by* (4) *is power convergent, then* Fix$(F)$ *consists of quasi-regular points.*

First we observe that it is sufficient to establish this theorem for bounded domains. Indeed, if $a$ is a fixed point of $F$ (hence of $F_\lambda$), the hyperbolicity of $D$ implies the existence of a ball $\mathcal{B}(a,R)$ (with respect to the $K_D$ pseudo-distance) centered at $a$ which is bounded and strictly inside $D$. Since $F_\lambda$ is nonexpansive with respect to $K_D$, this ball is invariant under $F_\lambda$. Suppose now that (i) is established for that ball and let $B(c,r)$ be any ball (with respect to the norm) which is strictly inside $D$. Then the family $\{F_\lambda^n\}$ is bounded on the convex hull of $\mathcal{B}(a,R)\cup B(c,r)$ (which is also strictly inside $D$), and therefore $F_\lambda$ is power convergent on $B(c,r)$ by the Vitali property (see, for example, [9] and [14]).

*Proof of Theorem 1.* (i) Let $a \in \mathrm{Fix}(F)$ be a quasi-regular point, and let $A_\lambda = \lambda I + (1-\lambda)F'(a)$ for $\lambda \in [0,1)$. It is clear that $a \in \mathrm{Fix}(F_\lambda)$, where $F_\lambda$ is defined by (4), and

$$A_\lambda = (F_\lambda)'(a), \qquad \lambda \in [0,1),$$

with $A_0 = A = F'(a)$.

We intend to show that for each $\lambda \in (0,1)$ the mapping $F_\lambda$ satisfies the conditions of Theorem A. Actually, by Theorem E, instead of (**) it suffices to check condition (*). But this condition is obvious, because

(5) $$I - A_\lambda = (1-\lambda)A.$$

Now we must show that the set $\sigma(A_\lambda)\setminus\{1\}$ lies inside the open unit disk $\Delta$ in $\mathbb{C}$ for each $\lambda \in (0,1)$. Once again, the Cauchy integral formula shows that the operator $A_\lambda$ is power bounded. Suppose now that there exists $\zeta \in \partial\Delta\cap\sigma(A_\lambda)$ and $\zeta \neq 1$. Then we have for such $\zeta$,

$$\zeta I - A_\lambda = (1-\lambda)(tI - A),$$

where

(6) $$t = \frac{\zeta-\lambda}{1-\lambda} \in \sigma(A).$$

It is clear that $|t| \geq 1$. But on the other hand $|t| \leq 1$, since $\sigma(A) \subseteq \bar{\Delta}$ (see (1)). So, $|t| = 1$ and we have by (6),

$$\zeta = \lambda + (1-\lambda)t \in \partial\Delta.$$

But this is possible only if $\zeta = t = 1$. This contradiction proves our assertion.



(ii) Now, if for some $\lambda \in (0,1)$ the mapping $F_\lambda$ is power convergent, then it follows by the Cauchy inequalities that for each $a \in \mathrm{Fix}(F)$ the operator $A_\lambda = (F_\lambda)'(a)$ is also power convergent. It is known that this fact in turn implies condition (**) for $A_\lambda$ (see, for example, [25]). But since (**) is equivalent to (*), it follows by (5) that $a$ is quasi-regular. The proof is complete.

So, setting $\lambda = \frac{1}{2}$ in Theorem 1, we see that the Cesàro average (see (3))

$$C_2 = \tfrac{1}{2}(I+F)$$

is power convergent, i.e., it is sufficient to take $p = 2$ in Theorem C. As a matter of fact, it turns out that the Cesàro averages defined by (3) are power convergent for all $p = 2,3,\ldots$ Moreover, we are able to show that all proper convex combinations of the iterates of $F$ are also power convergent. We will now describe a general scheme for constructing a retraction onto Fix$(F)$.

REMARK 4. It follows by Theorems A and E (see also Theorem 2 in [25]) that $F \in \mathrm{Hol}(D,D)$ with $\mathrm{Fix}(F) \neq \emptyset$ is power convergent if and only if for some $a \in \mathrm{Fix}(F)$ the linear operator $A = F'(a)$ is power convergent. Thus to construct a retraction onto Fix$(F)$ by using a power convergent mapping, we must find a mapping $\Phi \in \mathrm{Hol}(D,D)$ such that $\mathcal{F} = \mathrm{Fix}(F) = \mathrm{Fix}(\Phi)$ and for some point $a \in \mathcal{F}$, $\Phi'(a)$ is power convergent.

LEMMA. *Let $A$ be a bounded linear operator in $X$, and let $\lambda_0 \in \sigma(A)$ be such that*

(7) $$\mathrm{Ker}(\lambda_0 I - A) \oplus \mathrm{Im}(\lambda_0 I - A) = X.$$

*Suppose that there exist a domain $\Omega \subset \mathbb{C}$ and a holomorphic function $f$ defined in a neighborhood $\widetilde{\Omega}$ of $\Omega$ with the following properties*:

(a) $\overline{\Omega} \supset \sigma(A)$ *and* $\lambda_0 \in \partial\Omega$;

(b) $f(\Omega) \subseteq \bar{\Delta}$;

(c) $\lambda_0$ *is a simple root of the equation* $f(\lambda_0) = 1$;

(d) $|f(\lambda)| \neq 1$ *for all* $\lambda \in \partial\Omega$, $\lambda \neq \lambda_0$.

*Then the linear operator $B = f(A) : X \to X$ defined by the formula*

(8) $$B = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda I - A)^{-1}\,d\lambda,$$

*where $\Gamma \subset \widetilde{\Omega}$ is a closed path around $\sigma(A)$, is power convergent.*

Proof. It follows by Dunford's Spectral Mapping Theorem that

(9) $$\sigma(B) = f(\sigma(A)).$$

Hence, conditions (a), (b) and (d) imply that

$$\sigma(B)\setminus\{1\}\subset\Delta. \tag{10}$$

Consider now the function

$$g(\lambda)=[1-f(\lambda)](\lambda_0-\lambda)^{-1}. \tag{11}$$

Condition (c) implies that $g(\lambda)$ is holomorphic in a neighborhood of $\lambda_0\in\mathbb{C}$, and $g(\lambda_0)\neq 0$. In addition, $g(\lambda)\neq 0$ for all $\lambda\in\sigma(A)$, by condition (d), and therefore the operator $C=g(A)$ defined by the formula

$$C=\frac{1}{2\pi i}\int_\Gamma g(\lambda)(\lambda I-A)^{-1}\,d\lambda \tag{12}$$

is invertible in $X$. Furthermore, it follows from the multiplicative property of the calculus of $L(X)$-valued functions defined by (8) and (12) (see [33]) and from formula (11) that

$$I-B=C(\lambda_0 I-A)=(\lambda_0 I-A)C.$$

This implies the equalities $\mathrm{Ker}(I-B)=\mathrm{Ker}(\lambda_0 I-A)$ and $\mathrm{Im}(I-B)=\mathrm{Im}(\lambda_0 I-A)$. Now (7) and (10) imply that $B$ is power convergent. The Lemma is proved.

THEOREM 2. *Let $D$ be a hyperbolic convex domain in $X$, and let $F\in\mathrm{Hol}(D,D)$ with $\mathcal{F}=\mathrm{Fix}(F)\neq\emptyset$. If $\mathcal{F}$ contains a quasi-regular point $a\in D$, then each mapping $\Phi$ of the form $\Phi=\sum_{k=0}^{p}\alpha_k F^k$, where $\sum_{k=0}^{p}\alpha_k=1$ and $0\le\alpha_k\neq 1$ for all $k$, is power convergent.*

Proof. As in the proof of Theorem 1, we may assume that $D$ is bounded. Let $\Phi\in\mathrm{Hol}(D,D)$ be defined by

$$\Phi=\sum_{k=0}^{p}\alpha_k F^k, \tag{13}$$

where $\sum_{k=0}^{p}\alpha_k=1$ and $0\le\alpha_k\neq 1$ for all $k=0,1,\ldots,p$.

Consider the holomorphic function (polynomial) $f:\bar{\Delta}\to\bar{\Delta}$ defined by

$$f(\lambda)=\sum_{k=0}^{p}\alpha_k\lambda^k,\qquad \lambda\in\bar{\Delta}.$$

It is clear that $f$ satisfies the conditions (a)–(d) of the Lemma, with $\Omega=\Delta$, and therefore the operator $B$ defined by (8) is power convergent to a projection onto $\mathrm{Ker}(I-A)$, where $A=F'(a)$. But it follows from the chain rule that $\Phi'(a)=B=f(A)$. Hence, $\Phi$ is power convergent onto $\mathrm{Fix}(\Phi)$ (see Remark 4). Furthermore, (13) implies that $\mathrm{Fix}(F)\subseteq\mathrm{Fix}(\Phi)$. At the same time these sets are connected submanifolds in $D$ tangent to the same subspace $\mathrm{Ker}(I-A)$. Therefore they coincide in $D$. We are done.



REMARK 5. If $D\ni 0$, then it can be shown that each infinite convex combination

$$\Phi=\sum_{k=0}^{\infty}\alpha_k F^k, \tag{14}$$

where $\sum_{k=0}^{\infty}\alpha_k=1$ and $0\le\alpha_k\neq 1$ for all $k$, belongs to $\mathrm{Hol}(D,D)$ and is also power convergent to a retraction onto $\mathrm{Fix}(F)$.

We now describe an implicit method for approximating fixed points of holomorphic mappings which has been used many times in the theory of nonexpansive mappings (see, for example, [10]).

Let $D$ be a bounded convex domain in $X$, and let $F$ belong to $\mathrm{Hol}(D,D)$. For $t\in[0,1)$ and a fixed $y\in D$, consider the mapping $\Phi(x)=tF(x)+(1-t)y$, $x\in D$. Since $\Phi$ maps $D$ strictly inside itself, the Earle–Hamilton theorem [7] implies that there exists a unique fixed point $z=z_t(y)\in D$ of the mapping $\Phi$. Moreover,

$$z=\underset{n\to\infty}{\text{T-lim}}\,\Phi^n. \tag{15}$$

The $X$-valued function $z_t(y)$, $0\le t<1$, is called an *approximating curve*. It was shown in [22] that if $X=\mathbb{C}^n$ and $\mathrm{Fix}(F)\neq\emptyset$, then there is a sequence $t_n\in(0,1)$, $t_n\to 1$, such that for each $y\in D$ the sequence $z_{t_n}(y)$ converges to a fixed point of $F$ in $D$.

At the same time, changing our point of view, for each $t\in[0,1)$, $z_t=z_t(y)$ holomorphically depends on $y\in D$, by (15). We denote this mapping by $\mathcal{T}_t$. In other words, $\mathcal{T}_t$ is the unique solution of the nonlinear operator equation

$$\mathcal{T}_t=tF\circ\mathcal{T}_t+(1-t)I. \tag{16}$$

The mapping $\mathcal{T}_t$ belongs to $\mathrm{Hol}(D,D)$ and it is easy to check that

$$\mathrm{Fix}(\mathcal{T}_t)=\mathrm{Fix}(F)\qquad\text{for all } t\in(0,1).$$

THEOREM 3. *Let $D$ be a bounded convex domain in a complex Banach space $X$, and let $F\in\mathrm{Hol}(D,D)$ with $\mathcal{F}=\mathrm{Fix}(F)\neq\emptyset$. If $\mathcal{F}$ contains a quasi-regular point in $D$, then the mapping $\mathcal{T}_t$ defined by (16) is power convergent to a retraction onto $\mathcal{F}$.*

Proof. Let $a\in\mathcal{F}$, and let $B=(\mathcal{T}_t)'(a)$, $t\in(0,1)$. Then $B$ satisfies the operator equation

$$B=tA\circ B+(1-t)I$$

where $A=F'(a)$.

Setting $r=(1-t)/t$, we have

$$B=[I+r(I-A)]^{-1}. \tag{17}$$

Therefore $B$ can be defined by formula (8) with

$$f(\lambda)=\frac{1}{1+r(1-\lambda)},\quad \lambda\in\bar{\Delta}.$$

Since $r>0$, this function maps $\overline{\Omega}=\bar{\Delta}$ into itself and satisfies all the conditions of the Lemma. Thus by that lemma, $B$ is power convergent to a projection onto $\operatorname{Ker}(I-A)$ and so $\mathcal{T}_t$ is power convergent to a retraction onto $\operatorname{Fix}(F)$. The theorem is proved.

REMARK 6. Note that it follows from the Neumann series representation of the operator B in (17), that the mapping $\Phi_t$ defined by (14),

$$\Phi_t=\sum_{k=0}^{\infty}\alpha_k F^k,$$

with $\alpha_k=t(1-t)^k$, $t\in(0,1)$, has $B=[I+r(I-A)]^{-1}$ as its Fréchet derivative at the point $a$. However, generally speaking, $\Phi_t$ and $\mathcal{T}_t$ are different in the nonlinear case.

Observe that for $F\in\operatorname{Hol}(D,D)$ the family $\{F^n\}_{n=0}^{\infty}$ of the iterates of $F$ can be considered a "discrete time" one-parameter semigroup, and $\operatorname{Fix}(F)=\bigcap_{n>0}\operatorname{Fix}(F^n)$. The question is what can be said about the common fixed point set of a continuous semigroup. *Is it also a submanifold of $D$ and is there a retraction onto it?*

More precisely, let us say that a family $S=\{F_t\}_{t>0}$, $F_t\in\operatorname{Hol}(D,D)$, is a *T-continuous one-parameter semigroup* if

$$F_{t+s}=F_t\circ F_s,\qquad t,s>0,$$

and

$$\text{(18)}\qquad \operatorname*{T-lim}_{t\to0^+}F_t=I.$$

Since $\{F_t\}$ is a commutative family, it is known that for a bounded convex domain in a finite-dimensional space the common fixed point set of this family is a holomorphic retract of $D$ (if it is not empty, of course) (see [1]). But even in this case, it is unknown how to construct a retraction onto this set. The situation becomes more complicated in the infinite-dimensional case. However, for semigroups we can establish a continuous analog of Theorem 2.

Let $D$ be a convex domain in a Banach space $X$, and let $S=\{F_t\}_{t>0}$ be a T-continuous semigroup of holomorphic self-mappings $F_t$ of $D$ with $\mathcal{F}=\bigcap_{t>0}\operatorname{Fix}F_t\neq\emptyset$.

Pick $a\in\mathcal{F}$ and set $A_t=(F_t)'(a)$, $t>0$. It follows from (18) and the Cauchy inequalities that $\{A_t\}_{t>0}$ is a uniformly continuous semigroup of bounded linear operators, i.e., $A_t$ converges to the identity in the operator



norm as $t\to0^+$. Therefore it has an infinitesimal generator $B:X\to X$, i.e.,

$$\text{(19)}\qquad B=\lim_{t\to0^+}\frac{I-A_t}{t}.$$

Moreover, it can be shown (see [31]) that the nonlinear semigroup $\{F_t\}_{t>0}$ also has an infinitesimal generator $f:D\to X$ defined by the formula

$$\text{(20)}\qquad f=\operatorname*{T-lim}_{t\to0^+}\frac{I-F_t}{t},$$

and $f'(a)=B$. In addition, $F_t(x)$ is the unique solution of the Cauchy problem

$$\text{(21)}\qquad \begin{cases}\dfrac{\partial F_t(x)}{\partial t}+f(F_t(x))=0,\\ \lim\limits_{t\to0^+}F_t(x)=x.\end{cases}$$

Now we can state our assertion.

THEOREM 4. *Let $D$ and $S=\{F_t\}_{t>0}$ be as above. Suppose that for some $a\in\mathcal{F}=\bigcap_{t>0}\operatorname{Fix}(F_t)$ the following condition holds:*

$$\text{(22)}\qquad \operatorname{Ker}B\oplus\operatorname{Im}B=X,$$

*where $B$ is defined by (19) with $A_t=(F_t)'(a)$. Then, for each $t>0$, the continuous Cesàro average*

$$\text{(23)}\qquad \Phi_t=\frac{1}{t}\int_0^t F_s\,ds$$

*is power convergent to a retraction onto $\mathcal{F}$. Thus $\mathcal{F}$ is a connected submanifold of $D$.*

Proof. It follows from (23) that $\Phi_t(a)=a$ for each $t>0$, and hence $\Phi_t\in\operatorname{Hol}(D,D)$. In addition, $(\Phi_t)'(a)=C_t$, where $C_t$ is defined by

$$\text{(24)}\qquad C_t=\frac{1}{t}\int_0^t A_s\,ds=\frac{1}{t}\int_0^t e^{-Bs}\,ds.$$

Therefore, if we set in our Lemma

$$\Omega=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\},\quad \lambda_0=0,$$

and

$$f_t(\lambda)=\frac{1-e^{-t\lambda}}{t\lambda},\quad t>0,$$

we get by (24),

$$C_t=\frac{1}{2\pi i}\int_{\Gamma}f(\lambda)(\lambda I-B)^{-1}\,d\lambda,$$

where $\Gamma$ is a contour that surrounds $\sigma(B)$ in $\Omega_\varepsilon = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\varepsilon\}$ for some $\varepsilon > 0$. Furthermore, it follows by the maximum modulus principle that $f(\overline{\Omega} \setminus \{0\}) \subset \Delta$ and hence $\lambda = 0$ is the unique and simple root of the equation $f(\lambda) = 1$ in $\overline{\Omega}$. So, by the Lemma, $C_t$ is power convergent to a projection onto $\operatorname{Ker} B$. But it follows from (21) and the uniqueness property that $\mathcal{F} = \bigcap_{t>0} \operatorname{Fix}(F_t)$ coincides with the null point set of $f$ in $D$, where $f$ is defined by (20) and $f'(a) = B$. Since $(\Phi_t)'(a) = C_t$, $\Phi_t$ is power convergent to a retraction onto $\operatorname{Fix}(\Phi_t)$, which is tangent to $\operatorname{Ker} B$. Together with the inclusion $\mathcal{F} \subseteq \operatorname{Fix}(\Phi_t)$ this implies the equality $\mathcal{F} = \operatorname{Fix}(\Phi_t)$, which proves our assertion.

**Acknowledgements.** The research of the first author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund – M. and M.L. Bank Mathematics Research Fund. Both authors thank the referee for many helpful comments and corrections.

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*Received August 20, 1997* (3940)
*Revised version February 20, 1998*


STUDIA MATHEMATICA 130 (3) (1998)

# Reverse-Hölder classes in the Orlicz spaces setting

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**Abstract.** In connection with the $A_\phi$ classes of weights (see [K-T] and [B–K]), we study, in the context of Orlicz spaces, the corresponding reverse-Hölder classes $RH_\phi$. We prove that when $\phi$ is $\Delta_2$ and has lower index greater than one, the class $RH_\phi$ coincides with some reverse-Hölder class $RH_q$, $q > 1$. For more general $\phi$ we still get $RH_\phi \subset A_\infty = \bigcup_{q>1} RH_q$ although the intersection of all these $RH_\phi$ gives a proper subset of $\bigcap_{q>1} RH_q$.

**1. Introduction.** By a *weight* $w$ we mean a non-negative and locally integrable function on $\mathbb{R}^n$. As is well known, a weight $w$ is said to belong to the *reverse-Hölder class* with exponent $q$, $RH_q$, if it satisfies the inequality

$$\left(\frac{1}{|Q|}\int_Q w(x)^q\,dx\right)^{1/q} \le C\frac{1}{|Q|}\int_Q w(x)\,dx$$

for any cube $Q \subset \mathbb{R}^n$ with sides parallel to the axes; here $|Q|$ denotes the Lebesgue measure of $Q$. These classes appeared in connection with the $A_p$ classes of Muckenhoupt which characterize the weights such that the Hardy–Littlewood maximal operator is bounded on $L^p(w)$, $1 < p < \infty$. To be precise, the $A_p$ weights are defined as those weights such that

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C$$

for any cube $Q \subset \mathbb{R}^n$. The limiting case $p = 1$, $A_1$, is defined as the weights satisfying

$$m_Q(w) \le C \inf_{x\in Q} w(x)$$

for any cube $Q \subset \mathbb{R}^n$, where $m_Q(w)$ denotes the average of $w$ over $Q$. For $p = \infty$, $A_\infty$ consists of the weights $w$ such that for any $Q$ and any measurable

1991 *Mathematics Subject Classification*: Primary 42B25.
*Key words and phrases*: reverse-Hölder class, $A_p$ classes of Muckenhoupt, Orlicz spaces.
The authors were supported by the Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.