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On complex interpolation and spectral continuity

by

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Abstract. Let $[X_0, X_1]_t$, $0 \le t \le 1$, be Banach spaces obtained via complex interpolation. With suitable hypotheses, linear operators T that act boundedly on both X_0 and X_1 will act boundedly on each $[X_0, X_1]_t$. Let T_t denote such an operator when considered on $[X_0, X_1]_t$, and $\sigma(T_t)$ denote its spectrum. We are motivated by the question of whether or not the map $t \to \sigma(T_t)$ is continuous on $(0,1)$; this question remains open. In this paper, we study continuity of two related maps: $t \to (\sigma(T_t))^\wedge$ (polynomially convex hull) and $t \to \partial_e(\sigma(T_t))$ (boundary of the polynomially convex hull). We show that the first of these maps is always upper semicontinuous, and the second is always lower semicontinuous. Using an example from [5], we now have definitive information: $t \to (\sigma(T_t))^\wedge$ is upper semicontinuous but not necessarily continuous, and $t \to \partial_e(\sigma(T_t))$ is lower semicontinuous but not necessarily continuous.

Setting. Assume that $[X_0, X_1]$ is an interpolation pair in the sense of Calderón [2] and that $X_0 \cap X_1$ is dense in both X_0 and X_1. Further, assume that

$$T : X_0 \cap X_1 \to X_0 \cap X_1$$

is continuous with respect to the norm on X_0 and with respect to the norm on X_1. Then T induces, for each $t \in [0,1]$, a bounded linear operator T_t on the interpolation Banach space $[X_0, X_1]_t$. Let $\sigma(T_t)$ denote the spectrum of the operator T_t in the Banach algebra of bounded linear operators on $[X_0, X_1]_t$. We consider the map $t \to \sigma(T_t)$ assigning to each value of t in $[0,1]$ the compact set $\sigma(T_t)$. As is usual in this context, the collection of all compact subsets of the complex plane is endowed with the Hausdorff metric topology.

It is known that the map $t \to \sigma(T_t)$ can be discontinuous at $t = 0$ and at $t = 1$, but the question of whether or not it is continuous on $(0,1)$ remains open. When attempting to answer this question, two other maps, and hence questions about their continuity, arise naturally. These two maps are:

$$t \to (\sigma(T_t))^\wedge \quad \text{and} \quad t \to \partial_e(\sigma(T_t)),$$

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where the image under the first map is the polynomially convex hull of the spectrum and the image under the second map is the exterior boundary of the spectrum.

To get a feel for what is going on here, we begin by looking at an example of interpolated spectra. Consider $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ to be Hilbert spaces each with orthogonal basis $\{e_n\}_{n\in\mathbb{Z}}$ and

$$\Big\|\sum a_n e_n\Big\|_0 = \sqrt{\sum (R^n a_n)^2}, \qquad \Big\|\sum a_n e_n\Big\|_1 = \sqrt{\sum (r^n a_n)^2},$$

for some choice $0 < r < R$. Let T be the (left) shift $Te_n = e_{n-1}$. Then T_0 is unitarily equivalent to $\frac{1}{R}U$ and T_1 is unitarily equivalent to $\frac{1}{r}U$, where U is the adjoint of the bilateral shift on $\ell^2(\mathbb{Z})$. It can be shown that the intermediate space $[X_0, X_1]_t$ is a Hilbert space with the same basis and that

$$\Big\|\sum a_n e_n\Big\|_t = \sqrt{\sum ((R^{1-t}r^t)^n a_n)^2}.$$

Further, T_t is unitarily equivalent to $\frac{1}{R^{1-t}r^t}U$. Since

$$\sigma_{\ell^2(\mathbb{Z})}(U) = \partial B_1(0),$$

it follows that

$$\sigma_{\ell^2(\mathbb{Z})}\left(\frac{1}{R^{1-t}r^t}U\right) = \partial B_{\frac{1}{R^{1-t}r^t}}(0),$$

and hence

$$\sigma(T_t) = \partial B_{\frac{1}{R^{1-t}r^t}}(0).$$

We now choose $R = 4$ and $r = 1$. Then, clearly,

$$\sigma(T_t) = \{\lambda : |\lambda| = 4^{t-1}\}.$$

Set $S = e^{2\pi i T}$; then $S_t = e^{2\pi i T_t}$ and

$$\sigma(S_t) = \{e^{2\pi i\lambda} : |\lambda| = 4^{t-1}\}.$$

Thus $\sigma(S_t)$ is a "croissant" for $0 \le t < 1/2$, and $\sigma(S_t)$ is a "doughnut" for $1/2 \le t \le 1$.

The example of the preceding paragraph was given in [5] to show that $t \to (\sigma(T_t))^\wedge$ need not be continuous. In fact, this example shows that neither $t \to (\sigma(T_t))^\wedge$ nor $t \to \partial_{\mathrm{e}}(\sigma(T_t))$ is necessarily continuous. Specifically, it shows that $t \to (\sigma(T_t))^\wedge$ fails to be lower semicontinuous, and that $t \to \partial_{\mathrm{e}}(\sigma(T_t))$ fails to be upper semicontinuous. Can a different example be found where $t \to (\sigma(T_t))^\wedge$ fails to be upper semicontinuous? Can a different example be found where $t \to \partial_{\mathrm{e}}(\sigma(T_t))$ fails to be lower semicontinuous? We show that the answer to both questions is no. That is, the map $t \to (\sigma(T_t))^\wedge$ must always be upper semicontinuous, and the map $t \to \partial_{\mathrm{e}}(\sigma(T_t))$ must always be lower semicontinuous.

The former of these two facts follows in a straightforward manner (by applying Lemma 2 below) from Shneĭberg's remarkable result in [7] that $t \to \sigma(T_t)$ is always upper semicontinuous. The latter will be deduced from the former; this will be done in the final section of the paper. This is a strong statement about interpolated spectra, as it is extremely easy to find a function $t \to \Gamma(t)$ mapping $t \in [0,1]$ to a compact subset $\Gamma(t) \subseteq \mathbb{C}$ with $t \to (\Gamma(t))^\wedge$ upper semicontinuous yet $t \to \partial_{\mathrm{e}}(\Gamma(t))$ not lower semicontinuous.

Definitions and notations. In this section we provide information about the Hausdorff metric topology and about polynomially convex hulls that is hard to find in textbooks, and also establish notation to be used. Aside from this topological material, we assume that the reader is familiar with the basic ideas of complex interpolation theory. The reader unfamiliar with these ideas may want to consult [1] or [2].

For a compact set $\Gamma \subseteq \mathbb{C}$ we let $\Gamma^\wedge$ denote the polynomially convex hull of Γ and $\partial_{\mathrm{e}}(\Gamma) \equiv \partial(\Gamma^\wedge)$ denote the *exterior boundary* of Γ. Let $\mathcal{K}$ denote the family of all compact subsets of $\mathbb{C}$ endowed with the Hausdorff metric Δ. For $r > 0$ and $\lambda \in \mathbb{C}$ we use $B_r(\lambda) \equiv \{\mu \in \mathbb{C} : |\lambda - \mu| < r\}$ to denote the open ball of radius r centered at λ. For $\Gamma \in \mathcal{K}$ and $\varepsilon > 0$ let

$$\Gamma_\varepsilon \equiv \Gamma + \overline{B_\varepsilon(0)} = \{\lambda \in \mathbb{C} : d(\lambda, \Gamma) \le \varepsilon\}.$$

Then, for $\Gamma_1, \Gamma_2 \in \mathcal{K}$,

$$\Delta(\Gamma_1, \Gamma_2) \le \varepsilon \quad \text{if and only if} \quad \Gamma_2 \subseteq (\Gamma_1)_\varepsilon \text{ and } \Gamma_1 \subseteq (\Gamma_2)_\varepsilon.$$

In particular, $t \to \Gamma(t)$ is continuous at t_0 if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\Gamma(t) \subseteq (\Gamma(t_0))_\varepsilon \quad \text{and} \quad \Gamma(t_0) \subseteq (\Gamma(t))_\varepsilon$$

whenever $t \in (0,1)$ and $|t - t_0| < \delta$. Then $t \to \Gamma(t)$ is *upper semicontinuous* at $t_0 \in (0,1)$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\Gamma(t) \subseteq (\Gamma(t_0))_\varepsilon$$

whenever $t \in (0,1)$ and $|t - t_0| < \delta$. Likewise, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\Gamma(t_0) \subseteq (\Gamma(t))_\varepsilon$$

whenever $t \in (0,1)$ and $|t - t_0| < \delta$ then $t \to \Gamma(t)$ is *lower semicontinuous* at $t_0 \in (0,1)$.

We conclude this section with two topological lemmas.

LEMMA 1. *Let $\Gamma \in \mathcal{K}$. Then*

(a) $\Gamma = \bigcap_{n=1}^\infty \Gamma_{1/n}$.
(b) *For $\lambda \in \partial\Gamma$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(\lambda, \mathbb{C}\setminus\Gamma_\delta) < \varepsilon$.*
(c) $\Gamma^\wedge = \bigcap_{n=1}^\infty [(\Gamma_{1/n})^\wedge]$.

Proof. The results of (a) and (b) are clear. One containment of (c) is obvious. To see the other containment, suppose that $\lambda \notin \Gamma^\wedge$. Then λ is in the unbounded component of $\mathbb{C} \setminus \Gamma$ and so we can choose a curve γ connecting λ to ∞ such that $\gamma \cap \Gamma^\wedge = \emptyset$. Now, if for each n there exists $\lambda_n \in \Gamma_{1/n} \cap \gamma$ then $\{\lambda_n\}_{n=1}^\infty \subseteq \Gamma_1$ and hence $\{\lambda_n\}_{n=1}^\infty$ has a convergent subsequence with limit which we will call μ. We may assume that $\lambda_n \to \mu$ as $n \to \infty$. Now γ closed implies that $\mu \in \gamma$. Given $\varepsilon > 0$ we may assume that $|\lambda_n - \mu| < \varepsilon$ and so, for any n,

$$d(\mu, \Gamma) \le d(\mu, \lambda_n) + d(\lambda_n, \Gamma) < \varepsilon + 1/n.$$

Since n was arbitrary and Γ is closed, $\mu \in \Gamma$, contradicting $\gamma \cap \Gamma^\wedge = \emptyset$. Therefore, there exists N such that $\Gamma_{1/N} \cap \gamma = \emptyset$ and thus λ is in the unbounded component of $\mathbb{C} \setminus \Gamma_{1/N}$. That is, $\lambda \notin (\Gamma_{1/N})^\wedge$ and so $\lambda \notin \bigcap_{n=1}^\infty [(\Gamma_{1/n})^\wedge]$. ■

LEMMA 2. *If $t \to \Gamma(t)$ from $(0,1)$ into $\mathcal{K}$ is upper semicontinuous, then so is $t \to (\Gamma(t))^\wedge$.*

Proof. Suppose to the contrary that there exists $t \in (0,1)$, $t_n \to t$ and an open set U such that $(\Gamma(t_n))^\wedge \not\subseteq U$ yet $(\Gamma(t))^\wedge \subseteq U$. Thus, for each n there exists $\lambda_n \in (\Gamma(t_n))^\wedge$ such that $\lambda_n \in \mathbb{C} \setminus U$. For each k, upper semicontinuity of $t \to \Gamma(t)$ implies that there exists n_k such that

$$\Gamma(t_n) \subseteq (\Gamma(t))_{1/k} \quad \text{for all } n \ge n_k.$$

So, in particular,

$$\lambda_{n_k} \in (\Gamma(t_{n_k}))^\wedge \subseteq [(\Gamma(t))_{1/k}]^\wedge \quad \text{for each } k.$$

Note that $j \ge k$ implies $[(\Gamma(t))_{1/j}]^\wedge \subseteq [(\Gamma(t))_{1/k}]^\wedge$. Hence, $\lambda_{n_k} \in [(\Gamma(t))_1]^\wedge$ for all k. Since $[(\Gamma(t))_1]^\wedge$ is compact, $\{\lambda_{n_k}\}_{k=1}^\infty$ has a convergent subsequence with limit which we will call λ. We may assume that $\lambda_{n_k} \to \lambda$ as $k \to \infty$. Since $\lambda_{n_k} \in \mathbb{C} \setminus U$ for all k and $\mathbb{C} \setminus U$ is closed, $\lambda \in \mathbb{C} \setminus U$. However, if we fix k, then

$$\lambda_{n_j} \in [(\Gamma(t))_{1/j}]^\wedge \subseteq [(\Gamma(t))_{1/k}]^\wedge \quad \text{for all } j \ge k.$$

Since $[(\Gamma(t))_{1/k}]^\wedge$ is closed, $\lambda \in [(\Gamma(t))_{1/k}]^\wedge$. Since k was arbitrary, part (c) of Lemma 1 implies that $\lambda \in (\Gamma(t))^\wedge$, contradicting the fact that $\lambda \in U^c$. ■

Lower semicontinuity of the exterior boundary of the spectrum. Up until this point, our results have been purely topological. We now assume that the compact sets we are considering are interpolated spectra, that is, $\Gamma(t) = \sigma(T_t)$.

THEOREM. *The map $t \to \partial_e(\sigma(T_t))$ is lower semicontinuous.*

The main ingredient in our proof is:

COROLLARY 6 OF [6]. *Let λ be any complex number that is contained in the resolvent set of every T_t, $t \in [0,1]$. Then the function $t \to d(\lambda, \sigma(T_t))$ is continuous on the open interval $(0,1)$.*

This result follows from the fact that the spectral radius of T_t is a continuous function of t (Theorem 4 of [6]). Halberg, in [4], first proved this when the interpolation spaces are the ℓ^p-spaces. The general result uses Cwikel's [3] improvement of Calderón's Reiteration Theorem [2].

If we step back from the spectrum for a minute, and consider arbitrary functions $t \to \Gamma(t)$ mapping the unit interval $[0,1]$ into the collection of all compact subsets of $\mathbb{C}$, there are eight implications that might possibly hold between upper semicontinuity and lower semicontinuity of one of the functions $t \to (\Gamma(t))^\wedge$ and $t \to \partial_e(\Gamma(t))$ and upper semicontinuity and lower semicontinuity of the other. Only one of these holds for completely arbitrary maps $t \to \Gamma(t)$: upper semicontinuity of $t \to \partial_e(\Gamma(t))$ always implies upper semicontinuity of $t \to (\Gamma(t))^\wedge$. As $(\Gamma(t))^\wedge = [\partial_e(\Gamma(t))]^\wedge$ this follows immediately from Lemma 2. Before returning to our specific spectral setting, we emphasize that upper semicontinuity of $t \to (\Gamma(t))^\wedge$ does not generally imply lower semicontinuity of $t \to \partial_e(\Gamma(t))$. This can be seen easily. One could, for example, set

$$\Gamma(t) = \begin{cases} \{z : |z| = 1,\ \mathrm{Re}(z) \ge 0\} & \text{for } 0 \le t < 1/2, \\ \{z : |z| = 1\} & \text{for } 1/2 \le t \le 1. \end{cases}$$

We now return to the spectral setting, $\Gamma(t) = \sigma(T_t)$. Recall that both $t \to \sigma(T_t)$ and $t \to (\sigma(T_t))^\wedge$ are upper semicontinuous (Shneĭberg [7] and Lemma 2 above). We now proceed to the

Proof of the Theorem. Suppose, to the contrary, that $t \to \partial_e(\Gamma(t))$ is not lower semicontinuous. Then for some t_0, a sequence $t_n \to t_0$, and some $\varepsilon_0 > 0$, we have

$$\partial_e(\Gamma(t_0)) \not\subseteq [\partial_e(\Gamma(t_n))]_{\varepsilon_0} \quad \text{for all } n.$$

So there exists for each n an element $\lambda_n \in \partial_e(\Gamma(t_0))$ such that

$$d(\lambda_n, \partial_e(\Gamma(t_n))) \ge \varepsilon_0.$$

Now $\{\lambda_n\}_{n=1}^\infty$ has a convergent subsequence with limit $\lambda \in \partial_e(\Gamma(t_0))$ and we may assume that $\lambda_n \to \lambda$ as $n \to \infty$. We may also assume that $d(\lambda, \lambda_n) < \varepsilon_0/2$ for all n and hence that

$$d(\lambda, \partial_e(\Gamma(t_n))) \ge \varepsilon_0/2 \quad \text{for all } n.$$

For each n either

(a) $B_{\varepsilon_0/2}(\lambda) \subseteq (\Gamma(t_n))^\wedge$, or
(b) $B_{\varepsilon_0/2}(\lambda) \subseteq \mathbb{C} \setminus (\Gamma(t_n))^\wedge$.

Say there exists a subsequence $\{n_k\}$ with (a) holding. Given m there exists n_{k_m} such that

$$(\Gamma(t_{n_{k_m}}))^\wedge \subseteq [(\Gamma(t_0))^\wedge]_{1/m}$$

by the upper semicontinuity of $t \to (\Gamma(t))^\wedge$. Thus, $B_{\varepsilon_0/2}(\lambda) \subseteq [(\Gamma(t_0))^\wedge]_{1/m}$ for all m and hence

$$B_{\varepsilon_0/2}(\lambda) \subseteq \bigcap_{m=1}^{\infty} [(\Gamma(t_0))^\wedge]_{1/m} = (\Gamma(t_0))^\wedge$$

(the last equality holds since $(\Gamma(t_0))^\wedge$ is closed), which contradicts the fact that $\lambda \in \partial_{\mathrm{e}}(\Gamma(t_0))$. So only a finite number and hence we may assume that none of the n's satisfy (a). Thus, for all n we have

$$d(\lambda, (\Gamma(t_n))^\wedge) \geq \varepsilon_0/2.$$

Hence,

$$d(\lambda, \Gamma(t_n)) \geq \varepsilon_0/2$$

for all n.

By part (b) of Lemma 1, there exists a $\delta > 0$ and a $\mu \in \mathbb{C} \setminus (\Gamma(t_0))_\delta$ such that $d(\lambda, \mu) < \varepsilon_0/16$. Also, via Shneĭberg, there exists $\eta > 0$ such that $\Gamma(t) \subseteq (\Gamma(t_0))_\delta$ and hence $\mu \notin \Gamma(t)$ for all $t \in [t_0 - \eta, t_0 + \eta]$. By Corollary 6 of [6] (this is the point of the proof at which it is crucial that $\Gamma(t)$ is the spectrum $\sigma(T_t)$ of interpolated operators), $t \to d(\mu, \Gamma(t))$ is continuous on $(t_0 - \eta, t_0 + \eta)$. For sufficiently large n, we have $t_n \in (t_0 - \eta, t_0 + \eta)$, and so

$$d(\mu, \Gamma(t_n)) \to d(\mu, \Gamma(t_0)) \leq d(\mu, \lambda) < \varepsilon_0/16$$

since $\lambda \in \partial(\Gamma(t_0)) \subseteq \Gamma(t_0)$. On the other hand,

$$d(\Gamma(t_n), \mu) \geq d(\Gamma(t_n), \lambda) - d(\lambda, \mu) \geq \varepsilon_0/2 - \varepsilon_0/16 > \varepsilon_0/16,$$

a contradiction. Therefore, $t \to \partial_{\mathrm{e}}(\Gamma(t))$ is lower semicontinuous. ∎

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