[GLV] C. Godrèche, J. M. Luck and F. Vallet, *Quasiperiodicity and types of order: a study in one dimension*, J. Phys. A 20 (1987), 4483–4499.
[K] A. Khinchin, *Continued Fractions*, The Univ. of Chicago Press, Chicago and London, 1964.
[Ke] H. Kesten, *On a conjecture by Erdős and Szüsz related to uniform distribution modulo 1*, Acta Arith. 12 (1966), 193–212.
[KN] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York, 1974.
[L] P. Liardet, *Regularities of distribution*, Compositio Math. 61 (1987), 267–293.
[LV] P. Liardet and D. Volný, *Sums of continuous and differentiable functions in dynamical systems*, Israel J. Math. 98 (1997), 29–60.
[Pa] D. A. Pask, *Skew products over the irrational rotation*, ibid. 69 (1990), 65–74.
[P1] K. Petersen, *Ergodic Theory*, Cambridge Univ. Press, Cambridge, 1983.
[P2] —, *On a series of cosecants related to a problem in ergodic theory*, Compositio Math. 26 (1973), 313–317.

UFR de Mathématiques et URA GAT
Université des Sciences et Technologies de Lille
59655 Villeneuve d'Ascq Cedex, France
E-mail: debievre@gat.univ-lille1.fr

Department of Mathematics
Princeton University
Princeton, New Jersey 08544
U.S.A.
E-mail: gforni@math.princeton.edu

Received September 5, 1996
Revised version January 12, 1998
(3733)

STUDIA MATHEMATICA 130 (3) (1998)

Singular integral models for p-hyponormal operators and the Riemann–Hilbert problem

by

MUNEO CHŌ (Yokohama), TADASI HURUYA (Niigata)
and MASUO ITOH (Tokyo)

Dedicated to Professor Isao Miyadera
in celebration of his having been honoured as
an emeritus Professor of Waseda University

Abstract. The purpose of this paper is to give singular integral models for p-hyponormal operators and apply them to the Riemann–Hilbert problem.

1. Introduction. Prof. D. Xia, in [4], studied the singular integral models of semi-hyponormal operators and showed many useful results for such operators. In this paper we first introduce the singular integral models of p-hyponormal operators for $0 < p < 1/2$ and next apply them to the Riemann–Hilbert problem.

Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$. If $p = 1$, then T is called *hyponormal*, and if $p = \frac{1}{2}$, then T is called *semi-hyponormal*. The set of all semi-hyponormal operators in $B(\mathcal{H})$ is denoted by SH.

The set of all p-hyponormal operators in $B(\mathcal{H})$ is denoted by p-H. Let SHU and p-HU denote the sets of all operators in SH and in p-H with equal defect and nullity ([4], p. 4), respectively. Hence we may assume that the operator U in the polar decomposition $T = U|T|$ is unitary if $T \in \mathrm{SHU} \cup p\text{-HU}$. Throughout this paper, let p satisfy $0 < p < 1/2$.

Let A be a contraction and $T \in B(\mathcal{H})$. Define

$$A^{[n]} = \begin{cases} A^n, & n \geq 0, \\ (A^*)^{-n}, & n < 0. \end{cases}$$

1991 *Mathematics Subject Classification*: Primary 47B20; Secondary 47A10.
Key words and phrases: Hilbert space, p-hyponormal operator, singular integral model, Riemann–Hilbert problem.

If

$$\mathcal{S}_A^{\pm}(T) = \operatorname*{s\text{-}lim}_{n\to\pm\infty} A^{[-n]}TA^{[n]}$$

exists, then the operator $\mathcal{S}_A^{\pm}(T)$ is called the *polar symbol* of T related to A. If an operator $T = U|T|$ is semi-hyponormal, then $\mathcal{S}_U^{\pm}(T)$ exists and, for $0 \le k \le 1$,

$$T_{(k)} = (1-k)\mathcal{S}_U^{-}(T) + k\mathcal{S}_U^{+}(T)$$

are called the *general polar symbols* of T. The following property holds: If $T = U|T|$ with unitary U is p-hyponormal, then

$$\mathcal{S}_U^{-}(|T|^{2p}) \le |T|^{2p} \le \mathcal{S}_U^{+}(|T|^{2p}).$$

See Xia ([4]) for details.

Let $\mathbf{T} = \{e^{i\theta} \mid 0 \le \theta < 2\pi\}$, Σ be the set of all Borel sets in $\mathbf{T}$, m be a measure on the measurable space $(\mathbf{T}, \Sigma)$ such that $dm(e^{i\theta}) = \frac{1}{2\pi}d\theta$ and $\mathcal{D}$ be a separable Hilbert space. In the sequel we write w for $e^{i\theta}$.

The Hilbert space of all vector-valued, strongly measurable and square-integrable functions on $\mathbf{T}$ with values in $\mathcal{D}$ and with inner product

$$(f,g) = \int_{\mathbf{T}} (f(w), g(w))_{\mathcal{D}}\, dm$$

is denoted by $L^2(\mathcal{D})$; the Hardy space is denoted by $H^2(\mathcal{D})$, and the projection from $L^2(\mathcal{D})$ to $H^2(\mathcal{D})$ by $\mathcal{P}$. If $f \in L^2(\mathcal{D})$, then

$$(\mathcal{P}(f))(w) = \lim_{r\to 1-0} \frac{1}{2\pi i} \int_{|z|=1} f(z)(z-rw)^{-1}\, dz. \tag{1}$$

Let ν be a singular measure on $(\mathbf{T}, \Sigma)$, and $F \in \Sigma$ be a set such that $\nu(\mathbf{T} \setminus F) = 0$ and $m(F) = 0$. Put $\mu = m + \nu$. Let $R(\cdot)$ be a standard operator-valued strongly measurable function defined on $\Omega = (\mathbf{T}, \Sigma, \mu)$ with values being projections in $\mathcal{D}$, let $L^2(\Omega, \mathcal{D})$ be the Hilbert space of all $\mathcal{D}$-valued strongly measurable and square-integrable functions on Ω with inner product $(f,g) = \int_{\mathbf{T}} (f(w), g(w))_{\mathcal{D}}\, d\mu$, and

$$\widehat{H} = \{f \mid f \in L^2(\Omega, \mathcal{D}),\ R(w)f(w) = f(w),\ w \in \mathbf{T}\}.$$

Then $\widehat{H}$ is a subspace of $L^2(\Omega, \mathcal{D})$.

We define an operator $\mathcal{P}_0$ from $L^2(\Omega, \mathcal{D})$ to $\mathcal{D}$ as follows:

$$\mathcal{P}_0(f) = \int f(w)\, dm.$$

Then $\mathcal{P}_0$ is the projection from $L^2(\Omega, \mathcal{D})$ to $\mathcal{D}$ ([3], p. 707, or [4], p. 50). Let $\alpha(\cdot)$ and $\beta(\cdot)$ be operator-valued, uniformly bounded, and strongly measurable functions on Ω such that $\alpha(w)$ and $\beta(w)$ are linear operators in $\mathcal{D}$,

satisfying

$$R(w)\alpha(w) = \alpha(w)R(w) = \alpha(w),$$
$$R(w)\beta(w) = \beta(w)R(w) = \beta(w)$$

and $\beta(w) \ge 0$.

Furthermore, suppose that $\alpha(w) = 0$ if $w \in F$. We define $(\alpha f)(w) = \alpha(w)f(w)$.

Xia ([3]) constructed two operators $\widehat{U}$ and $\widehat{T}$ in $\widehat{\mathcal{H}}$ as follows:

$$(\widehat{U}f)(w) = wf(w), \tag{2}$$

$$(\widehat{T}f)(w) = w[\alpha(w)^*(\mathcal{P}(\alpha f))(w) + \beta(w)f(w)]. \tag{3}$$

He showed the following theorems.

THEOREM A ([3], Theorem 6). *Let $\widehat{U}$ and $\widehat{T}$ be operators defined by (2) and (3), respectively. Then the operator $\widehat{T}$ is semi-hyponormal and the corresponding polar symbols of $\widehat{T}$ are*

$$(\mathcal{S}_{\widehat{U}}^{\pm}(\widehat{T})f)(w) = \mathcal{S}_{\widehat{U}}^{\pm}(\widehat{T})(w)f(w),$$

where $\mathcal{S}_{\widehat{U}}^{+}(\widehat{T})(w) = w(\alpha(w)^*\alpha(w) + \beta(w))$ and $\mathcal{S}_{\widehat{U}}^{-}(\widehat{T})(w) = w\beta(w)$. The corresponding polar difference operator $\widehat{Q} = |\widehat{T}| - \widehat{U}|\widehat{T}|\widehat{U}^*$ is $(\widehat{Q}f)(w) = \alpha(w)^*\mathcal{P}_0(\alpha f)$.

THEOREM B ([3], Theorem 7). *Let $T = U|T|$ be a semi-hyponormal operator in $\mathcal{H}$ such that U is unitary. Then there are a Hilbert space $\widehat{\mathcal{H}}$, operators $\widehat{T}$ and $\widehat{U}$ as in Theorem A and a unitary operator W from $\mathcal{H}$ to $\widehat{\mathcal{H}}$ such that*

$$WTW^{-1} = \widehat{T} \quad \text{and} \quad WUW^{-1} = \widehat{U}.$$

The operators $\widehat{U}$ and $\widehat{T}$ in Theorem B are called *singular integral models* (*or function models*) of U and T, respectively.

2. Singular integral models for p-hyponormal operators. First we introduce the general polar symbols for $T \in p$-HU. For $0 \le k \le 1$, the general polar symbols $T_{[k]}$ of an operator $T = U|T|$ in p-HU are defined as follows:

$$T_{[k]} = U[(1-k)\mathcal{S}_U^{-}(|T|^{2p}) + k\mathcal{S}_U^{+}(|T|^{2p})]^{1/(2p)}$$

(see [1] for details).

Since $\beta(w) \ge 0$ and $\mathcal{P}$ is a projection, we have $(\alpha(w)^*(\mathcal{P}(\alpha f))(w) + \beta(w)f(w), f(w))_{\mathcal{D}} \ge 0$. Therefore, we can define the operator $[\alpha^*\mathcal{P}\alpha + \beta]^{1/(2p)}$.

THEOREM 1. *Let $\widehat{\mathcal{H}}$ and $\widehat{U}$ be a Hilbert space and a unitary operator on $\widehat{\mathcal{H}}$ given in Introduction, respectively. Let $\widetilde{T}$ be an operator in $\widehat{\mathcal{H}}$ defined by*

$$(\widetilde{T}f)(w) = w(Af)(w),$$

where $(A^{2p}f)(w) = \alpha(w)^*(\mathcal{P}(\alpha f))(w) + \beta(w)f(w)$. Then $\widetilde{T}$ is p-hyponormal and the corresponding polar symbols of $\widetilde{T}$ are

$$(\mathcal{S}^+_{\widehat{U}}(\widetilde{T})f)(w) = w[\alpha(w)^*\alpha(w) + \beta(w)]^{1/(2p)}f(w)$$

and

$$(\mathcal{S}^-_{\widehat{U}}(\widetilde{T})f)(w) = w[\beta(w)]^{1/(2p)}f(w).$$

The corresponding polar difference operator $\widetilde{Q}_p = |\widetilde{T}|^{2p} - \widehat{U}|\widetilde{T}|^{2p}\widehat{U}^*$ is

$$(\widetilde{Q}_pf)(w) = \alpha(w)^*\mathcal{P}_0(\alpha f). \tag{4}$$

Moreover, the corresponding general polar symbols $\widetilde{T}_{[k]}$ are

$$(\widetilde{T}_{[k]}f)(w) = w[\beta(w) + k\alpha(w)^*\alpha(w)]^{1/(2p)}f(w). \tag{5}$$

Proof. Since

$$(A^{2p}f)(w) = (|\widetilde{T}|^{2p}f)(w),$$

it follows that

$$(|\widetilde{T}|^{2p}f)(w) = \alpha(w)^*(\mathcal{P}(\alpha f))(w) + \beta(w)f(w). \tag{6}$$

Therefore, as in [4], Theorem III.1.3, we have the corresponding polar difference operator such that

$$(\widetilde{Q}_pf)(w) = \alpha(w)^*\mathcal{P}_0(\alpha f).$$

So we have

$$\begin{aligned}(\widetilde{Q}_pf, f) &= \int_{\mathbf{T}}(\alpha(w)^*\mathcal{P}_0(\alpha f)(w), f(w))_{\mathcal{D}}\,d\mu \\ &= \int_{\mathbf{T}}(\mathcal{P}_0(\alpha f)(w), \alpha(w)f(w))_{\mathcal{D}}\,d\mu = (\mathcal{P}_0(\alpha f), \alpha f) \ge 0\end{aligned}$$

since $\mathcal{P}_0$ is a projection. Therefore, $\widetilde{Q}_p \ge 0$ and $\widetilde{T}$ is p-hyponormal in $\widehat{\mathcal{H}}$. Again as in [4], we have

$$\begin{aligned}(\mathcal{S}^+_{\widehat{U}}(|\widetilde{T}|^{2p})f)(w) &= [\alpha(w)^*\alpha(w) + \beta(w)]f(w), \\ (\mathcal{S}^-_{\widehat{U}}(|\widetilde{T}|^{2p})f)(w) &= \beta(w)f(w).\end{aligned}$$

Hence

$$\begin{aligned}(\mathcal{S}^+_{\widehat{U}}(\widetilde{T})f)(w) &= \widehat{U}[\mathcal{S}^+_{\widehat{U}}(|\widetilde{T}|^{2p})]^{1/(2p)}f(w) \\ &= w[\alpha(w)^*\alpha(w) + \beta(w)]^{1/(2p)}f(w), \\ (\mathcal{S}^-_{\widehat{U}}(\widetilde{T})f)(w) &= \widehat{U}[\mathcal{S}^-_{\widehat{U}}(|\widetilde{T}|^{2p})]^{1/(2p)}f(w) = w[\beta(w)]^{1/(2p)}f(w).\end{aligned}$$

Next, since

$$\begin{aligned}(\widetilde{T}_{[k]}f)(w) &= \widehat{U}[(1-k)\mathcal{S}^-_{\widehat{U}}(|\widetilde{T}|^{2p}) + k\mathcal{S}^+_{\widehat{U}}(|\widetilde{T}|^{2p})]^{1/(2p)}f(w) \\ &= w[\beta(w) + k\alpha(w)^*\alpha(w)]^{1/(2p)}f(w),\end{aligned}$$

we have

$$(\widetilde{T}_{[k]}f)(w) = w[\beta(w) + k\alpha(w)^*\alpha(w)]^{1/(2p)}f(w).$$

This completes the proof.

COROLLARY 2. *Under the assumptions of Theorem 1, the operators $\widetilde{T}_{[k]}$ are normal in $\widehat{\mathcal{H}}$.*

Proof. This follows from (5).

THEOREM 3. *Let $T = U|T|$ be a p-hyponormal operator in $\mathcal{H}$ such that U is unitary. Then there exist a function space $\widehat{\mathcal{H}}$ and operators $\widetilde{T}$ and $\widehat{U}$ in $\widehat{\mathcal{H}}$ which have the singular integral model in Theorem 1 such that*

$$WTW^{-1} = \widetilde{T} \quad \textit{and} \quad WUW^{-1} = \widehat{U},$$

where W is a unitary operator from $\mathcal{H}$ to $\widehat{\mathcal{H}}$. Moreover, the function α in this model satisfies $\alpha(\cdot) \ge 0$.

Proof. Put $S = U|T|^{2p}$. Since $S \in \mathrm{SHU}$, by Theorem B there exist a function space $\widehat{\mathcal{H}}$, operators $\widehat{S}$, $\widehat{U}$ and a unitary operator W from $\mathcal{H}$ to $\widehat{\mathcal{H}}$ such that

$$WSW^{-1} = \widehat{S} \quad \text{and} \quad WUW^{-1} = \widehat{U}.$$

Hence the model $\widehat{S}$ of S is

$$(\widehat{S}f)(w) = w[\alpha(w)(\mathcal{P}(\alpha f))(w) + \beta(w)f(w)]$$

and

$$(\widehat{U}f)(w) = wf(w).$$

Since we also have

$$\alpha = W(\mathcal{S}^+_U(|T|^{2p}) - \mathcal{S}^-_U(|T|^{2p}))^{1/2}W^{-1} \quad \text{and} \quad \beta = W(\mathcal{S}^-_U(|T|^{2p}))W^{-1},$$

it follows that $\alpha(\cdot) \ge 0$ and

$$(|\widehat{S}|f)(w) = \alpha(w)(\mathcal{P}(\alpha f))(w) + \beta(w)f(w). \tag{7}$$

Since

$$|\widehat{S}| = ((WSW^{-1})^*(WSW^{-1}))^{1/2} \quad \text{and} \quad |S| = |T|^{2p},$$

we have

$$W|T|W^{-1} = (W|T|^{2p}W^{-1})^{1/(2p)} = (W|S|W^{-1})^{1/(2p)} = |\widehat{S}|^{1/(2p)}. \tag{8}$$

Here we put $A = |\widehat{S}|^{1/(2p)}$ and define $\widetilde{T} = \widehat{U}A$. Then by (8) we have

$$(9)\qquad \begin{aligned}(WTW^{-1}f)(w) &= (WU|T|W^{-1}f)(w) = (WUW^{-1}W|T|W^{-1}f)(w)\\ &= (WUW^{-1}(W|T|^{2p}W^{-1})^{1/(2p)}f)(w)\\ &= (\widehat{U}|\widehat{S}|^{1/(2p)}f)(w) = (\widehat{U}Af)(w).\end{aligned}$$

From Theorem 1 and (9), $\widetilde{T}$ is a singular integral model of $T(\in p\text{-HU})$. This completes the proof.

THEOREM 4. *Let $T \in p$-HU. If T is completely p-hyponormal, then the measure ν, appearing in the singular integral model corresponding to T and $\mathcal{H}$, satisfies $\nu \equiv 0$.*

Proof. Suppose $\nu \not\equiv 0$ and ν is concentrated in $F \in \Sigma$ and $m(F) = 0$. Put $\mathcal{M} = \{f \mid f \in \widehat{\mathcal{H}},\ f(w) = 0 \text{ for } w \notin F\}$. Then, as in [4], Corollary III.3.3, we have

$$(\widetilde{T}f)(w) = w[\beta(w)]^{1/(2p)}f(w),$$
$$(\widetilde{T}^*f)(w) = w^{-1}[\beta(w)]^{1/(2p)}f(w)$$

for $f \in \mathcal{M}$. Therefore $\mathcal{M}$ is a reducing subspace for $\widetilde{T}$ of $\widehat{\mathcal{H}}$ and it is evidently non-trivial, which contradicts the fact that $\widetilde{T}$ is completely p-hyponormal. This ends the proof.

3. Riemann–Hilbert problem. Throughout this section, we use the same notation as in Sections 1 and 2.

We only consider the singular integral model $\widetilde{T} = \widehat{U}|\widetilde{T}|$ of a p-hyponormal operator $T = U|T|$ given in Theorem 1. In particular, from Theorem 3, we may assume that α is positive. For an operator S, $\varrho(S)$ denotes the resolvent set of S. Recall that $w = e^{i\theta}$ throughout.

We define a bounded linear operator K from $\mathcal{D}$ to $\widehat{\mathcal{H}}$ as follows:

$$(Ka)(w) = \alpha(w)a, \qquad a \in \mathcal{D}.$$

The dual operator K^*, as an operator from $\widehat{\mathcal{H}}$ to $\mathcal{D}$, is

$$K^*f = \frac{1}{2\pi}\int \alpha(w)f(w)\,d\theta, \qquad f \in \widehat{\mathcal{H}}.$$

Then, from (4) and the definition of $\mathcal{P}_0$, we have $KK^* = \widetilde{Q}_p$.

Next, we define $R_{\widetilde{T}}(w,l)$ for any $l \in \varrho(\mathcal{S}^-_{\widehat{U}}(|\widetilde{T}|^{2p}))$ as follows:

$$R_{\widetilde{T}}(w,l) = I + \alpha(w)(\beta(w) - l)^{-1}\alpha(w).$$

Then we have the following properties.

(i) Since $\mathcal{S}^{\pm}_{\widehat{U}}(|\widetilde{T}|^{2p} - l) = (\mathcal{S}^{\pm}_{\widehat{U}}(|\widetilde{T}|))^{2p} - l$ ([4], Cor. II.1.4), it follows that, for $l \in \mathbb{C}$,

$$(\mathcal{S}^+_{\widehat{U}}(|\widetilde{T}|^{2p} - l)f)(w) = (\alpha(w)^2 + \beta(w) - l)f(w),$$
$$(\mathcal{S}^-_{\widehat{U}}(|\widetilde{T}|^{2p} - l)f)(w) = (\beta(w) - l)f(w).$$

(ii) From (i), for $l \in \varrho(\mathcal{S}^-_{\widehat{U}}(|\widetilde{T}|^{2p}))$, we have

$$R_{\widetilde{T}}(w,l)\alpha(w) = \alpha(w)(\mathcal{S}^-_{\widehat{U}}((|\widetilde{T}|^{2p} - l)^{-1}))(w)(\mathcal{S}^+_{\widehat{U}}(|\widetilde{T}|^{2p} - l))(w),$$
$$\alpha(w)R_{\widetilde{T}}(w,l) = (\mathcal{S}^+_{\widehat{U}}(|\widetilde{T}|^{2p} - l))(w)(\mathcal{S}^-_{\widehat{U}}((|\widetilde{T}|^{2p} - l)^{-1}))(w))\alpha(w).$$

(iii) From (ii), for $l \in \varrho(\mathcal{S}^+_{\widehat{U}}(|\widetilde{T}|^{2p})) \cap \varrho(\mathcal{S}^-_{\widehat{U}}(|\widetilde{T}|^{2p}))$, the operator $R_{\widetilde{T}}(w,l)$ is invertible and

$$R_{\widetilde{T}}(w,l)^{-1} = I - \alpha(w)(\beta(w) + \alpha(w)^2 - l)^{-1}\alpha(w).$$

Furthermore, we define $E_{\widetilde{T}}(z,l)$ as follows:

$$(10)\qquad E_{\widetilde{T}}(z,l) = I - K^*(\widehat{U} - z)^{-1}\widehat{U}(|\widetilde{T}|^{2p} - l)^{-1}K$$

if $z \in \varrho(\widehat{U})$ and $l \in \varrho(|\widetilde{T}|^{2p})$. And, for $a \in \mathcal{D}$ and $l \in \varrho(|\widetilde{T}|^{2p})$, we also define $f(w;l,a)$ as follows:

$$f(w;l,a) = (|\widetilde{T}|^{2p} - l)^{-1}(Ka)(w).$$

Then the following theorem holds.

THEOREM 5. *Let $E_{\widetilde{T}}(z,l)$ be the operator of* (10). *Then, for $a \in \mathcal{D}$, $z \in \varrho(\widehat{U})$ and $l \in \varrho(|\widetilde{T}|^{2p})$,*

$$E_{\widetilde{T}}(z,l)a = a - \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{\alpha(w)f(w;l,a)}{w - z}\,dw.$$

Proof. This follows from

$$E_{\widetilde{T}}(z,l)a = a - \frac{1}{2\pi}\int_0^{2\pi} \frac{w\alpha(w)f(w;l,a)}{w - z}\,d\theta.$$

Let $\mathcal{P}_- = \mathcal{P}$ and $\mathcal{P}_+ = I - \mathcal{P}$. We note that our $\mathcal{P}_-$ means $\mathcal{P}_+$ of [2] (p. 161). The boundary functions $E_{\widetilde{T}}(w(1 \pm 0), l)a$ are defined by

$$\lim_{r \to 1\pm 0} \int_0^{2\pi} \|E_{\widetilde{T}}(rw,l)a - E_{\widetilde{T}}(w(1 \pm 0),l)a\|^2_{\mathcal{D}}\,d\theta = 0.$$

Then we have the following theorems.

THEOREM 6. *Let $E_{\widetilde{T}}(z,l)$ be the operator of* (10). *Then, for $a \in \mathcal{D}$ and $l \in \varrho(|\widetilde{T}|^{2p})$,*

$$E_{\widetilde{T}}(w(1-0),l)a = a - \mathcal{P}_-(\alpha(\cdot)f(\cdot;l,a))(w).$$

Proof. Since, by (1),

$$(\mathcal{P}_- f)(w) = \lim_{r\to 1-0} \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{f(e^{i\eta})}{e^{i\eta}-rw}\, de^{i\eta},$$

we have, from Theorem 5,

$$\begin{aligned} E_{\widetilde{T}}(rw,l)a &= a - \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{\alpha(e^{i\eta})f(e^{i\eta};l,a)}{e^{i\eta}-rw}\, de^{i\eta} \\ &\to a - \mathcal{P}_-(\alpha(\cdot)f(\cdot;l,a))(w) \quad (r\to 1-0). \end{aligned}$$

This completes the proof.

THEOREM 7. *Let $E_{\widetilde{T}}(z,l)$ be the operator of* (10). *Then, for $a \in \mathcal{D}$ and $l \in \varrho(|\widetilde{T}|^{2p})$,*

$$E_{\widetilde{T}}(w(1+0),l)a = a + \mathcal{P}_+(\alpha(\cdot)f(\cdot;l,a))(w).$$

Proof. By Privalov's theorem ([2], Th. 5.5 of p. 161), any $f \in L^2(\mathcal{D})$ has non-tangential limit

$$\lim_{r\to 1+0} \int_{\mathbf{T}} \frac{f(w)}{e^{i\eta}-rw}\, de^{i\eta}$$

at almost every point w. Put

$$\widetilde{P}(f)(w) = \lim_{r\to 1+0} \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{f(e^{i\eta})}{e^{i\eta}-rw}\, de^{i\eta}.$$

The Sokhotskiĭ–Plemelj–Privalov formula ([2], p. 162) implies that

$$\mathcal{P}_- - \widetilde{P} = I.$$

Hence $\widetilde{P} = -(I-\mathcal{P}_-) = -\mathcal{P}_+$. Therefore

$$\begin{aligned} E_{\widetilde{T}}(w(1+0),l)a &= a - \lim_{r\to 1+0}\frac{1}{2\pi i}\int_{\mathbf{T}} \frac{\alpha(e^{i\eta})f(e^{i\eta};l,a)}{e^{i\eta}-rw}\, de^{i\eta} \\ &= a - \widetilde{P}(\alpha(\cdot)f(\cdot;l,a))(w) = a + \mathcal{P}_+(\alpha(\cdot)f(\cdot;l,a))(w). \end{aligned}$$

This completes the proof.

From the above assertions, we have the following theorem. The proof is analogous to that of Theorem 1.1 of [4], p. 101, but we give it for completeness.

THEOREM 8. *Let $\widetilde{T} = \widehat{U}|\widetilde{T}| \in$ p-HU. Then, for $l \in \varrho(|\widetilde{T}|^{2p})$, $E_{\widetilde{T}}(\cdot,\cdot)$ is the solution of the following Riemann–Hilbert problem:*

$$R_{\widetilde{T}}(w,l)E_{\widetilde{T}}(w(1-0),l) = E_{\widetilde{T}}(w(1+0),l).$$

Proof. Since $(|\widetilde{T}|^{2p}-l)f(w;l,a) = \alpha(w)a$ for $a \in \mathcal{D}$, by (7) we have

$$(\beta(w)-l)f(w;l,a) + \alpha(w)\mathcal{P}_-(\alpha(\cdot)f(\cdot;l,a))(w) = \alpha(w)a,$$

and hence

$$\begin{aligned} \alpha(w)f(w;l,a) + \alpha(w)(\beta(w)-l)^{-1}\alpha(w)\mathcal{P}_-(\alpha(\cdot)f(\cdot;l,a))(w) \\ = \alpha(w)(\beta(w)-l)^{-1}\alpha(w)a. \end{aligned}$$

From (iii),

$$(I-\mathcal{P}_-)(\alpha(\cdot)f(\cdot;l,a))(w) = R_{\widetilde{T}}(w,l)(a - \mathcal{P}_-(\alpha(\cdot)f(\cdot;l,a))(w)) - a.$$

Since $I - \mathcal{P}_- = \mathcal{P}_+$, we have

$$a + \mathcal{P}_+(\alpha(\cdot)f(\cdot;l,a))(w) = R_{\widetilde{T}}(w,l)(a - \mathcal{P}_-(\alpha(\cdot)f(\cdot;l,a))(w)).$$

Hence the theorem follows from Theorems 6 and 7.

Acknowledgements. We would like to express our thanks to the referee for his useful advice.

References

[1] M. Chō and M. Itoh, *On spectra of p-hyponormal operators*, Integral Equations Operator Theory 23 (1995), 287–293.

[2] M. Martin and M. Putinar, *Lectures on Hyponormal Operators*, Birkhäuser, Basel, 1989.

[3] D. Xia, *On the non-normal operators—semi-hyponormal operators*, Sci. Sinica 23 (1980), 700–713.

[4] —, *Spectral Theory of Hyponormal Operators*, Birkhäuser, Basel, 1983.

Department of Mathematics
Kanagawa University
Yokohama 221, Japan
E-mail: m-cho@cc.kanagawa-u.ac.jp

Faculty of Education
Niigata University
Niigata 950-21, Japan
E-mail: huruya@ed.niigata-u.ac.jp

Asuka Senior High School
Ōji 6-8-8, Kita-ku
Tokyo 114, Japan
E-mail: mitoh@mxk.meshnet.or.jp

Received December 23, 1996
Revised version January 9, 1998
(3809)