Almost 1-1 extensions of Furstenberg–Weiss type and applications to Toeplitz flows

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Abstract. Let \((Z, T_Z)\) be a minimal non-periodic flow which is either symbolic or strictly ergodic. Any topological extension of \((Z, T_Z)\) is Borel isomorphic to an almost 1-1 extension of \((Z, T_Z)\). Moreover, this isomorphism preserves the affine-topological structure of the invariant measures. The above extends a theorem of Furstenberg–Weiss (1989). As an application we prove that any measure-preserving transformation which admits infinitely many rational eigenvalues is measure-theoretically isomorphic to a strictly ergodic Toeplitz flow.

Introduction. In 1989, Furstenberg and Weiss proved a theorem [F-W, Theorem 1] which can be informally expressed as follows: every topological point-transitive flow \((X, T_X)\) which is an extension of a minimal non-periodic flow \((Z, T_Z)\) is in some sense equivalent to a minimal flow \((Y, T_Y)\) which is an almost 1-1 extension of \((Z, T_Z)\). The equivalence is given by a Borel measurable injective map \(\phi\) defined on a subset \(X' \subset X\) whose mass is 1 for any \(T_X\)-invariant probability measure carried by \(X\). Such a Borel embedding provides a 1-1 affine map \(\phi^*\) (defined as the adjoint map on measures) from the set \(P(X)\) of all \(T_X\)-invariant probability measures carried by \(X\) into the set \(P(Y)\) defined analogously for the flow \((Y, T_Y)\). Moreover, for every \(\mu \in P(X)\), \(\phi^*\) is a measure-theoretic isomorphism between the measure-preserving transformations \((X, B_X, \mu, T_X)\) and \((Y, B_Y, \phi^*(\mu), T_Y)\) (here \(B_X\) and \(B_Y\) denote the \(\sigma\)-fields of Borel measurable sets in \(X\) and \(Y\), respectively).

In this paper we improve the Furstenberg–Weiss theorem. By the methods of symbolic dynamics we obtain a stronger isomorphism under even

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weaker assumptions: dropping transitivity of \((X,T_X)\) we construct \((Y,T_Y)\)
and \(\phi\) such that in addition to all previous properties, \(\phi^*\) is a homeomor-
phism between \(P(X)\) and \(P(Y)\) for the weak* topology of measures. This
is partially achieved by obtaining the image \(Y' = \phi(X')\) of mass 1 for every
\(\nu \in P(Y)\), and partially by controlling the frequencies with which blocks oc-
cur in sequences. This kind of isomorphism (which we call "Borel*") is proba-
bly the best one can expect to exist between a nearly arbitrary topological
flow (the only restriction is that it admits a minimal non-periodic factor) and
a minimal one. Clearly, obtaining a topological isomorphism is impossible,
nevertheless, our isomorphism behaves like one at the level of invariant
measures. In particular, in virtue of the variational principle, topological entropy is
preserved (this was not guaranteed by the original version of the theorem).

Our proof is based on combinatorial constructions for symbolic flows
(subshifts). Most operations have their direct translations to the general
topological case, for instance observing repeating blocks along a sequence
corresponds to finding return times of an orbit to a fixed open set. Some
tricks, however, like replacing each occurrence of a block by another block
of the same length, or permuting certain letters within a block, might lead
to a rather complicated description when translated to the general topolog-
ical language. This is why we decided to state the main result for subshifts.
Later we discuss the possibility of extending it to the general case. An ad-
ditional advantage of such a formulation is that obtaining \((Y,T_Y)\) symbolic in
the case of \((X,T_X)\) symbolic (and transitive) does not follow directly from the
original Furstenberg–Weiss theorem (it can be derived from it via a theorem
of Denker–Keane [D-K, Theorem 20], but then it works for a fixed measure
on \((X,T_X)\) only).

At the end of the section devoted to the symbolic case we make a digres-
sion on the special type of codes we exploit.

In the last section, as an application of the results obtained, we present
a measure-theoretic (and Borel*) characterization of Toeplitz flows. In par-
icular, some previous results on possible point spectra of Toeplitz flows
obtained in [I-L], [I], and [D-L] are recovered.

The symbolic theorem. By a topological dynamical system (flow) we
mean a pair \((X,T_X)\), where \(X\) is a compact metrizable space and \(T_X\) is a
homeomorphism of \(X\) onto itself. We will denote by \(P(X)\) the collec-
tion of all \(T_X\)-invariant Borel probability measures on \(X\). It is known that this set
is convex (even a simplex) and compact for the weak* topology of measures.
A Borel subset \(X' \subset X\) is called a full set if every measure \(\mu \in P(X)\) assigns
mass 1 to it.

By a Borel* isomorphism between two flows \((X,T_X)\) and \((Y,T_Y)\) we
understand a Borel measurable invertible map \(\phi : X' \rightarrow Y'\) between full
sets \(X' \subset X\) and \(Y' \subset Y\) such that \(\phi \circ T_X = T_Y \circ \phi\) and the adjoint
map \(\phi^* : P(X) \rightarrow P(Y)\), defined by

\[\phi^*(\mu)(A) = \mu(\phi^{-1}(A))\]

(for any Borel set \(A \subset Y\), is an affine homeomorphism for the weak* to-

In symbolic dynamics one considers subshifts, i.e., shifts \((Z,S)\), where
\(S\) denotes the shift transformation on \(A^Z\) and \(Z\) is a shift-invariant closed
subset of \(A^Z\). The set \(A\) is called the alphabet. Unless otherwise specified,
we assume the alphabets appearing in this paper to be finite.

A block over the alphabet \(A\) is a \(k\)-tuple \(B = (\lambda_0, \lambda_1, \ldots, \lambda_{k-1}) \in A^k\).
We denote by \([B]\) the length \(k \in \mathbb{N}\) of the block \(B\). We say that \(B\) occurs in
a sequence \(x \in A^\mathbb{N}\) if \(x(n), x(n+1), \ldots, x(n+k-1) = B\) for some \(n \in \mathbb{Z}\).
The integer interval \([n, n+k) = \{n, n+1, \ldots, n+k-1\}\) is then called
the domain of the occurrence. Given finitely many blocks \(B_1, \ldots, B_n\), we can
build their concatenation, i.e., the block \(B = B_1 \ldots B_n\). We say that a block
\(C\) starts with \(B\) if \(C = B\) or \(C = BD\) for some block \(D\).

It is well known that for a minimal subshift \((Z,S)\) every block which
occurs in some \(x \in Z\) occurs in each element of \(Z\) syntactically, i.e., it occurs
arbitrarily far in both directions and the distances between consecutive
occurrences are bounded. We say that a block \(B\) has non-overlapping occurrences
if for any \(x \in Z\) the domains of any two different occurrences of \(B\) in
\(x\) are disjoint:

\[x = \ldots B \ldots B \ldots B \ldots B \ldots B \ldots B \ldots B \ldots \]

Clearly, all blocks of length 1 have this property. If \(B\) has non-overlapping occurrences then by a \(B\)-block we mean any block \(B\) which starts with \(B\), and such that:

\bullet \(B\) cannot be written as a concatenation involving two occurrences of
\(B\), and
\bullet \(B \ldots B\) occurs in some \(x \in Z\).

By minimality, the lengths of all \(B\)-blocks are bounded, hence the col-
clection of all \(B\)-blocks is finite. Every \(x \in Z\) can be represented in a unique
way as a concatenation of \(B\)-blocks.

The following fact is the starting point of our construction:

**Lemma.** Assume \((Z,S)\) is a minimal non-periodic subshift. Let \(B\) be
a block having non-overlapping occurrences in \(Z\). Then there exist arbitrarily
long blocks starting with \(B\) and having non-overlapping occurrences.

**Proof.** Let \(B_\ldots B\) denote a fixed \(B\)-block. Suppose \(B \ldots B\) has overlapping
occurrences. This implies that \(B \ldots B \ldots B\) occurs in \(Z\). If the last block
has overlapping occurrences then $B \ldots B \ldots B \ldots B$ occurs in $Z$, and so on. By minimality and non-periodicity, some block $C = B \ldots B \ldots B \ldots \ldots B \ldots B$ (essentially longer than $B$) has non-overlapping occurrences. Repeating the same argument for $C$, and so on, we can obtain arbitrarily long blocks of the required form.

It is important to note that

1. if $C$ starts with $B$ and is sufficiently long then it starts with a concatenation of $B$-blocks, while
2. each $C$-block is a concatenation of $B$-blocks.

Recall that a factor map between two flows $(X, T_X)$ and $(Z, T_Z)$ is a continuous surjective map $\pi : X \to Z$ such that $\pi \circ T_X = T_Z \circ \pi$. For a given factor map $\pi$, its fibers are the preimages of points. We say that $\pi$ provides an almost 1-1 extension if the subset of points of $Z$ having one-point fibers is residual. If $(Z, T_Z)$ is minimal then to establish that the extension is almost 1-1 it suffices to show that a one-point fiber exists. Almost 1-1 extensions play an important role in topological dynamics. Many topological properties pass to almost 1-1 extensions (for instance see [A] for topological disjointness).

The main result of this paper is the following symbolic version of the Furstenberg–Weiss theorem:

**Theorem.** Let $(Z, S)$ be a minimal non-periodic subshift over an alphabet $\Lambda$, and let $(X, S)$ be a subshift over an alphabet $\Sigma$. Suppose there exists a factor map $\pi_X : X \to Z$. Then there exists a minimal subshift $(Y, S)$ (over a new alphabet $\Sigma$) and a commutative diagram

$$
\begin{array}{cccc}
X & \phi & Y \\
\pi_X & \downarrow & \pi_Y \\
Z & & &
\end{array}
$$

where $\phi$ is a Borel* isomorphism, and $\pi_Y$ provides an almost 1-1 extension.

**Proof.** By Lemma 1 and by (1), we can choose inductively two sequences of blocks $B_t$ and $C_t$ (over $\Lambda$) appearing in $Z$ such that for each $t \geq 1$,

1. $B_t$ and $C_t$ have non-overlapping occurrences in $Z$,
2. $C_t$ starts with a concatenation of $2t + 2 B_t$-blocks, where $l_t$ is the length of the initial $B_t$-block in $C_t$,
3. $B_{t+1}$ starts with a concatenation of $C_t$-blocks so long that every existing $C_t$-block (in $Z$) is used in it at least $r_t = (\#\Sigma(\#\Sigma + 1))^{m_t}$ times, where $m_t$ is the maximal length of a $C_t$-block.

We denote by $B_{t::}$ the $B_t$-block with which $C_t$ starts (hence each $C_t$-block also starts with $B_{t::}$):

$$
\begin{array}{cccccccccccc}
\cdots & B_{t::} B_{t:} B_t & \cdots & B_{t::} B_{t:} B_t & \cdots & B_{t::} B_{t:} B_t & \cdots & B_{t::} B_{t:} B_t & \cdots & B_{t::} B_{t:} B_t & \cdots & B_{t::} B_{t:} B_t & \cdots & C_t
\end{array}
$$

For fixed $t \geq 1$, every $z \in Z$ can be represented in a unique way as an infinite concatenation of $C_t$-blocks, each decomposing into at least $2t + 2 B_t$-blocks.

Let $\Sigma = \Sigma \times (\Sigma \cup \{\square\}) \times \Lambda$, where $\square$ is an additional symbol. By *letters* we mean the elements of $\Sigma$. We view the elements of $\Sigma$ as columns of height 3, hence the sequences over $\Sigma$ are represented as three sequences (rows): the top row containing letters, the central row containing letters and squares, and the bottom row containing elements of $\Lambda$. The positions in the central row will be called *cells*. A cell can be *occupied* or *empty* depending on whether it contains a letter or a square. To start the construction, we treat each element $x \in X$ as the top row and we add two rows below it: the central row consisting entirely of empty cells, and the bottom row identical with $\pi_X(x)$. Since $\pi_X$ is continuous, this procedure yields a topologically isomorphic representation of $(X, S)$ as a subshift over the alphabet $\Sigma$. From now on $(X, S)$ stands for this representation.

$$
\begin{array}{cccccccccccc}
\cdots & \sigma_{-1} & \sigma_0 & \sigma_1 & \sigma_2 & \cdots & \cdots & \square & \square & \square & \square & \cdots & \cdots & \cdots & \cdots
\end{array}
\Longleftrightarrow \begin{array}{cccccccccccc}
\text{top row, letters, } x \\
\text{middle row, cells}
\end{array}
\begin{array}{cccccccccccc}
\cdots & \lambda_{-1} & \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \cdots & \pi_X(x)
\end{array}
$$

Consider a block over the alphabet $\Sigma$ such that its bottom row is a $C_t$-block. We call every such block a *t-train*. Observe that

1. there exist not more than $r_t$ different $t$-trains with a common bottom row.

We have the following decomposition of the $t$-trains:

1. each $t$-train decomposes into a concatenation of a *locomotive* having $B_{t::}$ in the bottom row (hence of length $l_t$), followed by at least $2t + 1$ *wagons* (having further $B_t$-blocks in the bottom row).
We must remember that, except for \( t = 1 \),

(8) the locomotive and the wagons are concatenations of \((t - 1)\)-trains.

It follows immediately from the representation of \((X, S)\) as a subshift over \(\Sigma\) that for each \( t \geq 1 \) every \( x \in X \) can be decomposed in a unique way as an infinite concatenation of \( t \)-trains. Obviously, by the construction, the positioning of the component \( t \)-trains, their locomotives and wagons is determined by the third row \( \pi_X(x) \). The \( t \)-trains occurring in \( X \) will be called original \( t \)-trains.

We will soon define a sequence of maps \( \phi_t \) on \( X \) into some subshifts over \( \Sigma \). Each of the maps \( \phi_t \) will be obtained by a code replacing consecutively the original \( t \)-trains by other \( t \)-trains. At most coordinates, \( \phi_{t+1}(x) \) coincides with \( \phi_t(x) \). The only differences are due to the modifications described below. The idea is to introduce certain syntactically repeating new blocks (modification (B)) without forgetting the letters which these blocks would overwrite. To achieve this goal we first have to “memorize” these letters by copying them into the empty cells in the middle row (modification (A)).

**Step 1.** Let \( W_1 \) be an arbitrary block over the alphabet \( \Sigma \) (\( W_1 \) need not occur in \( X \)) having \( B_1::: \) in the bottom row (hence of length \( l_1 \)). The \( 1 \)-code is defined as a transformation of the original \( 1 \)-trains by applying the following two modifications:

(A) using consecutively all the letters occurring in the top row of the locomotive we fill in the terminal empty cells in each of the next \( l_1 \) wagons,

(B) we replace the locomotive by the new locomotive \( W_1 \).

By regular \( 1 \)-trains we mean the images of the original \( 1 \)-trains under the \( 1 \)-code.

---

\[
\begin{array}{cccccccc}
\sigma_1 \sigma_2 \cdots \sigma_t & \square & \square & \cdots & \square & \square & \cdots & \square \\
B_1::: & B_1 & B_1 & \cdots & B_1 & B_1 & \cdots & B_1 \\
\end{array}
\]

original \( 1 \)-train

\[
\begin{array}{cccccccc}
\square & \square & \cdots & \square & \square & \cdots & \square & \square & \cdots & \square & \square & \cdots & \square \\
B_1::: & B_1 & B_1 & \cdots & B_1 & B_1 & \cdots & B_1 & B_1 & \cdots & B_1 & B_1 & \cdots & B_1 \\
\end{array}
\]

regular \( 1 \)-train

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Note that the \( 1 \)-code does not affect the terminal wagon of the \( 1 \)-train and thus the terminal cell of each regular \( 1 \)-train remains empty. Next, since \( W_1 \) has \( B_1::: \) in the bottom row, the entire bottom row of each \( 1 \)-train remains unchanged. Finally, observe that the original \( 1 \)-train can be reconstructed from its image by removing the upper two rows of the locomotive and filling the top row by the letters occupying the terminal cells of the next \( l_1 \) wagons (at the same time we empty those cells). Thus, it is clear that the \( 1 \)-code is a \( 1 \)-1 correspondence between the original and regular \( 1 \)-trains.

**Inductive assumption.** Let \( t \in N \) and suppose that

(9) a \( t \)-code has been defined as a \( 1 \)-1 correspondence between the original \( t \)-trains and their images called regular \( t \)-trains,

(10) the bottom row is unchanged by the \( t \)-code,

(11) the terminal cell of each regular \( t \)-train is empty.

**Step \( t + 1 \).** We create a block \( W_{t+1} \) over the alphabet \( \Sigma \) (\( W_{t+1} \) not necessarily occurring in \( \phi_t(x) \)) so that:

(12) the bottom row of \( W_{t+1} \) is \( B_{t+1}::: \),

(13) \( W_{t+1} \) is a concatenation of regular \( t \)-trains,

(14) every regular \( t \)-train is used at least once in the above concatenation (this is possible by (5) and (6)).

We define the \((t+1)\)-code on the original \((t+1)\)-trains in the following way: we first replace all the original \( t \)-trains into which the given original \((t+1)\)-train decomposes (see (8)) by their images under the \( t \)-code. The \((t+1)\)-train so obtained will be called the \( t \)-coded \((t+1)\)-train. Next we apply the following two modifications:

(A) using consecutively all the letters and squares occurring in the top and middle rows of the locomotive of the \( t \)-coded \((t+1)\)-train we fill in the terminal empty cells in each of the next \( 2l_{t+1} \) wagons (by (7) there are enough wagons; observe that each wagon of a \( t \)-coded \((t+1)\)-train is a concatenation of regular \( t \)-trains, hence, by (11) its terminal cell is empty),

(B) we replace the locomotive by \( W_{t+1} \).

Note that the above modifications do not affect the terminal wagon of the \( t \)-coded \((t+1)\)-train. Thus the terminal cell remains empty, as required in (11). Clearly, by (12), the bottom row is unchanged, as required in (10). We can reverse the modifications (A) and (B) by emptying the two upper rows of the locomotive and filling them back with the letters and squares appearing in the terminal cells of the next \( 2l_{t+1} \) wagons (at the same time we empty these cells). Next, the original \((t+1)\)-train can be recovered from the \( t \)-coded \((t+1)\)-train by reversing the \( t \)-code (use the inductive assumption (9)). Thus the \((t+1)\)-code is a \( 1 \)-1 correspondence, as required in (9).

**End of induction.** The following obvious observations are important:

(15) the modification (B) replaces regular \( t \)-trains by other regular \( t \)-trains preserving the bottom row (see (13) and (12)),

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(16) the modification (B) preserves all the locomotives \( W_s \) with \( s \leq t \) introduced by the \( t \)-code

(because the distribution of these locomotives within a regular \( t \)-train depends only on the third row, and after step \( t \) there are no locomotives other than \( W_s \)).

During the modifications (A) and (B) in step \( t + 1 \) each (regular) \( t \)-train of the \( t \)-coded \((t + 1)\)-train can be either left unaffected, or replaced by another regular \( t \)-train (modification (B)), or it can happen that a letter will be inserted into its terminal cell (modification (A)). A \( t \)-train differing from a regular one by having the terminal cell occupied will be called an **irregular \( t \)-train**. Thus,

(17) every regular \((t + 1)\)-train is a concatenation of regular and irregular \( t \)-trains.

Later we also use a reverse procedure. A given regular \( t \)-train can be \( t \)-decoded, i.e., replaced by its (unique) preimage under the \( t \)-code. Moreover, we can also \( t \)-decode an irregular \( t \)-train simply disregarding the letter occupying the terminal cell. For instance, we can \( t \)-decode a regular (or irregular) \((t + 1)\)-train by \( t \)-decoding all component \( t \)-trains. Comparing the \( t \)-decoded \((t + 1)\)-train with the original \([(t + 1)\]-decoded) \((t + 1)\)-train we can see that the differences result from applying or not applying the reversed modifications (A) and (B), hence

(18) the \( t \)-decoded \((t + 1)\)-train differs from the original \((t + 1)\)-train only in having a different locomotive

(in the first case we have \( t \)-decoded the component \( t \)-trains of \( W_{t+1} \), while in the second case \( W_{t+1} \) has been removed and the original locomotive has been recovered from the terminal cells of the wagons—these cells have been ignored in the first case).

**The maps \( \phi_t \).** Fix \( x \in X \) and \( t \geq 1 \). As noticed before, \( x \) decomposes in a unique way into an infinite concatenation of original \( t \)-trains. We define \( \phi_t(x) \) as the sequence obtained from \( x \) by replacing each original \( t \)-train in \( x \) by its image under the \( t \)-code. Note that the locomotives of all the \( t \)-trains of \( \phi_t(x) \) are \( W_t \), hence

(19) \( W_t \) occurs in \( \phi_t(x) \) syndetically.

It is easily seen that \( \phi_t \) is continuous, injective and commutes with the shift transformation. Thus \( \phi_t \) provides a topological isomorphism between \((X, S)\) and \((\phi_t(X), S)\). We will be using the following facts:

(20) \( \phi_t(x) \) is a concatenation of regular \( t \)-trains,

(21) for each \( s > t \), \( \phi_s(x) \) is a concatenation of regular and irregular \( t \)-trains.

The last statement is obvious for \( s = t + 1 \) (see (17)). For larger \( s \) use an inductive argument and the observation that the modification (B) replaces regular \((s - 1)\)-trains by other regular \((s - 1)\)-trains (see (15)), while modification (A) inserts letters into terminal cells of some wagons of the \((s - 1)\)-trains, and these cells are terminal with respect to \( t \)-trains, so some more irregular \( t \)-trains are produced.

**The map \( \phi \) and its domain \( X' \).** Let

\[
Z_{t,n} = \{ z \in Z : n \in \text{domain of the starting } B_t \text{-block } B_{t+1} \ldots \text{in a } C_t \text{-block of } z \}.
\]

Estimating the mass which an invariant probability measure assigns to a set by the maximal frequency at which this set is visited by some trajectory, we can see that each such measure assigns to \( Z_{t,n} \) a mass of at most \( 1/(2t) \). It follows easily from the construction that \( t \) and \( t \) grow exponentially, hence

\[
\sum_{t} 1/t < \infty \quad \text{and} \quad \sum_{t} 1/r_t < \infty.
\]

Thus

\[
X' = \{ x \in X \text{ : each coordinate falls into the domain of a locomotive of a } t \text{-train for at least finitely many indices } t \}.
\]

Observe that

(22) if \( x \in X' \) then every block of \( x \) is subject to at most finitely many modifications during the construction of the sequence \((\phi_t(x)) \) (see argument below).

Modification (B) affects only the locomotives. Modification (A) in step \( t + 1 \) alters the terminal letters of some wagons. By (8), every such letter is followed by a locomotive of a \( t \)-train. Thus if a letter were modified infinitely many times, then this or the following coordinate would fall into the domain of a locomotive for infinitely many indices.

It is now clear that the maps \( \phi_t \) converge (coordinatewise) on \( X' \). Thus the map

\[
\phi = \lim_{t \to \infty} \phi_t
\]

is well defined on the full set \( X' \). Obviously, \( \phi \) is Borel measurable and commutes with the shift. Note that, by (21),
(24) for each $x \in X'$ and $t \geq 1$, $\phi(x)$ is a concatenation of regular and irregular $t$-trains.

**Minimal almost 1-1 extension.** We first need to prove that $\phi$ provides a Borel isomorphism between $(X, S)$ and a subshift $(Y, S)$ which is a minimal almost 1-1 extension of $(Z, S)$.

Consider an $x \in X'$. Set $y = \phi(x)$. Let $B = y[n, m]$ be a block of $y$. By the definition of $X'$, we can find $t$ large enough so that

(25) $[n, m]$ does not intersect any domain of a locomotive for any $s > t$.

Moreover, by (23), we can assume that

$$\phi_{t+1}(x)[n, m] = \phi(x)[n, m] = B.$$ 

It is now seen that $B$ is a part of a regular $(t+1)$-train in $\phi_{t+1}(x)$, thus, due to (14), it occurs as a part of $W_{t+2}$. This implies that $B$ is introduced in a syndetic way in $\phi_{t+2}(x)$ (see (19)). By (16), all these occurrences remain unaltered in further steps, so they occur in $y$. We have proved that $y$ satisfies the well known criterion for having a minimal orbit-closure. Moreover, by the above argument, any block occurring in $y$ also occurs in $y' = \phi(x')$ for any $x' \in X'$ (because it occurs in $W_t$ for some $t$). Hence $\phi(X')$ is contained in one minimal subshift $(Y, S)$ over the alphabet $\Sigma$. It is obvious (by (10)) that each element $y$ of $(Y, S)$ has a subshift $(Z, S)$ in the bottom row, thus $(Y, S)$ is an extension of $(Z, S)$. Denote by $\pi_Y : Y \rightarrow Z$ the projection on the bottom row in $Y$. By minimality of $(Z, S)$, this projection is surjective.

At this point, we can note that the diagram in the assertion of the theorem commutes, because $\phi$ preserves the bottom row.

We now prove that $\pi_Y$ provides an almost 1-1 extension. Let $y = \phi(x)$ ($x \in X'$) and let $z = \pi_Y(y)$. Note that every block of $y$ having $C_t$ in the bottom row starts with $W_t$. By minimality of $(Y, S)$, this property passes to all elements of $Y$. Recall that $C_t$ occurs many (more than 3) times in $B_{t+1}$. Thus we can find a $z_0 \in Z$ such that for each $t \geq 1$ the zero coordinate is contained in the domain of a non-extreme (neither initial nor terminal) occurrence of $C_t$ in the starting $B_{t+1}$ of some occurrence of $C_{t+1}$ in $z_0$.

\[ \begin{array}{cccc}
0 \text{ coordinate} \\
\hline
\cdot & C_t & C_t & C_t \\
\hline
C_{t+1} & C_{t+1} & C_{t+1} & C_{t+1} \\
B_{t+1} & B_{t+1} & B_{t+1} & B_{t+1} \\
\hline
\end{array} \]

It is now seen that any preimage by $\pi_Y$ of $z_0$ has the block $W_{t+1}$ around zero coordinate, and the domains of these blocks expand in both directions as $t \to \infty$. This determines that the preimage is unique, and the almost 1-1 extension is established.

Define $Y' = \pi_Y^{-1}(Z')$. Being the preimage of a full set, $Y'$ is a full set in $(Y, S)$. We show that $\phi$ is an invertible map from $X'$ onto $Y'$. To prove this we construct a map $\psi$ inverse to $\phi$ on $Y'$. First note that

(26) $\phi(X')$ is dense in $Y$.

Obviously, since $\phi$ preserves the bottom row, we have $\phi(X') \subset Y'$. Let $y \in Y$ be such that $\pi_Y(y) \in Z'$. For each $t$, $y$ can be decomposed as a concatenation of regular and irregular $t$-trains (use (24) and (26)). For each $t \geq 1$ define $\omega_t$ as the element obtained by $t$-decoding all $t$-trains of $y$. Compare $\omega_t$ with $\omega_{t+1}$. The differences may occur only in the locomotives of the $(t+1)$-trains (see (18)). On the other hand, since the bottom row is an element of $Z'$, and since the distribution of the locomotives depends only on the bottom row, $y$ satisfies the condition that every coordinate $n$ falls into the domain of a locomotive for at most finitely many indices $t$. Combining the last two statements we find that

(27) $\omega_t$ converges coordinate-wise to some $x$.

We define $\psi(y) = x$. Consider an interval $[n, m]$. For $t$ large enough, $[n, m]$ is contained in the domain of a single $t$-train (satisfying (25) is possible whenever the bottom row belongs to $Z'$). Thus the corresponding block of $x_t$ occurs in $X$ (as a part of an original $t$-train). This implies that $x \in X$, because $X$ is closed. The bottom row of $x$ is the same as $\pi_Y(y) \in Z'$, hence $x \in X'$.

Now, check $\phi(x)$ at a coordinate $n$. As before, by the definition of $Z'$, we can choose $t$ so large that:

(28) for every $s \geq t$ neither $n$ nor $n+1$ are in the domain of the locomotive of an $s$-train.

Thus the regular $t$-train of $\phi(x)$ whose domain $C$ contains $n$ (see (20)) coincides with the corresponding $t$-train of $\phi(x)$, except perhaps for its last cell (by (28) this regular $t$-train is not a part of a larger locomotive, hence when applying the $s$-codes for $s > t$ only modification (A) can affect it).

Hence

(29) the original $t$-train of $x$ with domain $C$ can be obtained by $t$-decoding the corresponding (regular or irregular) $t$-train of $\phi(x)$.

On the other hand, let $t' \geq t$ be so large that

(30) $\pi_{t'}$ coincides with $x$ on $C$.

By the definition of $\pi_{t'}$, the $t'$-train of $\pi_{t'}$ whose domain $C$ is obtained by $t'$-decoding the corresponding $t'$-train of $y$. Since $C$ is not in the domain of any locomotive for any indices between $t$ and $t'$, the $t'$-decoded
Almost 1-1 extensions

For every \( x \geq t \) the upper density of the set of coordinates where \( \phi_t(x) \) differs from \( \phi_n(x) \) is hence less than \( \varepsilon / k(\varepsilon) \) (because \( \phi_t(x) \) differs from \( \phi_n(x) \) along at most \( 3l_{t+2} \) coordinates within the domain of each \( (t+1) \)-train and the length of each \( (t+1) \)-train is at least \( r_{t+1} l_{t+1} \)). This implies that

\[
\text{(34)} \quad \text{the frequencies with which a block } B \in B^t \text{ occurs in } \phi_n(x) \text{ and in } \phi_t(x) \text{ may differ by at most } k(\varepsilon) \varepsilon / k(\varepsilon) = \varepsilon.
\]

Combining (32), (33), the definition of \( B^t \), and (34), we obtain

\[
d^* (\phi_t(\mu), \phi_n(\mu)) < 2 \varepsilon.
\]

This yields the desired uniform convergence for ergodic measures, which, by convexity of the metric \( d^* \), extends to all invariant measures. The map \( \phi^* \) has been proved continuous, hence, as an invertible map between compact sets, it is a homeomorphism. This completes the proof of Theorem 1.

**Remarks on reducing the alphabet.** The size of the alphabet used to define the flow \( (Y, S) \) is \( \# \Sigma (\# \Sigma + 1) \# A \). It might be interesting to note that the flow \( (Y, S) \) can be represented as a subshift over the same alphabet \( \Sigma \) as originally used by \((X, S)\). This is possible thanks to the power of a Borel* isomorphism, more precisely, by the fact that it preserves topological entropy. Namely, we have the following

**Lemma 2.** The subshift \((Y, S)\) of Theorem 1 is topologically isomorphic to a subshift over the alphabet \( \Sigma \).

**Proof.** Set \( p = \# \Sigma \). Consider the following two cases:

(a) the topological entropy \( h(\mu) \) of \((Y, S)\) is equal to \( \ln p \),

(b) \( h(\mu) < \ln p \).

The case (a) is trivial: the flow \((X, S)\) is the full shift over \( \Sigma \) (apply [D-G-S, Theorem 20.11] and some standard arguments). Such a flow contains fixed points, thus it admits no minimal topological factors except for the one-point flow, hence our theory does not apply.

Assume (b). By a well known formula, we have

\[
h(\mu) = \lim_{n \to \infty} \frac{\ln \# B_n}{n},
\]

where \( B_n \) denotes the collection of all blocks of length \( n \) occurring in \( Y \). Because \( h(Y) = h(\mu) < \ln p \), an easy calculation shows that

\[
\# B_{n_0} \leq p^{n_0} - 1 \quad \text{and} \quad \# B_{n_0+1} \leq p^{n_0+1}
\]

for some sufficiently large \( n_0 \). Let \( t_0 \) be such that \( |B_{t_0}| \geq n_0 \) and \( |C_{t_0}| \geq n_0^2 \) (we refer to the objects defined in the proof of Theorem 1). Then every \( C_{t_0} \)-block has length at least \( n_0^2 \). Every block that long can be decomposed as a concatenation of subblocks whose lengths are either \( n_0 \) or \( n_0+1 \). Fix one
such decomposition starting with a subblock of length \( n_0 \) for each \( C_{t_0} \)-block, so that all \( C_{t_0} \)-blocks of the same length are cut in the same places. This induces a decomposition of all \( t_0 \)-trains of \( Y \). The starting subblock of each \( t_0 \)-train consists of the initial \( n_0 \) symbols of the locomotive \( W_{t_0} \), hence is common for all \( t_0 \)-trains. By \((35)\), there exist 1-1 correspondences \( \psi \) from \( B_{n_0} \) into all blocks of length \( n_0 \) over the alphabet \( \Sigma \), and \( \psi_1 \) from \( B_{n_0+1} \) into all blocks of length \( n_0 + 1 \) over \( \Sigma \). Moreover, there remains at least one block \( B_0 \) of length \( n_0 \) over \( \Sigma \) unused as a \( \psi \)-image.

The desired topological isomorphism between \((Y,S)\) and a subshift over \( \Sigma \) is obtained by a code replacing each subblock of each \( t_0 \)-train in \( Y \) by its image under \( \psi \) or \( \psi_1 \) (depending on whether its length is \( n_0 \) or \( n_0 + 1 \)), except for the starting subblock of each \( t_0 \)-train which we replace by \( B_0 \). It is clear that the above code yields a continuous map commuting with the shift. Its injectivity is immediate: we can determine the positioning of the \( t_0 \)-trains in the preimage from the positioning of the occurrences of \( B_0 \) in the image. Then, knowing the lengths of consecutive \( t_0 \)-trains, we can determine where they are cut into subblocks. Finally, reversing \( \psi \) and \( \psi_1 \), we can determine the preimage.

We take this opportunity to make a general comment concerning block codes. In 1969 G. Hedlund proved that every factor map \( \pi : X \to Z \) between two subshifts (the first over \( \Sigma \), the second over \( \Lambda \)) is induced by a block code, i.e., there exists a map \( \Pi : \Sigma^{2r+1} \to \Lambda \) such that \( \pi(x) \) at position \( n \) is equal to \( \Pi(x[n-r, n+r]) \). The parameter \( r \) is often called the radius of the code. Because all the codes appearing in this paper have a slightly different form, it might be interesting to see how general this form is.

**Definition 1.** By a length-preserving code we mean any function \( \psi \) defined on some finite collection \( B \) of blocks over \( \Sigma \) into the blocks over \( \Lambda \) such that \( |\psi(B)| = |B| \) for each \( B \in B \).

We say that a map \( \pi \) between two subshifts \((X,S)\) and \((Z,S)\) is induced by a length-preserving code if there exists a length-preserving code \( \psi \) such that

- each \( x \in X \) can be decomposed in a unique way as an infinite concatenation of blocks belonging to the domain of \( \psi \), and
- \( x = \pi(z) \) coincides with the sequence obtained from \( z \) by replacing all blocks in the above concatenation by their images under \( \psi \).

It is not hard to see that any map induced by a length-preserving code is continuous and commutes with the shift transformation, hence is a factor map. It is not true that every factor map between two subshifts is induced by a length-preserving code. However, we now prove that it is always so whenever the factor is minimal. A similar result has been obtained for factor maps between Toeplitz flows in [D-K-L, Theorem 1].

**Proposition.** Let \( \pi : X \to Z \) be a factor map between two subshifts, \((X,S)\) and \((Z,S)\), where \((Z,S)\) is minimal and non-periodic. Then \( \pi \) is induced by a length-preserving code \( \psi \).

**Proof.** Let \( \Pi \) be the classical block code inducing \( \pi \) and let \( r \) denote its radius. Let \( B \) be a block of length at least \( 2r \), having non-overlapping occurrences in \( Z \) (see Lemma 1). For each \( x \in X \) let \( (n_i(x))_{i \in \mathbb{Z}} \) denote the starting positions of the consecutive occurrences in \( B \) of \( \pi(x) \). Consider the family \( \mathcal{B} \) of blocks occurring in \( Z \) as \( x[n_0(x) + r, n_{i+1}(x) - r] \) for some \( x \in X \) and \( i \in \mathbb{Z} \). Since \( B \) occurs syntactically in \( Z \), this family is finite. To define the length-preserving code \( \psi \) on \( B \) we first apply the block code \( \Pi \), from which we can determine all letters of the image blocks except for the extreme \( r \) positions at both ends. But we know that each of these image blocks ends with the initial subblock of \( B \) of length \( r \) and starts with the remaining part of \( B \) of length \( |B| - r \geq r \). This covers the missing \( r \) positions on both sides, thus the image blocks are fully determined. It is obvious that the length-preserving code \( \psi \) so obtained induces \( \pi \), as desired.

**The general case.** The construction used in the proof of Theorem 1 can be easily generalized to the case where \((X,T_X)\) is an arbitrary (non-symbolic) flow. Temporarily, we maintain the assumption that \((Z,S)\) is a subshift. Clearly, the resulting almost 1-1 extension, \((Y,T_Y)\), will no longer be symbolic.

**Theorem 2.** Let \((X,T_X)\) be an arbitrary extension of a minimal non-periodic subshift \((Z,S)\). Then \((X,T_X)\) is Borel* isomorphic to some minimal almost 1-1 extension \((Y,T_Y)\) of \((Z,S)\). The corresponding diagram commutes (see formulation of Theorem 1).

**Proof.** We represent \((X,T_X)\) as a subshift over the infinite alphabet \( X \), i.e., we identify each \( x \in X \) with the sequence \((\ldots, T_X^{-2}(x), T_X^{-1}(x), T_X^0(x), \ldots) \in X^\mathbb{Z} \). Obviously, such a representation is a topological isomorphism. From this point on, we repeat the whole proof of Theorem 1, which leads to obtaining \((Y,S)\) as a subshift over the infinite alphabet \( E = X \times (X \cup \{\text{closed}\}) \times \Lambda \). The minor differences are the following:

- Before we start, we fix a sequence \( (\varepsilon_i) \) decreasing to zero. The number of all possible regular \( t \)-trains is infinite, nevertheless, by compactness of \( \Phi_1(X) \), there exists a finite collection \( T_t \) of regular \( t \)-trains such that every regular \( t \)-train is close to some of the \( t \)-trains from \( T_t \), where by “close” we understand that the distance at each coordinate is less than \( \varepsilon_i \). In (5) we define \( r_t = T_t \).
We can now apply our Theorem 2 to the extension \( \pi_2 : \widetilde{X} \to \widetilde{Z} \) (or Theorem 1 if \( X \) is a subshift, because then \( \widetilde{X} \) is obviously also a subshift). Let \( \widetilde{Y} \), \( \pi_2 : \widetilde{Y} \to \widetilde{Z} \), and \( \phi : \widetilde{X} \to \widetilde{Y} \) denote the flow, the almost 1-1 extension, and the Borel* isomorphism obtained, respectively.

(b) Let \( Y = \{(y, z) : z \in Z', \pi_2(y) = \varphi(z)\} \subset \widetilde{Y} \times Z \). Now, \( \pi_1 \) provides a Borel* isomorphism between \( Y \) and \( \widetilde{Y} \) (use the same argument as for \( \widetilde{X} \) and \( X \), but this time with continuity of \( \varphi^{-1} \)). On the other hand, \( \pi_2 \) provides an almost 1-1 extension of \( Z \). Indeed, if \( z \in Z' \) is such that \( \varphi(z) \) has a one-point fiber for \( \pi_1 \), then \( z \) has a one-point fiber for \( \pi_2 \) (similar argument again). It could be proved that the \( Y \) so defined is minimal, but we can avoid proving this by letting \( Y \) be a minimal subset of the set previously defined. By minimality of \( \widetilde{Y} \) and \( Z \), both projections remain onto, hence their required properties remain satisfied.

Once this is done, our assertion holds for \( X \), \( Z \), and \( Y \) with \( \phi \) defined on (an appropriate subset of) \( X \) as \( \pi_1^{-1} \circ \phi \circ \pi_1^{-1} \) (see diagram).

A similar method involving infinite products leads to the following

**Theorem 4.** Theorem 2 also holds if \( (Z, S) \) is a subshift over the countable alphabet \( \mathbb{N} \cup \{\infty\} \) (this time we do not assume strict ergodicity).

**Proof.** Let \( z \in Z \). For each \( n \in \mathbb{N} \) denote by \( x_n \) the sequence over the finite alphabet \( \{1, \ldots, n\} \) obtained from \( z \) by replacing all letters of the alphabet \( \mathbb{N} \cup \{\infty\} \) which are larger than \( n \) (including \( \infty \)) by \( n \). Let \( Z_n \) be the corresponding factor of \( Z \).

If for each \( n \), \( Z_n \) is periodic, then \( Z \) represents the rotation of a compact monothetic group (so-called \( p \)-adic adding machine, see next section on Toeplitz flows). Such a flow has topological entropy zero and is strictly ergodic. This case has been dealt with in Theorem 3.

So suppose the flows \( Z_n \) are non-periodic (for \( n \) sufficiently large). We can apply Theorem 2 to \( X \) and each \( Z_n \), which produces a sequence of flows \( Y_n \). The flow \( Y \) will be defined as an appropriate joining within the infinite product \( \prod Y_n \). We omit the details of the definition of \( Y \), and proving its required properties. The arguments are similar to those used for joinings in the proof of Theorem 3. •
Theorem 5. Theorem 3 also holds if $(Z, T_2)$ is strictly ergodic and has infinite entropy.

Proof. By [D-K, Theorem 18], we can find a subshift $(\bar{Z}, S)$ over the countable alphabet $\mathbb{N} \cup \{\infty\}$, finitarily (and hence Borel*) isomorphic to $(Z, T_2)$. The assertion follows by the same proof as in Theorem 3, with Theorem 4 applied instead of Theorem 2. [1]

The problem with generalizing our Theorem 3 (and 5) to the non-strictly ergodic case lies in finding an appropriate symbolic representation $(\bar{Z}, S)$ for $(Z, T_2)$.

Question. Let $(Z, T_2)$ be a minimal non-periodic topological flow with finite topological entropy. Does there exist a subshift representation $\varphi : (Z, T) \rightarrow (\bar{Z}, S)$ which is both a Borel* isomorphism and a universally (for each invariant measure) finitary isomorphism?

Comment. The starting point in the quoted construction of [D-K] is finding a Rokhlin tower with an open base $U$ whose boundary $\partial U$ is a null set. This is done for a single invariant measure. It can also be easily done for at most countably many ergodic measures. But even then we do not know whether the universally finitary isomorphism obtained induces a continuous map on invariant measures (this problem does not appear in the case of finitely many ergodic measures, because any affine map defined on a finite-dimensional simplex is continuous). Also without strict ergodicity there is a danger that some unwanted invariant measures might be supported by $\bar{Z} \setminus \bar{Z}'$.

Remark 1. Theorem 1 can also be proved for $\mathbb{Z}^2$-actions. A proof based on the same principles works in the case where both horizontal and vertical shifts on $\mathbb{Z}$ are minimal non-periodic. As a consequence, theorems analogous to Theorems 2 through 5 are valid. Our Theorems 2 through 5 may be useful in producing concrete examples of topological dynamical systems with prescribed properties, for instance, as was done in [B-G-K].

Characterization of Toeplitz flows. Toeplitz sequences have been introduced in 1969 by Jacobs and Keane [J-K], although particular examples were known much earlier (see e.g. [G-H], [O], [G-H]). Some general topological dynamical properties such as minimality and strict ergodicity (for the regular case) were established in these earlier works. The maximal equicontinuous factor was identified in [E] (1970) for regular Toeplitz flows. Topological characterization of all Toeplitz flows as minimal almost $1$-$1$ symbolic extensions over the so-called $p$-adic adding machines is stated (without proof) in [J-M] (1979). Because of the importance of this characterization for our further investigations, a simple proof of this fact is presented below.

Since 1984 there have appeared various constructions of Toeplitz flows exhibiting a variety of topological and, to some extent, spectral invariants, such as the set of invariant measures, topological centralizer, topological entropy, topological coalescence, point spectrum (see e.g. [Wi], [D1,2,3], [B-KL,2], [D-I], [D-KL], [I-L], [I], [D-L]).

Much less was known about possible realizations within this class of measure-theoretic invariants such as rank, covering number, spectral multiplicity or order of the quotient group (of the measure-theoretic centralizer). However, multiple realizations of these invariants were obtained in a larger class of flows including Morse sequences and other extensions over the rational point spectrum (see e.g. [I-L], [I-F-M], [I-L]).

Our original desire, motivated by several discussions with some other mathematicians interested in this subject, was to fully characterize Toeplitz flows from the measure-theoretic point of view. The missing link was a "symbolic version" of the Furstenberg-Weiss theorem. In view of the results of the preceding sections, such a characterization is now possible even at the level of a Borel* isomorphism.

Definition 2 [J-K]. A Toeplitz sequence is a non-periodic element $x \in \mathbb{Z}^2$ such that

$$(\forall n \in \mathbb{Z})(\exists p \in \mathbb{N})(\forall k \in \mathbb{Z})\ x(kp + n) = x(n),$$

i.e., each position in $x$ is a periodic position.

A subshift $(X, S)$ is called a Toeplitz flow if it is the orbit-closure of some Toeplitz sequence. Toeplitz flows are well known to be minimal.

The topological maximal equicontinuous factor (see e.g. [A] for the definition) of a Toeplitz flow is known to have the form of a so-called $p$-adic adding machine $(G_p, 1)$ (see e.g. [Wi]). One of the possible ways of viewing the group $G_p$ is the following: its elements are sequences $(z_i)_{i \geq 1} \in \prod_{i \geq 1} \{0, \ldots, p-1\}$ such that for each $t, z_{i+1} \equiv z_i \mod p_i$, where $(p_i)_{i \geq 1}$ is a fixed increasing sequence of positive integers satisfying $p_i | p_{i+1}$. Addition is defined coordinatewise modulo $p_i$. Then $1 := (1, 1, 1, \ldots)$ is a topological generator of the compact monothetic group $G_P$ (by the same letter 1 we also denote the rotation by the generator 1 in $G_P$; see [H-R] for more details on $G_P$). We view $G_p$ as a compactification of $(\mathbb{Z}_+, +)$ by writing $k$ instead of $k1$ (multiplication by integers is well defined in $G_p$). Recall that the sets $p_iG_p$ form a base for the topology at 0 in $G_p$. The flow $(G_p, 1)$ is strictly ergodic and the invariant measure is the Haar measure $\lambda$.

Theorem 6 [M-P]. A subshift $(X, S)$ is a Toeplitz flow if and only if it is a minimal almost $1$-$1$ extension of some $p$-adic adding machine.

Proof. If $(X, S)$ is a Toeplitz flow then the required properties are fulfilled for the maximal equicontinuous factor $(G_p, 1)$ of $(X, S)$ (see [Wi]).
Conversely, suppose \((X, S)\) satisfies the above conditions with some \((G_p, 1)\) and let \(x\) be a one-point fiber of the factor \(\pi_X : X \to G_p\). We show that \(x\) itself is a Toeplitz sequence (by minimality, this will complete the proof). Suppose the converse, i.e., that there exists a non-periodic position \(n\) in \(x\). In particular, for each \(t\) we can find a \(k_t\) such that

\[ x(k_t p_t + n) \neq x(n). \]

Choosing if necessary a convergent subsequence we define

\[ x' = \lim_{t \to \infty} S^{k_t p_t} x. \]

Of course, \(x \neq x'\), because they differ at position \(n\). On the other hand,

\[ \pi_X(x') = \lim k_t p_t + \pi_X(x), \]

by the properties of \(\pi_X\). But \(k_t p_t\) converges to 0 in \(G_p\), from which it follows that \(\pi_X(x') = \pi_X(x)\), a contradiction.

The above characterization allows us to apply the theorems previously obtained on almost 1-1 extensions:

**Theorem 7.** Any symbolic topological extension \((X, S)\) of a \(p\)-adic adding machine is Borel* isomorphic to a Toeplitz flow.

**Proof.** Use Theorem 3.

**Remark 2.** A similar passage (in a very particular case) can be found in [D2], where a Borel* representation in the form of a Toeplitz flow is constructed for a (non-transitive) flow obtained as the closure of a union of many Toeplitz flows factoring to the dyadic integers.

**Remark 3.** The statement reversing Theorem 7 is false. For example, there exists a strictly ergodic flow having an adding machine as a measurable but not topological factor. Such a flow is measure-theoretically isomorphic to a Toeplitz flow (see Theorem 8). By strict ergodicity, this isomorphism is Borel*.

Finally, we state the measure-theoretic characterization of Toeplitz flows, as a consequence of which all the measure-theoretic information that was known for systems factoring to some \(p\)-adic adding machines is now known to be realizable within the class of Toeplitz flows. To pass from measure-preserving transformations to topological flows we apply a strengthening of the famous Jewett–Krieger Theorem, due to Weiss (1985), in which the entire diagram of a measure-theoretic factor is replaced by a strictly ergodic topological model.

**Theorem 8.** An ergodic dynamical system \((X, \mu, T_X)\) is measure-theoretically isomorphic to a strictly ergodic Toeplitz flow if and only if it has finite entropy and its set of eigenvalues contains infinitely many rationals.
Two-sided estimates for the approximation numbers of Hardy-type operators in $L^\infty$ and $L^1$

by

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Abstract. In [2] and [3] upper and lower estimates and asymptotic results were obtained for the approximation numbers of the operator $T : L^p(\mathbb{R}^+) \to L^{p'}(\mathbb{R}^+)$ defined by $Tf(x) = v(x) \int_0^x u(t)f(t) \, dt$ when $1 < p < \infty$. Analogous results are given in this paper for the case $p = 1$, $\infty$ not included in [2] and [3].

1. Introduction. In [2] and [3] the operator $T : L^p(\mathbb{R}^+) \to L^{p'}(\mathbb{R}^+)$ defined by

$$
Tf(x) = v(x) \int_0^x u(t)f(t) \, dt
$$

was studied in the case $1 < p < \infty$, with $u, v$ real-valued functions and $u \in L^p_{\text{loc}}(\mathbb{R}^+), v \in L^{p'}(\mathbb{R}^+)$, $p' = p/(p-1)$. Estimates for the approximation numbers $\alpha_n(T)$ of $T$ were obtained in [2], but the procedure for extracting the upper and lower bounds from the results is rather cumbersome to apply. This deficiency was overcome in [3] where asymptotic bounds for the approximation numbers which are easy to check in practice were determined. Specifically, it was proved that:

$$
\lim_{n \to \infty} \frac{1}{n} \int_0^1 |u(t)v(t)| \, dt
$$

when $p = 2$; and when $p \neq 2$,

$$
\frac{1}{4} \alpha_p \int_0^1 |u(t)v(t)| \, dt \leq \liminf_{n \to \infty} \frac{1}{n} \int_0^1 |u(t)v(t)| \, dt
$$

$$
\leq \limsup_{n \to \infty} \frac{1}{n} \int_0^1 |u(t)v(t)| \, dt
$$

for some constant $\alpha_p$ depending on $p$. Further in [3], two-sided estimates

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