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On $(C, 1)$ summability of integrable functions with respect to the Walsh–Kaczmarz system

by

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Abstract. Let G be the Walsh group. For $f \in L^1(G)$ we prove the a.e. convergence $\sigma_n f \to f$ $(n \to \infty)$, where σ_n is the nth $(C,1)$ mean of f with respect to the Walsh–Kaczmarz system. Define the maximal operator $\sigma^* f := \sup_n |\sigma_n f|$. We prove that σ^* is of type (p,p) for all $1 < p \le \infty$ and of weak type $(1,1)$. Moreover, $\|\sigma^* f\|_1 \le c\||f|\|_H$, where H is the Hardy space on the Walsh group.

Introduction and the main results. This paper is devoted to the problem of a.e. convergence of the $(C,1)$ means of integrable functions with respect to the Walsh–Kaczmarz system. The Walsh system in the Kaczmarz enumeration was studied by a lot of authors (see [SCH1], [SCH2], [SK1], [SK2], [BAL], [SWS], [WY]). In [SH] it was pointed out that the behavior of the Dirichlet kernel of the Walsh–Kaczmarz system is worse than of the kernel of the Walsh–Paley system considered more often. Namely, for the Dirichlet kernel $D_n(x)$ of the Walsh–Kaczmarz system the inequality $\limsup_{n\to\infty} D_n(x)/\log n \ge C > 0$ holds a.e. This "dispersion" of the system makes it easier to construct examples of divergent Fourier series [BAL]. A number of pathological properties are due to this "dispersion" property of the kernel. For example, for Fourier series with respect to the Walsh–Kaczmarz system it is impossible to establish any local test for convergence at a point or on an interval, since the localization principle does not hold for this system.

On the other hand, the global behavior of the Fourier series with respect to this system is similar in many respects to the case of the Walsh–Paley system. Schipp [SCH1] and Wo-Sang Young [WY] proved that the Walsh–Kaczmarz system is a convergence system. Skvortsov proved, for continuous functions f, that the Fejér means converge uniformly to f. In this paper

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we prove for integrable functions that the Fejér means (with respect to the Walsh–Kaczmarz system) converge almost everywhere to the function.

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$ the set of nonnegative integers and Z_2 the discrete cyclic group of order 2. That is, $Z_2 = \{0,1\}$, the group operation is addition mod 2 and every subset is open. The Haar measure is such that the measure of a singleton is 1/2. Let

$$G := \mathop{\times}_{k=0}^{\infty} Z_2$$

be the complete direct product. Thus, every $x \in G$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in \{0,1\}$ $(i \in \mathbb{N})$. The group operation on G is coordinatewise addition (which is the so-called logical addition), the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the *Walsh group*. Let $e_i := (0,0,\ldots,1,0,0,\ldots) \in G$ have all coordinates zero except the ith which is 1.

A neighborhood base for G can be given as follows:

$$I_0(x) := G, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G : y_i = x_i \text{ for } i < n\}$$

for $x \in G$ and $n \in \mathbb{P}$. Let $0 = (0, i \in \mathbb{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ $(n \in \mathbb{N})$. Let $\mathcal{I} := \{I_n(x) : x \in G,\ n \in \mathbb{N}\}$. The elements of $\mathcal{I}$ are called the *dyadic intervals* of G. Furthermore, let $L^p(G)$ $(1 \le p \le \infty)$ denote the usual Lebesgue spaces (and $\|\cdot\|_p$ the corresponding norms) on G, $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x)$ $(x \in G)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$.

Define the *Hardy space* H^1 as follows. Let $f^* := \sup_{n\in\mathbb{N}} |E_n f|$ be the maximal function of the integrable function $f \in L^1(G)$. Then

$$H^1(G) := \{f \in L^1(G) : f^* \in L^1(G)\};$$

endowed with the norm $\|f\|_{H^1} := \|f^*\|_1$, H^1 is a Banach space. Another definition is common: $a \in L^\infty(G)$ is called an *atom* if either $a = 1$ or a has the following properties: $\operatorname{supp} a \subseteq I_a$, $\|a\|_\infty \le 1/\mu(I_a)$, $\int_{I_a} a = 0$, for some $I_a \in \mathcal{I}$. We say that the function f belongs to the Hardy space $H(G)$ if f can be represented as $f = \sum_{i=0}^\infty \lambda_i a_i$, where a_i's are atoms and the scalar coefficients λ_i $(i \in \mathbb{N})$ satisfy $\sum_{i=0}^\infty |\lambda_i| < \infty$. It is known that $H(G)$ is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^\infty \lambda_i a_i$ as above. Moreover (cf. Theorem 3.6 of [SWS]), $H^1(G) = H(G)$ and

$$\|f\|_{H^1} \sim \|f\|_H.$$

In the proof of some lemmas we will use the following. Let $f \in H$ and $z \in G$. Then $f(\cdot + z) \in H$ and $\|f\|_{H^1} \sim \|f(\cdot + z)\|_{H^1}$. Indeed, if $f = \sum_{i=0}^\infty \lambda_i a_i \in H$, where the a_i are atoms, then so are the functions $a_i(\cdot + z)$. That is, $f(\cdot + z) = \sum_{i=0}^\infty \lambda_i a_i(\cdot + z) \in H$, which after some elementary considerations gives $\|f\|_H = \|f(\cdot + z)\|_H$. Thus,

$$\|f\|_{H^1} \sim \|f(\cdot + z)\|_{H^1}$$

for all $z \in G$.

Let $n \in \mathbb{N}$ have base 2 expansion $n = \sum_{i=0}^\infty n_i 2^i$, where $n_i \in \{0,1\}$. Define $|n| := \max(j \in \mathbb{N} : n_j \ne 0)$, that is, $2^{|n|} \le n < 2^{|n|+1}$. The *Rademacher functions* are defined as

$$r_n(x) := (-1)^{x_n} \quad (x \in G,\ n \in \mathbb{N}).$$

The *Walsh–Paley system* $\omega := (\omega_n, n \in \mathbb{N})$ is defined as the set of *Walsh–Paley functions*

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \quad (x \in G,\ n \in \mathbb{N}).$$

The nth *Walsh–Kaczmarz function* is

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}$$

for $n \in \mathbb{P}$, and $\kappa_0(x) := 1$, $x \in G$. The *Walsh–Kaczmarz system* $\kappa := (\kappa_n, n \in \mathbb{N})$ can be obtained from the Walsh–Paley system by renumbering the functions within the dyadic "block" with indices from the segment $[2^n, 2^{n+1} - 1]$. That is, $\{\kappa_n : 2^k \le n < 2^{k+1}\} = \{\omega_n : 2^k \le n < 2^{k+1}\}$ for all $k \in \mathbb{N}$, $\kappa_0 = \omega_0$.

By means of the transformation $\tau_A : G \to G$,

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots) \in G,$$

which is clearly measure-preserving and such that $\tau_A(\tau_A(x)) = x$, we have

$$\kappa_n(x) = r_{|n|}(x)\omega_n(\tau_{|n|}(x)) \quad (n \in \mathbb{N}).$$

Let us consider the Dirichlet and Fejér kernel functions:

$$D_n^\alpha := \sum_{k=0}^{n-1} \alpha_k, \quad K_n^\alpha := \frac{1}{n}\sum_{k=1}^{n} D_k^\alpha, \quad D_0^\alpha, K_0^\alpha := 0,$$

where α is either κ or ω. The nth Fourier coefficient, the nth partial sum of the Fourier series and the nth $(C,1)$ mean of $f\in L^1(G)$ are, respectively,

$$\widehat{f}^{\alpha}(n):=\int_G f(x)\alpha_n(x)\,d\mu(x)\quad (n\in\mathbb{N}),$$

$$S_n^{\alpha}f(y):=\sum_{k=0}^{n-1}\widehat{f}^{\alpha}(k)\alpha_k(y)=\int_G f(x+y)D_n^{\alpha}(x)\,d\mu(x)\quad (n\in\mathbb{P},\ S_0^{\alpha}f=0),$$

$$\sigma_n^{\alpha}f(y):=\frac{1}{n}\sum_{k=1}^{n}S_k^{\alpha}f(y)=\int_G f(x+y)K_n^{\alpha}(x)\,d\mu(x)\quad (n\in\mathbb{P},\ \sigma_0^{\alpha}f=0,\ y\in G),$$

where α is either κ or ω. Define the maximal operator

$$\sigma^*f:=\sup_{n\in\mathbb{P}}|\sigma_n^{\kappa}f|\quad (f\in L^1(G)).$$

We say that the operator $T:L^1\to L^0$ is *of type* (p,p) if $\|Tf\|_p\le c_p\|f\|_p$ for some constant c_p for all $f\in L^p(G)$ $(1\le p\le\infty)$; T is *of type* (H^1,L^1) if $\|Tf\|_1\le c\|f\|_{H^1}$ for all $f\in H^1(G)$; finally, T is *of weak type* $(1,1)$ if there exists a $c>0$ such that $\mu(\{y\in G:Tf(y)>\lambda\})\le c\|f\|_1/\lambda$ for all $\lambda>0$ and $f\in L^1(G)$.

Set $S^*f:=\sup_{n\in\mathbb{P}}|S_n^{\alpha}f|$ for $f\in L^1$, where α is ω or κ or any piecewise linear rearrangement of the Walsh–Paley system (κ is of this kind; for the notion of piecewise linear rearrangement see [SWS]). Then S^* is of type (p,p) for all $p\ge 2$ and for $f\in L^p$ $(p\ge 2)$ it follows $S_nf\to f$ a.e. [SWS, Theorem 6.10]. Moreover, if $\alpha=\kappa$ and $f\in L^1(\log^+L)^2$ (in particular, if $f\in L^p$ for any $p>1$), then the Walsh–Kaczmarz–Fourier series of f converges to f a.e. on G (cf. Theorem 6.11 of [SWS]).

The main aim of this paper is to prove

THEOREM 1. *$\sigma_n^{\kappa}f\to f$ $(n\to\infty)$ almost everywhere for all $f\in L^1(G)$.*

THEOREM 2. *The operator σ^* is of type (p,p) for all $1<p\le\infty$ and of weak type $(1,1)$. Moreover, $\|\sigma^*f\|_1\le c\||f\||_{H^1}$.*

Theorems 1 and 2 for the Walsh–Paley system can be found in [SWS, Corollary 6.2]. Corollary 6.2 of [SWS] states for the Walsh–Paley system even more. Namely, the maximal operator σ^* is of type (H^1,L^1), i.e. $\|\sigma^*f\|_1\le c\|f\|_{H^1}$ $(f\in H^1(G))$. Skvortsov [SK1] proved the uniform $(C,1)$ summability of the Fourier series of a continuous function with respect to the Walsh–Kaczmarz system. For more details on the systems ω and κ see e.g. [WY, SK1, SK2, SWS, SCH1].

In this paper c denotes an absolute constant which may not be the same at different occurrences, and similarly for c_p which depends on p $(p\in\mathbb{R})$.

The proofs. In order to prove Theorems 1 and 2 we need some lemmas.

LEMMA 3. *Let $f\in L^1(G)$ and $l\in\mathbb{N}$. Then the operator*

$$T_lf(y):=\sup_{A\ge l}\Big|2^{A-l}\int_{\{x\in G:x_l=x_{l+1}=\ldots=x_{A-1}=0\}}f(x+y)\,d\mu(x)\Big|$$

$(y\in G)$ is of type (H^1,L^1), (p,p) for all $1<p\le\infty$ and of weak type $(1,1)$ (uniformly in l).

Proof. Set

$$g(z):=2^{-l}\sum_{a_i\in\{0,1\},\,i\in\{0,1,\ldots,l-1\}}f(a_0e_0+\ldots+a_{l-1}e_{l-1}+z),\qquad z\in G.$$

Then

$$E_ng(y)=2^n\int_{I_n}g(x+y)\,d\mu(x)=2^{n-l}\sum_{a_i\in\{0,1\},\,i\in\{0,1,\ldots,l-1\}}\int_{I_n}f(a_0e_0+\ldots+a_{l-1}e_{l-1}+x+y)\,d\mu(x)=2^{n-l}\int_{\{x\in G:x_l=x_{l+1}=\ldots=x_{n-1}=0\}}f(x+y)\,d\mu(x).$$

This implies

$$T_lf(y)=\sup_{n\ge l}|E_ng(y)|\le g^*(y).$$

Since the operator $f^*(y):=\sup_{n\in\mathbb{N}}|E_nf(y)|$ is of type (p,p) for all $1<p\le\infty$, and of type (H^1,L^1) and weak type $(1,1)$ (see e.g. [SWS]), we have

$$\|T_lf\|_p\le\|g^*\|_p\le c_p\|g\|_p\le c_p\|f\|_p,$$
$$\|T_lf\|_1\le\|g^*\|_1\le c\|g\|_{H^1}\le c\|f\|_{H^1},$$

for all $1<p\le\infty$, and

$$\mu(T_lf>\lambda)\le\mu(g^*>\lambda)\le c\|g\|_1/\lambda\le c\|f\|_1/\lambda.\ ■$$

LEMMA 4. *Let $f\in L^1(G)$ and $l,t\in\mathbb{N}$, $l<t$. Then the operator*

$$T_{l,t}f(y):=\sup_{A\ge t}\Big|2^{A-l}\int_{\{x\in G:x_l=x_{l+1}=\ldots=x_{t-1}=0,x_t=1,x_{t+1}=\ldots=x_{A-1}=0\}}f(x+y)\,d\mu(x)\Big|$$

$(y\in G)$ is of type (H^1,L^1), (p,p) for all $1<p\le\infty$, and of weak type $(1,1)$ (uniformly in l,t).

Proof. By Lemma 3 the inequality $T_{l,t}f(y)\le T_lf(y+e_t)$ and the equality of the appropriate norms of f and $f(\cdot+e_t)$ (equivalence in the case of the H^1 norm) we have

$$\|T_{l,t}f(\cdot)\|_p\le c_p\|f(\cdot)\|_p$$

for each $1 < p \le \infty$,

$$\|T_{l,t}f(\cdot)\|_1 \le c\|f(\cdot)\|_{H^1} \quad \text{and} \quad \mu(T_{l,t}f > \lambda) \le c\|f(\cdot)\|_1/\lambda. \ \blacksquare$$

It is well known that ([SWS, p. 28])

$$(1) \qquad D^{\omega}_{2^n}(x) = D^{\kappa}_{2^n}(x) = D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{otherwise,} \end{cases}$$

$$(2) \qquad D^{\omega}_n(x) = \omega_n(x)\sum_{k=0}^{\infty} n_k(D_{2^{k+1}}(x) - D_{2^k}(x)) = \omega_n(x)\sum_{k=0}^{\infty} n_k(-1)^{x_k}D_{2^k}(x)$$

for $n \in \mathbb{N}$ and $x \in G$. Set

$$K^{\alpha}_{a,b} := \sum_{j=a}^{a+b-1} D^{\alpha}_j \qquad (a,b \in \mathbb{N},\ \alpha = \kappa,\omega)$$

and $n^{(s)} := \sum_{i=s}^{\infty} n_i 2^i$ $(n,s \in \mathbb{N})$. Recall that for $n \in \mathbb{N}$, $|n| := \max(j \in \mathbb{N} : n_j \ne 0)$, that is, $2^{|n|} \le n < 2^{|n|+1}$ (and e.g. $n^{(0)} = n$, $n^{(|n|+1)} = 0$). By elementary calculations we have

$$(3) \qquad nK^{\alpha}_n = \sum_{s=0}^{|n|} n_s K^{\alpha}_{n^{(s+1)},2^s} + D^{\alpha}_n \qquad (\alpha = \kappa,\omega,\ n \in \mathbb{P}).$$

LEMMA 5. *Suppose that $s,t,n \in \mathbb{N}$ and $x \in I_t \setminus I_{t+1}$. If $s \le t \le |n|$, then $|K^{\omega}_{n^{(s+1)},2^s}(x)| \le c2^{s+t}$, while if $t < s \le |n|$, then*

$$K^{\omega}_{n^{(s+1)},2^s}(x) = \begin{cases} 0 & \text{if } x - x_t e_t \notin I_s, \\ \omega_{n^{(s+1)}}(x)2^{s+t-1} & \text{if } x - x_t e_t \in I_s. \end{cases}$$

Proof. If $s \le t$, then for all $k \in \mathbb{N}$ by (1) and (2) we have $|D^{\omega}_k(x)| \le c\sum_{j=0}^{t} 2^j \le c2^t$, thus in this case $|K^{\omega}_{n^{(s+1)},2^s}(x)| \le c2^{s+t}$.

Now, let $|n| \ge s > t$. Then by (2) and (1),

$$D^{\omega}_{n^{(s+1)}+j}(x) = \omega_{n^{(s+1)}+j}(x)\sum_{k=0}^{t}(n^{(s+1)}+j)_k r_k(x)D_{2^k}(x) = \omega_{n^{(s+1)}+j}(x)\Big(\sum_{k=0}^{t-1} j_k 2^k - j_t 2^t\Big).$$

This implies that

$$K^{\omega}_{n^{(s+1)},2^s}(x) = \sum_{j=0}^{2^s-1} D^{\omega}_{n^{(s+1)}+j}(x) = \omega_{n^{(s+1)}}(x)\sum_{j=0}^{2^s-1}\omega_j(x)\sum_{k=0}^{t-1} j_k 2^k - \omega_{n^{(s+1)}}(x)\sum_{j=0}^{2^s-1}\omega_j(x)j_t 2^t =: \sum\nolimits^1 - \sum\nolimits^2.$$

We now prove that $\sum^1 = 0$. The proof is based on the fact that $\omega_{n^{(s+1)}}(x)$, $\sum_{k=0}^{t-1} j_k 2^k$ and $\prod_{l=0,l\ne t}^{s-1}(-1)^{j_l x_l}$ do not depend on j_t, while $\sum_{j_t=0}^{1}(-1)^{j_t x_t} = 0$ for $x_t = 1$. We have

$$\sum\nolimits^1 = \omega_{n^{(s+1)}}(x)\sum_{j_0,\dots,j_{s-1}}\omega_j(x)\sum_{k=0}^{t-1} j_k 2^k = \sum_{j_i=0,\,i\ne t,\,i=0,\dots,s-1}^{1}\ \prod_{l=0,\,l\ne t}^{s-1}(-1)^{j_l x_l}\sum_{k=0}^{t-1} j_k 2^k \sum_{j_t=0}^{1}(-1)^{j_t x_t} = 0,$$

since

$$\sum_{j_t=0}^{1}\omega_j(x) = \sum_{j_t=0}^{1}(-1)^{j_0x_0+\dots+j_{t-1}x_{t-1}+j_tx_tj_{t+1}x_{t+1}+\dots+j_{s-1}x_{s-1}} = 0.$$

That is,

$$K^{\omega}_{n^{(s+1)},2^s}(x) = -\omega_{n^{(s+1)}}(x)\sum_{j=0}^{2^s-1}\omega_j(x)j_t 2^t = \begin{cases} 0 & \text{if } x - x_t e_t \notin I_s, \\ \omega_{n^{(s+1)}}(x)2^{s+t-1} & \text{if } x - x_t e_t \in I_s. \end{cases} \ \blacksquare$$

COROLLARY 6. *Let $A,t \in \mathbb{N}$, $A > t$. Suppose that $x \in I_t \setminus I_{t+1}$. Then*

$$K^{\omega}_{2^A}(x) = \begin{cases} 0 & \text{if } x - x_t e_t \notin I_A, \\ 2^{t-1} & \text{if } x - x_t e_t \in I_A. \end{cases}$$

If $x \in I_A$ then $K^{\omega}_{2^A}(x) = 2^{A-1} + 1/2$.

Proof. If $x \in I_A$ then for $j \le 2^A$ we have $D^{\omega}_j(x) = j$, thus $K^{\omega}_{2^A}(x) = 2^{-A}\sum_{j=1}^{2^A} j = 2^{A-1} + 1/2$.

If $x \in I_t \setminus I_{t+1}$ for some $t < A$, then the assertion follows from

$$2^A K^{\omega}_{2^A}(x) = \sum_{j=1}^{2^A} D^{\omega}_j(x) = \sum_{j=1}^{2^A-1} D^{\omega}_j(x) + D^{\omega}_{2^A}(x) = K^{\omega}_{0,2^A}(x) = K^{\omega}_{(2^A)^{(A+1)},2^A}(x)$$

$(D^{\omega}_{2^A}(x) = 0$ since $x \notin I_A)$ and from Lemma 5 (with $s = A$). $\blacksquare$

LEMMA 7. *Set*

$$Lf(y) := \sup_{A\in\mathbb{N}} \Big| \int_G f(x+y) r_A(x) K^{\omega}_{2^A}(\tau_A(x))\, d\mu(x) \Big|$$

$(y\in G,\ f\in L^1(G))$. *Then the operator* L *is of type* (p,p) *for all* $1<p\le\infty$, *of weak type* $(1,1)$, *and* $\|Lf\|_1 \le c\||f|\|_{H^1}$.

Proof. For a given $A\in\mathbb{N}$, the integral over G splits into the sum of integrals over I_A and $G\setminus I_A$. Since by Corollary 6,

$$\sup_{A\in\mathbb{N}} \Big| \int_{I_A} f(x+y) r_A(x) K^{\omega}_{2^A}(\tau_A(x))\, d\mu(x) \Big|$$
$$= \sup_{A\in\mathbb{N}} (2^{A-1}+1/2) \Big| \int_{I_A} f(x+y) r_A(x)\, d\mu(x) \Big| \le c|f|^*(y),$$

we need to consider the integral over $G\setminus I_A$ only. We have

$$G\setminus I_A = \bigcup_{t=0}^{A-1} (I_t\setminus I_{t+1}).$$

Next, we decompose the set $I_t\setminus I_{t+1}$. For an integer $T>t$ set

$$I_t^T := \{x\in G : x_t = x_T = 1 \text{ and } x_i = 0 \text{ for } i<T,\ i\ne t\}.$$

Then $I_t\setminus I_{t+1}$ can be represented as the disjoint union

$$I_t\setminus I_{t+1} = \bigcup_{T=t+1}^{\infty} I_t^T \cup \{e_t\}.$$

Fix $t<A$ and $x\in I_t^T$ for some $T>t$.

If $T<A$, then among $(\tau_A(x))_i$, $i=0,1,\dots,A-1$, there are at least two indices equal to 1. Namely, $x_t = (\tau_A(x))_{A-1-t} = 1$ and $x_T = (\tau_A(x))_{A-1-T} = 1$. Corollary 6 gives $K^{\omega}_{2^A}(\tau_A(x)) = 0$ in this case. (More specifically, set $l := \max(j\in\mathbb{N} : x_j = 1,\ j<A)$. Then $l\ge T>t$. Consequently, $x_t = 1$, $x_l = 1$, $x_{l+1} = \dots = x_{A-1} = 0$. This gives $\tau_A(x)\in I_{A-l-2}\setminus I_{A-l-1}$, but $(\tau_A(x))_{A-t-1} = 1$ and consequently $\tau_A(x) - (\tau_A(x))_{A-l-1} e_{A-l-1} \notin I_A$.)

If $T\ge A$, then $x - x_t e_t \in I_A$, which means $\tau_A(x) - (\tau_A(x))_{A-1-t} e_{A-1-t} \in I_A$ and consequently, by Corollary 6, we have $K^{\omega}_{2^A}(\tau_A(x)) = 2^{A-t-2}$. It follows that for each $y\in G$,

$$Lf(y) \le \sup_{A\in\mathbb{N}} \Big| \int_{I_A} f(x+y) r_A(x) K^{\omega}_{2^A}(\tau_A(x))\, d\mu(x) \Big|$$
$$+ \sup_{A\in\mathbb{N}} \Big| \sum_{t=0}^{A-1} \int_{I_t\setminus I_{t+1}} f(x+y) r_A(x) K^{\omega}_{2^A}(\tau_A(x))\, d\mu(x) \Big|$$
$$\le c|f|^*(y) + \sum_{t=0}^{\infty} \sup_{A>t} \Big| \int_{I_t\setminus I_{t+1}} f(x+y) r_A(x) K^{\omega}_{2^A}(\tau_A(x))\, d\mu(x) \Big|$$

$$\le c|f|^*(y) + \sum_{t=0}^{\infty} \sup_{A>t} \sum_{T=A}^{\infty} \Big| \int_{I_t^T} f(x+y) r_A(x) K^{\omega}_{2^A}(\tau_A(x))\, d\mu(x) \Big|$$
$$= c|f|^*(y) + \sum_{t=0}^{\infty} \sup_{A>t} \sum_{T=A}^{\infty} \Big| \int_{I_t^T} f(x+y) r_A(x) 2^{A-t-2}\, d\mu(x) \Big|$$
$$\le c|f|^*(y) + \sum_{t=0}^{\infty} \sup_{A>t} 2^{A-t-2} \int_{I_A(e_t)} |f(x+y)|\, d\mu(x)$$
$$\le c|f|^*(y) + \sum_{t=0}^{\infty} 2^{-t} \sup_{A>t} 2^A \int_{I_A(e_t)} |f(x+y)|\, d\mu(x)$$
$$\le c|f|^*(y) + c\sum_{t=0}^{\infty} 2^{-t} T_{0,t}|f|(y).$$

Lemma 4 now gives

$$\|Lf\|_p \le c_p\||f|^*\|_p + c\sum_{t=0}^{\infty} 2^{-t}\|T_{0,t}|f|\|_p$$
$$\le c_p\|f\|_p + c_p\sum_{t=0}^{\infty} 2^{-t}\|f\|_p \le c_p\|f\|_p.$$

Moreover, also by Lemma 4,

$$\|Lf\|_1 \le c\||f|^*\|_1 + c\sum_{t=0}^{\infty} 2^{-t}\|T_{0,t}|f|\|_1$$
$$\le c\||f|\|_{H^1} + c\sum_{t=0}^{\infty} 2^{-t}\||f|\|_{H^1} = c\||f|\|_{H^1}.$$

Finally, for $\lambda>0$,

$$\mu(Lf > c\lambda) \le \mu(|f|^* > c\lambda) + \mu\Big(c\sum_{t=0}^{\infty} 2^{-t} T_{0,t}|f| > c\lambda\Big)$$
$$\le c\|f\|_1/\lambda + \mu\Big(\bigcup_{t=0}^{\infty}\{T_{0,t}|f| > 2^{t/2}c\lambda\}\Big)$$
$$\le c\|f\|_1/\lambda + \sum_{t=0}^{\infty} \mu(T_{0,t}|f| > 2^{t/2}c\lambda)$$
$$\le c\|f\|_1/\lambda + c\sum_{t=0}^{\infty} 2^{-t/2}\|f\|_1/\lambda \le c\|f\|_1/\lambda$$

by an application of Lemma 4. ■

For $f\in L^1(G)$ define the operator M as follows:

$$Mf(y):=\sup_{n,A\in\mathbb{N},\,|n|\le A}\Big|\int_G f(x+y)r_A(x)K_n^\omega(\tau_A(x))\,d\mu(x)\Big|\quad (y\in G).$$

LEMMA 8. *The operator M is of type (p,p) for all $1<p\le\infty$, of weak type $(1,1)$, and $\|Mf\|_1\le c\||f|\|_{H^1}$.*

Proof. Since $|n^{-1}D_n^\omega|\le 1$, from (3) it follows that we have to consider the modified kernel

$$n\widetilde{K}_n^\omega:=\sum_{s=0}^{|n|}n_sK^\omega_{n^{(s+1)},2^s}$$

and

$$\widetilde{M}f(y):=\sup_{n,A\in\mathbb{N},\,|n|\le A}\Big|\int_G f(x+y)r_A(x)\widetilde{K}_n^\omega(\tau_A(x))\,d\mu(x)\Big|\quad (y\in G).$$

For $A,t\in\mathbb{N}$, $A>t\ge 1$, set $J_t^A:=\{x\in G: x_{A-1}=\ldots=x_{A-t}=0,\ x_{A-t-1}=1\}$ and $J_0^A:=\{x\in G: x_{A-1}=1\}$ for $A\ge 1$. Then for every $1\le A\in\mathbb{N}$ we can decompose G as the (disjoint) union

$$G=I_A\cup\bigcup_{t=0}^{A-1}J_t^A.$$

Namely, if $x\in G\setminus I_A$, then there is an index $j\in\{0,1,\ldots,A-1\}$ for which $x_j=1$. Let $A-t-1$ be the maximal such index ($t\in\{0,1,\ldots,A-1\}$). Then $x\in J_t^A$. Split G as $G=I_A\cup(G\setminus I_A)$. If $x\in I_A$, then by the definition of $\widetilde{K}_n$ and (3) we have $|\widetilde{K}_n^\omega(\tau_A(x))|\le cn\le c2^A$. That is,

$$\sup_{n,A\in\mathbb{N},\,|n|\le A}\frac{n}{2^A}\Big|\int_{I_A} f(x+y)r_A(x)\widetilde{K}_n^\omega(\tau_A(x))\,d\mu(x)\Big| \tag{4}$$
$$\le \sup_{n,A\in\mathbb{N},\,|n|\le A}\frac{n}{2^A}c2^A\int_{I_A}|f(x+y)|\,d\mu(x)\le |f|^*(y).$$

By (3) and Lemma 5 we have

$$\sup_{n,A\in\mathbb{N},\,|n|\le A}\frac{n}{2^A}\Big|\int_{G\setminus I_A} f(x+y)r_A(x)\widetilde{K}_n^\omega(\tau_A(x))\,d\mu(x)\Big|$$
$$\le \sup_{n,A\in\mathbb{N},\,|n|\le A}\frac{1}{2^A}\int_{G\setminus I_A}|f(x+y)|\sum_{s=0}^{|n|}|K^\omega_{n^{(s+1)},2^s}(\tau_A(x))|\,d\mu(x)$$
$$\le \sup_{n,A\in\mathbb{N},\,|n|\le A}\frac{1}{2^A}\sum_{t=0}^{A-1}\sum_{s=0}^{|n|}\int_{J_t^A}|f(x+y)|\cdot|K^\omega_{n^{(s+1)},2^s}(\tau_A(x))|\,d\mu(x)$$

$$\le \sup_{n,A\in\mathbb{N},\,|n|\le A}\frac{1}{2^A}\sum_{t=0}^{A-1}\sum_{s=0}^{t}\int_{J_t^A}|f(x+y)|c2^{s+t}\,d\mu(x)$$
$$+\sup_{n,A\in\mathbb{N},\,|n|\le A}\frac{1}{2^A}\sum_{t=0}^{A-1}\sum_{s=t+1}^{A}\int_{J_t^A}|f(x+y)|\cdot|K^\omega_{n^{(s+1)},2^s}(\tau_A)|\,d\mu(x)$$
$$=: S^1+S^2.$$

Moreover, for $l:=A-t$ we have

$$S^1\le \sup_{n,A\in\mathbb{N},\,|n|\le A}\sum_{t=0}^{A-1}2^{2t-A}\int_{J_t^A}|f(x+y)|\,d\mu(x) \tag{5}$$
$$\le\sum_{l=1}^{\infty}2^{-l}\sup_{A\in\mathbb{N},\,A>l}2^{A-l}\int_{J_{A-l}^A}|f(x+y)|\,d\mu(x)=:\sum_{l=1}^{\infty}2^{-l}S_l^1.$$

Since $J_{A-l}^A=\{x\in G: x_{A-1}=\ldots=x_l=0,\ x_{l-1}=1\}$, the definition of the operator T_l gives

$$2^{A-l}\int_{J_{A-l}^A}|f(x+y)|\,d\mu(x)\le T_l|f|(y).$$

Thus by Lemma 4 and (5) we have $\|S^1\|_p\le c_p\|f\|_p$ for all $1<p\le\infty$, $\|S^1\|_1\le c\||f|\|_{H^1}$ and $\mu(S^1>\lambda)<c\|f\|_1/\lambda$ for all $\lambda>0$.

It remains to discuss S^2. Suppose that $x\in J_t^A$, $s>t$. This means that $\tau_A(x)\in I_t\setminus I_{t+1}$. Then from Lemma 5 it follows that $K^\omega_{n^{(s+1)},2^s}(\tau_A(x))$ differs from 0 only in the case when $x_{A-1}=\ldots=x_{A-t}=0$, $x_{A-t-1}=1$, $x_{A-t-2}=\ldots=x_{A-s}=0$, that is, when $\tau_A(x)-(\tau_A(x))_te_t\in I_s$. If x has this property, then also by Lemma 5 it follows that $|K^\omega_{n^{(s+1)},2^s}(\tau_A(x))|=2^{s+t-1}$. Thus,

$$S^2\le\sup_{A\in\mathbb{N}}\frac{1}{2^A}\sum_{t=0}^{A-1}\sum_{s=t+1}^{A}\int_{J_{t,s}^A}|f(x+y)|2^{s+t-1}\,d\mu(x),$$

where $J_{p,q}^A:=\{x\in G: x_{A-1}=\ldots=x_{A-p}=0,\ x_{A-p-1}=1,\ x_{A-p-2}=0,\ldots\ldots,x_{A-q}=0\}$. Set $l:=A-s$ and $m:=A-t$. Then

$$S^2\le\sup_{A\in\mathbb{N}}\frac{1}{2^A}\sum_{m=1}^{A}\sum_{l=0}^{m-1}2^{2A-l-m}\int_{J^A_{A-m,A-l}}|f(x+y)|\,d\mu(x)$$
$$\le c\sup_{A\in\mathbb{N}}\sum_{m=1}^{A}\sum_{l=0}^{m-1}2^{-m}T_{l,m-1}|f|(y)\le c\sum_{m=1}^{\infty}\sum_{l=0}^{m-1}2^{-m}T_{l,m-1}|f|(y).$$

Since by Lemma 4 the operator $T_{l,m-1}$ is of type (p,p) for all $1<p\le\infty$

uniformly in l,m and of type (H^1,L^1), it follows that

$$\|S^2\|_p \le c\sum_{m=1}^{\infty}\sum_{l=0}^{m-1}2^{-m}\|T_{l,m-1}|f|\|_p \le c_p\||f|\|_X, \tag{6}$$

where $X=p$ for $1<p\le\infty$ and $X=H^1$ for $p=1$. Moreover,

$$\begin{aligned}\mu(S^2>c\lambda) &\le \mu\Big(c\sum_{m=1}^{\infty}\sum_{l=0}^{m-1}2^{-m}T_{l,m-1}|f|>c\lambda\Big)\\ &\le \sum_{m=1}^{\infty}\sum_{l=0}^{m-1}\mu(T_{l,m-1}|f|>c\lambda 2^{m/2})\\ &\le \sum_{m=1}^{\infty}\sum_{l=0}^{m-1}\frac{c}{\lambda}2^{-m/2}\|f\|_1 \le c\|f\|_1/\lambda.\end{aligned}$$

Now, (4)–(6) show that $\widetilde{M}$ is of type (p,p) for all $1<p\le\infty$, of weak type $(1,1)$, and $\|\widetilde{M}f\|_1\le c\||f|\|_{H^1}$. The use of the definitions of M, $\widetilde{M}$ and $|D_n^\omega|/n\le 1$ completes the proof. ■

Skvortsov [SK1] proved that for $n\in\mathbb{P}$ and $x\in G$,

$$\begin{aligned}nK_n^\kappa(x) &= 1+\sum_{i=0}^{|n|-1}2^iD_{2^i}(x)+\sum_{i=0}^{|n|-1}2^ir_i(x)K_{2^i}^\omega(\tau_i(x))\\ &\quad+(n-2^{|n|})(D_{2^{|n|}}(x)+r_{|n|}(x)K_{n-2^{|n|}}^\omega)(\tau_{|n|}(x)).\end{aligned}\tag{7}$$

Proof of Theorem 2. By (7) we have

$$\begin{aligned}\|\sigma^*f\|_p &\le \Big\|\sup_{n\in\mathbb{P}}\Big|\int_G f(x+\cdot)\frac{1}{n}\,d\mu(x)\Big|\Big\|_p\\ &\quad+\Big\|\sup_{n\in\mathbb{P}}\sum_{i=0}^{|n|-1}\frac{2^i}{n}\Big|\int_G f(x+\cdot)D_{2^i}(x)\,d\mu(x)\Big|\Big\|_p\\ &\quad+\Big\|\sup_{n\in\mathbb{P}}\sum_{i=0}^{|n|-1}\frac{2^i}{n}\Big|\int_G f(x+\cdot)r_i(x)K_{2^i}^\omega(\tau_i(x))\,d\mu(x)\Big|\Big\|_p\\ &\quad+\Big\|\sup_{n\in\mathbb{P}}\Big(1-\frac{2^{|n|}}{n}\Big)\Big|\int_G f(x+\cdot)D_{2^{|n|}}(x)\,d\mu(x)\Big|\Big\|_p\\ &\quad+\Big\|\sup_{n\in\mathbb{P}}\Big(1-\frac{2^{|n|}}{n}\Big)\Big|\int_G f(x+\cdot)r_{|n|}(x)K_{n-2^{|n|}}^\omega(\tau_{|n|}(x))\,d\mu(x)\Big|\Big\|_p\\ &=: i_1+i_2+i_3+i_4+i_5.\end{aligned}$$

Let $1<p\le\infty$. Then, evidently, $i_1\le\|f\|_p$. For the maximal function f^* we see that the L^p norm $(1<p\le\infty)$ of f^* is bounded by $c\|f\|_p$. This implies

$$i_2\le\Big\|\sup_{n\in\mathbb{P}}\sum_{i=0}^{|n|-1}\frac{2^i}{n}|f^*(\cdot)|\Big\|_p\le c_p\|f\|_p.$$

By the definition of the operator L and Lemma 7 we get

$$i_3\le\Big\|\sup_{n\in\mathbb{P}}\sum_{i=0}^{|n|-1}\frac{2^i}{n}Lf(\cdot)\Big\|_p\le c_p\|f\|_p.$$

The definition of f^* obviously gives

$$i_4\le\Big\|\sup_{n\in\mathbb{P}}\Big(1-\frac{2^{|n|}}{n}\Big)|f^*(\cdot)|\Big\|_p\le c_p\|f\|_p.$$

Finally, we apply Lemma 8 to get an upper bound for i_5. Since

$$\Big|\int_G f(x+.)r_{|n|}(x)K_{n-2^{|n|}}^\omega)(\tau_{|n|}(x))\,d\mu(x)\Big|\le Mf(\cdot),$$

we have

$$i_5\le\Big\|\sup_{n\in\mathbb{P}}\Big(1-\frac{2^{|n|}}{n}\Big)Mf(\cdot)\Big\|_p\le c_p\|f\|_p.$$

That is, $\|\sigma^*f\|_p\le c_p\|f\|_p$ for all $f\in L^p(G)$ and $1<p\le\infty$. If $p=1$, then the same considerations as above give $\|\sigma^*f\|_1\le c\||f|\|_{H^1}$ for all $f\in H^1(G)$. On the other hand,

$$\begin{aligned}\mu(\sigma^*f>c\lambda) &\le \mu\Big(\sup_{n\in\mathbb{P}}\frac{1}{n}\Big|\int_G f\Big|>c\lambda\Big)\\ &\quad+\mu\Big(\sum_{i=0}^{|n|-1}\frac{2^i}{n}|f^*|>c\lambda\Big)+\mu\Big(\sum_{i=0}^{|n|-1}\frac{2^i}{n}Lf>c\lambda\Big)\\ &\quad+\mu\Big(\sup_{n\in\mathbb{P}}\Big(1-\frac{2^{|n|}}{n}\Big)|f^*|>c\lambda\Big)\\ &\quad+\mu\Big(\sup_{n\in\mathbb{P}}\Big(1-\frac{2^{|n|}}{n}\Big)Mf>c\lambda\Big)\\ &\le c\|f\|_1/\lambda. \ ■\end{aligned}$$

Proof of Theorem 1. The proof is based on the fact that the maximal operator σ^* is of weak type $(1,1)$ and on the standard density argument. Let $f\in L^1(G)$. Let $\varepsilon>0$ be arbitrary. Then there exists a Walsh–Kaczmarz polynomial P $(=\sum_{j=0}^k a_j\kappa_j$ for some $a_0,\dots,a_k\in\mathbb{C}$, $k\in\mathbb{N})$ for which $\|f-P\|_1<\varepsilon$. Consequently, since $\lim_n\sigma_n^\kappa P=P$ everywhere, the fact that

σ^* is of weak type $(1,1)$ implies for $\delta > 0$ that

$$\mu(\overline{\lim_{n\in\mathbb{P}}}|\sigma_n^\kappa f - f| > \delta) \le \mu(\overline{\lim_{n\in\mathbb{P}}}|\sigma_n^\kappa f - \sigma_n^\kappa P| > \delta/3) + \mu(\overline{\lim_{n\in\mathbb{P}}}|\sigma_n^\kappa P - P| > \delta/3) + \mu(\overline{\lim_{n\in\mathbb{P}}}|P - f| > \delta/3)$$
$$\le \frac{c}{\delta}\|f-P\|_1 + \mu(\sigma^*(f-P) > \delta/3) \le \frac{c}{\delta}\|f-P\|_1 \le \frac{c}{\delta}\varepsilon.$$

Letting $\varepsilon \to 0$ we have

$$\mu(\limsup|\sigma_n^\kappa f - f| > \delta) = 0$$

for any $\delta > 0$. This means that $\sigma_n^\kappa f \to f$ a.e. $(n \to \infty)$. ∎

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Almost 1-1 extensions of Furstenberg–Weiss type and applications to Toeplitz flows

by

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Abstract. Let (Z, T_Z) be a minimal non-periodic flow which is either symbolic or strictly ergodic. Any topological extension of (Z, T_Z) is Borel isomorphic to an almost 1-1 extension of (Z, T_Z). Moreover, this isomorphism preserves the affine-topological structure of the invariant measures. The above extends a theorem of Furstenberg–Weiss (1989). As an application we prove that any measure-preserving transformation which admits infinitely many rational eigenvalues is measure-theoretically isomorphic to a strictly ergodic Toeplitz flow.

Introduction. In 1989, Furstenberg and Weiss proved a theorem [F-W, Theorem 1] which can be informally expressed as follows: every topological point-transitive flow (X, T_X) which is an extension of a minimal non-periodic flow (Z, T_Z) is in some sense equivalent to a minimal flow (Y, T_Y) which is an almost 1-1 extension of (Z, T_Z). The equivalence is given by a Borel measurable injective map ϕ defined on a subset $X' \subset X$ whose mass is 1 for any T_X-invariant probability measure carried by X. Such a Borel embedding provides a 1-1 affine map ϕ^* (defined as the adjoint map on measures) from the set $P(X)$ of all T_X-invariant probability measures carried by X into the set $P(Y)$ defined analogously for the flow (Y, T_Y). Moreover, for every $\mu \in P(X)$, ϕ is a measure-theoretic isomorphism between the measure-preserving transformations $(X, \mathcal{B}_X, \mu, T_X)$ and $(Y, \mathcal{B}_Y, \phi^*(\mu), T_Y)$ (here $\mathcal{B}_X$ and $\mathcal{B}_Y$ denote the σ-fields of Borel measurable sets in X and Y, respectively).

In this paper we improve the Furstenberg–Weiss theorem. By the methods of symbolic dynamics we obtain a stronger isomorphism under even

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