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On Denjoy–Dunford and Denjoy–Pettis integrals

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Abstract. The two main results of this paper are the following: (a) If X is a Banach space and $f:[a,b]\to X$ is a function such that x^*f is Denjoy integrable for all $x^*\in X^*$, then f is Denjoy–Dunford integrable, and (b) There exists a Dunford integrable function $f:[a,b]\to c_0$ which is not Pettis integrable on any subinterval in $[a,b]$, while $\int_J f$ belongs to c_0 for every subinterval J in $[a,b]$. These results provide answers to two open problems left by R. A. Gordon in [4]. Some other questions in connection with Denjoy–Dunford and Denjoy–Pettis integrals are studied.

1. Introduction. Gordon introduced in [4] two extensions of the classical (real) Denjoy integral for Banach-valued functions: the Denjoy–Dunford and Denjoy–Pettis integrals. We solve here two problems left open by him and study quite thoroughfully these integrals. We show that the relationships between the Lebesgue and Denjoy integral on the one hand, and the Lebesgue and Dunford and Pettis integrals on the other, have a clear and natural translation when we consider Gordon's integrals. We think that this provides quite a complete picture of them and fills a gap between some parts of real analysis and Banach space theory. Techniques of both fields are used, and we have tried to make our paper easy to understand by specialists of any of them. Since this paper is a sort of continuation of [4], in case of any doubt the reader may consult that paper.

Let us begin with a glance at the Denjoy integral [4]. We will use this name but it should be observed that it is also called "Khintchine integral" (see [5]), "$\mathcal{D}$-integral", "Denjoy integral in the wide sense" or "Denjoy–Khintchine integral" (see [7]).

We do not wish to go into technical aspects (for this see [4], [5] or [7]), but let us recall a few fundamental facts. Perhaps one of the main features of this integral is that it generalizes the Lebesgue integral and provides a "good"

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fundamental theorem of calculus ("better" than the one for the Lebesgue integral). This is actually contained in the definition we give below. We have to comment first on two concepts appearing in this definition: ACG function and approximate derivative (for the precise definitions, see [5] or [7]). Concerning the first one, it will be enough to know that ACG functions are continuous functions which generalize absolutely continuous functions (in fact, "ACG" means "*absolutely continuous in the generalized sense*"), and that ACG functions in $[a,b]$ are ACG in every subinterval of $[a,b]$. Concerning the second concept let us just say that it is an extension of the usual one of derivative.

DEFINITION 1 (VIII.1 of [7], Definition 11 of [4], 15.1 of [5]). A function $f:[a,b]\to\mathbb{R}$ is *Denjoy integrable* on $[a,b]$ if there exists an ACG function $F:[a,b]\to\mathbb{R}$ such that $F'_{\mathrm{ap}}=f$ almost everywhere on $[a,b]$, where F'_{ap} denotes the approximate derivative of F. In this case, we write

$$\int_a^b f = F(b)-F(a).$$

We say that f is *Denjoy integrable on a subset* A of $[a,b]$ if $f\chi_A$ is Denjoy integrable on $[a,b]$, and in this case we write $\int_A f=\int_a^b f\chi_A$.

It can be shown that this integral has the "usual" properties of an integral and we do not even mention them. However, there are two theorems about it which deserve some attention. They reflect quite precisely the relationship between this integral and Lebesgue's, and are very characteristic of the former. They will also be particularly interesting for us. Roughly speaking, the first one says that the Denjoy integral is not too far from Lebesgue's (in fact, it asserts that any Denjoy integrable function is Lebesgue integrable in some, and therefore in "many", portions). The second one provides a method to construct Denjoy integrable functions which are not Lebesgue integrable (see Remark 1(c) below).

Recall that a *portion* of a subset A of $\mathbb{R}$ is any nonempty subset of A of the form $A\cap(\alpha,\beta)$, with $\alpha,\beta\in\mathbb{R}$, $\alpha<\beta$. If C is a closed subset of $\mathbb{R}$, and (α,β) is an open interval in $\mathbb{R}$ which meets C, then $(\alpha,\beta)\setminus C=\bigcup_n I_n$, where (I_n) is a sequence of disjoint open intervals. They are said to be *contiguous* to the portion $C\cap(\alpha,\beta)$. As usual, we denote by ω the *oscillation* of a function, that is, if g is a function defined on $[a,b]$ then

$$\omega(g,[a,b])=\sup\{|g(t_2)-g(t_1)|:t_1,t_2\in[a,b]\};$$

in particular, for the oscillation of the indefinite integral, we have

$$\omega\Big(\int_a^t f,\ [a,b]\Big)=\sup\Big\{\Big|\int_{t_1}^{t_2} f\Big|:t_1,t_2\in[a,b]\Big\}.$$

THEOREM 1 (Theorem 15.10 of [5], Theorem VIII(1.4) of [7]). *Let $f:[a,b]\to\mathbb{R}$ be Denjoy integrable on $[a,b]$. Let P be a closed set in $[a,b]$. Then there exists a portion P_0 of P such that f is Lebesgue integrable on P_0 and if $((a_k,b_k))_k$ is an enumeration of the intervals contiguous to P_0, then the series $\sum_k\int_{a_k}^{b_k}f$ is absolutely convergent and $\lim_k\ \omega(\int_{a_k}^t f,[a_k,b_k])=0$.*

THEOREM 2 (Theorem 15.13 of [5], Theorem VIII(5.1) of [7]). *Let E be a bounded, closed subset of $\mathbb{R}$ with bounds a and b and let $((a_k,b_k))_k$ be an enumeration of the intervals contiguous to E in (a,b). Suppose that $f:[a,b]\to\mathbb{R}$ is Denjoy integrable on E and on each interval $[a_k,b_k]$. If $\lim_k\ \omega(\int_{a_k}^t f,[a_k,b_k])=0$ and the series $\sum_k\int_{a_k}^{b_k}f$ is absolutely convergent, then f is Denjoy integrable on $[a,b]$ and*

$$\int_a^b f=\int_a^b f\chi_E+\sum_{k=1}^{\infty}\int_{a_k}^{b_k}f.$$

REMARK 1. (a) In the original statements of the preceding theorems sometimes one reads "perfect" instead of "closed". It is not difficult to see that this is irrelevant.

(b) Although, strictly speaking, none of the preceding theorems is a converse of the other, it is clear that there is a strong symmetry between them. We have included the assertion on the oscillations in our statement of Theorem 1 to stress the symmetry; however, very often in the literature, that assertion is not explicitly given. In any case, notice that clearly since we are assuming f is Denjoy integrable on $[a,b]$, the primitive F which appears in the definition is ACG and therefore (uniformly) continuous on $[a,b]$, hence $\lim_k\ \omega(\int_{\alpha_k}^t f,[\alpha_k,\beta_k])=\lim_k\omega(F(t),[\alpha_k,\beta_k])=0$ for any sequence $([\alpha_k,\beta_k])_k$ of intervals whose lengths go to zero. *A fortiori*, we have the following fact: *if f is Denjoy integrable on $[a,b]$, and $([\alpha_k,\beta_k])_k$ is a sequence of nonoverlapping intervals, then $\lim_k\ \omega(\int_{\alpha_k}^t f,[\alpha_k,\beta_k])=0$.*

(c) For any sequence $([\alpha_k,\beta_k])_k$ of nonoverlapping intervals, and $E=[a,b]\setminus\bigcup_k(\alpha_k,\beta_k)$, it is not difficult to construct functions f as in the hypothesis of Theorem 2, and such that $\sum_k\int_{a_k}^{b_k}|f|=\infty$. It is clear that they provide examples of Denjoy integrable functions which are not Lebesgue integrable.

The preceding theorem and the fact mentioned in Remark 1(b) above yield immediately the following well known result:

COROLLARY 1. *Let $f:[a,b]\to\mathbb{R}$ be Denjoy integrable on $[a,b]$ and let E be a closed set in $[a,b]$ such that f is Denjoy integrable on E. Let $((a_k,b_k))_k$ be an enumeration of the intervals contiguous to E in (a,b), and assume* *that the series* $\sum_k \int_{a_k}^{b_k} f$ *is absolutely convergent. Then*

$$\int_a^b f = \int_a^b f\chi_E + \sum_{k=1}^{\infty} \int_{a_k}^{b_k} f.$$

Before giving Gordon's definitions, let us recall the definitions of the Dunford and Pettis integrals (see for instance [2]). From now on X will be a Banach space.

A function $f : [a,b] \to X$ is said to be *Dunford integrable* on $[a,b]$ if for each $x^* \in X^*$ the function x^*f is Lebesgue integrable. In this case, as a consequence of the closed graph theorem, for every measurable subset A of $[a,b]$ there exists a vector x_A^{**} in X^{**} such that $\langle x^*, x_A^{**}\rangle = \int_A x^*f$ for all $x^* \in X^*$. The vector x_A^{**} is called the *Dunford integral* of f on A, and is denoted by

$$(\mathrm{D}) \int_A f.$$

A function $f : [a,b] \to X$ is said to be *Pettis integrable* on $[a,b]$ if it is Dunford integrable on $[a,b]$ and $x_A^{**} \in X$ for every measurable subset A of $[a,b]$.

The following definition extends the Denjoy integral to Banach-valued functions, exactly in the same way as the Dunford and Pettis integrals are extensions of the Lebesgue integral.

DEFINITION 2 (Gordon [4]). (a) The function $f : [a,b] \to X$ is *Denjoy–Dunford integrable* on $[a,b]$ if for each $x^* \in X^*$ the function x^*f is Denjoy integrable on $[a,b]$ and for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $\langle x^*, x_I^{**}\rangle = \int_I x^*f$ for all $x^* \in X^*$.

(b) The function $f : [a,b] \to X$ is *Denjoy–Pettis integrable* on $[a,b]$ if it is Denjoy–Dunford integrable on $[a,b]$ and $x_I^{**} \in X$ for every interval I in $[a,b]$.

The vector $x_{[a,b]}^{**}$ is called the *Denjoy–Dunford* (respectively, *Denjoy–Pettis*) *integral* of f on $[a,b]$, and is denoted by

$$(\mathrm{DD}) \int_a^b f \quad \Big(\text{respectively, } (\mathrm{DP}) \int_a^b f\Big).$$

A (real) Denjoy integrable function on $[a,b]$ is not necessarily integrable on all measurable subsets of $[a,b]$; in fact, it is only if the function is absolutely integrable, or equivalently, Lebesgue integrable [5, Theorem 15.9(c)], [7, Chapter VIII, Theorem (1.1)3 and Theorem (1.3)]. However, a Denjoy integrable function is of course Denjoy integrable on all subintervals of $[a,b]$ (we have implicitly used this fact in the preceding definition). The analogous result is also true for Denjoy–Dunford and Denjoy–Pettis vector-valued integrable functions on $[a,b]$: it is enough to notice that for each interval J in $[a,b]$ all subintervals of J are subintervals of $[a,b]$.

The first problem we consider is the following. Remember that given $f : [a,b] \to X$, Lebesgue integrability of each x^*f is enough to guarantee the existence of the Dunford integral. As Gordon points out (see [4, p. 80]), it seems natural to ask if the analogous result is true for the Denjoy–Dunford integral, that is, whether it is true that a function f is Denjoy–Dunford integrable whenever x^*f is Denjoy integrable for all $x^* \in X^*$. We show that the answer is affirmative (Theorem 3).

The second problem we consider is to generalize the crucial Theorems 1 and 2 for these vector-valued integrals. This will complete some work already done in [4]. We get very natural generalizations, showing that these integrals are related to Dunford and Pettis integrals exactly in the same way as Denjoy's is related to Lebesgue's. In this connection, we solve another problem posed by Gordon: we give an example showing that the natural extension of Theorem 1 does not hold for the Banach space c_0. It should be observed that it is actually an example on the classical Dunford and Pettis integrals: we construct a Dunford integrable function $f_0 : [a,b] \to c_0$ which is not Pettis integrable on any subinterval, while $(\mathrm{D})\int_J f_0$ belongs to c_0 for all subintervals J of $[a,b]$. We hope that our construction, which essentially relies on Lemma 3, could find further applications in real analysis.

2. Denjoy–Dunford integrability.

To prove our first main result (Theorem 3) we need some preliminaries. We begin with an easy proposition generalizing a well known result on the Denjoy integral (see for instance [5, Theorem 15.12]).

PROPOSITION 1. *Assume that* $f : [a,b] \to X$ *is Denjoy–Dunford integrable on* $[a,t]$ *for all* $t \in [a,b)$, *and for each* $x^* \in X^*$ *the limit* $\lim_{t\to b} \int_a^t x^*f$ *exists. Then* f *is Denjoy–Dunford integrable on* $[a,b]$, *and*

$$\Big\langle x^*, (\mathrm{DD}) \int_a^b f \Big\rangle = \lim_{t\to b} \Big\langle x^*, (\mathrm{DD}) \int_a^t f \Big\rangle$$

for each $x^* \in X^*$.

Proof. By [5, Theorem 15.12], x^*f is Denjoy integrable on $[a,b]$ for all $x^* \in X^*$. On the other hand, take $c \in [a,b)$ and any sequence (t_n) in $[a,b)$ convergent to b. Define

$$L_c(x^*) = \lim_n \int_c^{t_n} x^*f = \lim_n \Big\langle x^*, (\mathrm{DD}) \int_c^{t_n} f \Big\rangle.$$

The uniform boundedness principle guarantees that the linear functional L_c is continuous on X^*. Then it is immediate that f is Denjoy–Dunford integrable on $[a,b]$. Taking $c = a$, we get the desired equality. ■

The following lemma is the key to proving Theorem 3.

LEMMA 1. *Let $f : [a,b] \to X$ be such that x^*f is Denjoy integrable on $[a,b]$ for all $x^* \in X^*$. Let P be a closed subset of $[a,b]$ and assume that f is Denjoy–Dunford integrable on each open interval J disjoint from P. Then there exists a portion P_0 of P such that if (I_n) is an enumeration of the intervals contiguous to P_0 then the series*

$$\sum_n \int_{I_n} x^*f$$

is absolutely convergent for every $x^* \in X^*$.

Proof. Let (J_m) be an enumeration of all open intervals in $[a,b]$ with rational endpoints such that $J_m \cap P \neq \emptyset$. Let (K_n) be an enumeration of all open intervals contiguous to P in (a,b). For each $m \in \mathbb{N}$, the sequence $(J_m \cap K_n)_n$ is an enumeration of all open intervals contiguous to the portion $J_m \cap P$ (of course, in this enumeration some intervals may be empty). Therefore, to prove the result it is enough to show that there exists $m_0 \in \mathbb{N}$ such that

$$\sum_n \Big| \int_{J_{m_0} \cap K_n} x^*f \Big| < \infty$$

for all $x^* \in X^*$. Assume this is not true. For each $n \in \mathbb{N}$ the function f is Denjoy–Dunford integrable on K_n, since $K_n \cap P = \emptyset$. Therefore

$$X^* \to \mathbb{R}, \quad x^* \to \int_{J_m \cap K_n} x^*f,$$

defines a continuous linear functional for each $m \in \mathbb{N}$. So we conclude that for each $m, j \in \mathbb{N}$,

$$T_j^m : X^* \to \ell_1, \quad x^* \to \Big(\int_{J_m \cap K_1} x^*f, \ldots, \int_{J_m \cap K_j} x^*f, 0, 0, \ldots \Big),$$

defines a bounded linear operator. Our assumption means that for each $m \in \mathbb{N}$ there exists $x_m^* \in X^*$ such that

$$\lim_j \|T_j^m(x_m^*)\|_1 = \sum_n \Big| \int_{J_m \cap K_n} x^*f \Big| = \infty.$$

Then the theorem of condensation of singularities [3, p. 81] implies that there exists $x_0^* \in X^*$ such that

$$(1) \qquad \sum_n \Big| \int_{J_m \cap K_n} x_0^*f \Big| = \lim_j \|T_j^m(x_0^*)\|_1 = \infty$$

for all $m \in \mathbb{N}$. Finally, notice that each portion of P contains a portion of the form $P \cap J_m$ for some $m \in \mathbb{N}$, and for each $m \in \mathbb{N}$ the $J_m \cap K_n$'s are the intervals contiguous to P in J_m. Hence, (1) and Theorem 1 show that x_0^*f cannot be Denjoy integrable on $[a,b]$, which is a contradiction. ∎

THEOREM 3. *A function $f : [a,b] \to X$ is Denjoy–Dunford integrable on $[a,b]$ if and only if x^*f is Denjoy integrable on $[a,b]$ for all $x^* \in X^*$.*

Proof. Let us show the nontrivial implication. Let $f : [a,b] \to X$ be such that x^*f is Denjoy integrable for all $x^* \in X^*$. Let S be the set of all points $t \in [a,b]$ such that f is Denjoy–Dunford integrable on no neighbourhood of t.

CLAIM. *Let J be an open subinterval of $[a,b]$. Then f is Denjoy–Dunford integrable on J if and only if $J \cap S = \emptyset$.*

Proof. Necessity is obvious. Let us show sufficiency. Let $J = (c,d)$ be an open interval in $[a,b]$ which does not meet S. By compactness, it is clear that f is Denjoy–Dunford integrable on any closed subinterval $[c_1,d_1]$ of (c,d), and hence on (c,d) by Proposition 1. This completes the proof of the Claim.

If S is empty then we are done. Assume that S is nonempty; we will reach a contradiction. Gordon's Theorem 33 of [4] guarantees that under our hypothesis, each closed set in $[a,b]$ has a portion on which f is Dunford integrable. In particular, we get a portion $S_0 = S \cap (c_0,d_0)$ on which f is Dunford integrable. Now it is immediate that the closed set $\overline{S}_0$ satisfies the assumptions of the preceding lemma on $[c_0,d_0]$. So there exists a portion $S_1 = \overline{S}_0 \cap (c_1,d_1) = S \cap (c_1,d_1)$ of $\overline{S}_0$ (of course, S_1 is also a portion of S) on which f is Dunford integrable, and such that, if (I_n) is an enumeration of the intervals contiguous to S_1 in (c_1,d_1), then the series

$$\sum_n \int_{I_n} x^*f$$

is absolutely convergent for every $x^* \in X^*$.

To complete the proof it is enough to show that f must be Denjoy–Dunford integrable on (c_1,d_1): since (c_1,d_1) meets S, this will contradict the definition of S (or the Claim). For completeness we now give a direct proof of this fact; notice however, that it can be deduced from Theorem 31 of [4].

Let J be an interval in $[c_1,d_1]$ and let $x^* \in X^*$. Since f is Dunford integrable on S_1, x^*f is Lebesgue (and therefore Denjoy) integrable on S_1. On the other hand, the sequence $(I_n \cap J)_n$, in which we ignore the empty sets, is an enumeration of the intervals contiguous to $S \cap J$ in J, and notice that with the exception of at most two intervals, for all nonempty $I_n \cap J$'s we have $I_n \cap J = I_n$. So omitting at most two terms of the sequence $(\int_{I_n \cap J} x^*f)_n$, we can say that $\sum_n \int_{I_n \cap J} x^*f$ is a subseries of $\sum_n \int_{I_n} x^*f$, and so it is absolutely convergent. Hence, we can apply Corollary 1 to x^*f and $\overline{S}_1 \cap J = S \cap J$ on

J to deduce that

$$\int_J x^* f = \int_J x^* f \chi_{S \cap J} + \sum_n \int_{I_n \cap J} x^* f \tag{2}$$

for each $x^* \in X^*$.

For each $m \in \mathbb{N}$ define x_m^{**} by

$$x_m^{**}(x^*) = \int_J x^* f \chi_{S \cap J} + \sum_{n=1}^{m} \int_{I_n \cap J} x^* f.$$

Since f is Dunford integrable on $S \cap J \subset S \cap (c_1, d_1)$ and Denjoy–Dunford integrable on each $I_n \cap J \subset I_n$, the linear functionals x_m^{**} are continuous on X^*. Clearly, (2) means that

$$\int_J x^* f = \lim_m x_m^{**}(x^*)$$

for each $x^* \in X^*$. Therefore, the uniform boundedness principle guarantees that the linear functional x_J^{**} defined by

$$x_J^{**}(x^*) = \int_J x^* f$$

is continuous on X^*. Since this happens for all intervals J in $[c_1, d_1]$, we conclude that f is Denjoy–Dunford integrable on $[c_1, d_1]$. ∎

3. Extensions of Theorems 1 and 2. Let us now deal with the generalizations of Theorems 1 and 2 for Denjoy–Dunford and Denjoy–Pettis integrals. Gordon already gave the following generalizations of Theorem 1:

THEOREM 4 (Corollary 32 of [4]). *Assume that $f : [a, b] \to X$ is Denjoy–Dunford integrable on $[a, b]$, and let P be a closed set in $[a, b]$. Then there exists a portion P_0 of P such that f is Dunford integrable on P_0.*

THEOREM 5 (Theorem 38 of [4]). *Assume that X has no subspace isomorphic to c_0, and let $f : [a, b] \to X$ be Denjoy–Pettis integrable on $[a, b]$. Let P be a closed set in $[a, b]$. Then there exists a portion P_0 of P such that f is Pettis integrable on P_0.*

Now we wish to complete these theorems by describing also the behaviour of the series of integrals over the intervals contiguous to the closed set and the oscillations of the indefinite integrals in them. We also give the corresponding generalization of Theorem 2. We believe that in our generalizations the symmetry of the scalar results is preserved, which makes them the "right" generalizations.

Let us begin by recalling the most common notions of summability in Banach spaces (see [1], [6]). We say that the series $\sum x_n$ is *unconditionally convergent* if all its rearrangements converge. We say that the series $\sum x_n$ is *weakly unconditionally Cauchy* if for every $x^* \in X^*$ the series $\sum \langle x_n, x^* \rangle$ is absolutely convergent. In this last case, as an immediate consequence of the uniform boundedness principle, there exists $x^{**} \in X^{**}$ such that

$$\langle x^*, x^{**} \rangle = \sum_{n=1}^{\infty} \langle x_n, x^* \rangle$$

for all $x^* \in X^*$. Although it is not a standard notation, we denote this functional x^{**} by $w^*\text{-}\sum_{n=1}^{\infty} x_n$. In other words, $w^*\text{-}\sum_{n=1}^{\infty} x_n$ is the only functional in X^{**} such that

$$\Big\langle x^*, w^*\text{-}\sum_{n=1}^{\infty} x_n \Big\rangle = \sum_{n=1}^{\infty} \langle x_n, x^* \rangle \tag{3}$$

for all $x^* \in X^*$.

Of course, every unconditionally convergent series is weakly unconditionally Cauchy (and $\sum_{n=1}^{\infty} x_n = w^*\text{-}\sum_{n=1}^{\infty} x_n$). The converse is not true. The typical example of a non-unconditionally convergent weakly unconditionally Cauchy series is the canonical basis of c_0. However, one should recall that if the Banach space X has no subspace isomorphic to c_0 then every weakly unconditionally Cauchy series is unconditionally convergent; this is Bessaga and Pełczyński's classical theorem [1, Chapter V, Theorem 8].

We can now give our generalizations of Theorem 1 and 2 for Denjoy–Dunford integrals.

THEOREM 6. *Let $f : [a, b] \to X$ be Denjoy–Dunford integrable, and let P be a closed set in $[a, b]$. Then there exists a portion P_0 of P such that f is Dunford integrable on P_0 and if $((a_k, b_k))_k$ is an enumeration of the intervals contiguous to P_0 then the series $\sum_k (\mathrm{DD}) \int_{a_k}^{b_k} f$ is weakly unconditionally Cauchy and*

$$\lim_k \omega\Big(\int_{a_k}^{t} x^* f, [a_k, b_k] \Big) = 0$$

for each $x^* \in X^*$.

Proof. By Gordon's Theorem 4, P has a portion on which f is Dunford integrable, but Lemma 1 guarantees that this portion has itself a portion P_0 such that

$$\sum_n \Big| \Big\langle x^*, (\mathrm{DD}) \int_{I_n} f \Big\rangle \Big| = \sum_n \Big| \int_{I_n} x^* f \Big| < \infty$$

for all $x^* \in X^*$, where (I_n) is an enumeration of the intervals contiguous to P_0. Recall now that if $\sum x_n^*$ is a series in a dual X^* such that $\sum x_n^*(x)$ is absolutely convergent for each $x \in X$, then $\sum x_n^*$ is weakly unconditionally Cauchy (this is well known, and contained, for instance, in the proof of

Corollary 11 in Chapter V of [1]). Therefore, it is clear that the series

$$\sum_n (\mathrm{DD}) \int_{I_n} f$$

is weakly unconditionally Cauchy. Of course f is also Dunford integrable on P_0. Finally, given $x^* \in X^*$, since x^*f is Denjoy integrable, its indefinite integral is (uniformly) continuous in $[a,b]$. Then, using the fact that the lengths of the intervals (a_k, b_k) must go to zero, we conclude that $\omega(\int_{a_k}^t x^*f, [a_k, b_k])$ tends to 0. ■

THEOREM 7. *Let E be a bounded, closed subset of $\mathbb{R}$ with bounds a and b and let $((a_k, b_k))_k$ be an enumeration of the intervals contiguous to E in (a,b). Suppose that $f : [a,b] \to X$ is Denjoy–Dunford integrable on E and on each $[a_k, b_k]$. If $\lim_k \omega(\int_{a_k}^t x^*f, [a_k, b_k]) = 0$ for each $x^* \in X^*$ and the series $\sum_k \int_{a_k}^{b_k} f$ is weakly unconditionally Cauchy, then f is Denjoy–Dunford integrable on $[a,b]$ and*

$$(\mathrm{DD})\int_a^b f = (\mathrm{DD})\int_a^b f\chi_E + w^*\text{-}\sum_{k=1}^{\infty} (\mathrm{DD})\int_{a_k}^{b_k} f.$$

Proof. It is immediate that for each $x^* \in X^*$ the function x^*f is exactly as in the hypothesis of Theorem 2, and so we conclude that x^*f is Denjoy integrable and

$$\int_a^b x^*f = \int_a^b x^*f\chi_E + \sum_{k=1}^{\infty} \int_{a_k}^{b_k} x^*f.$$

Since x^*f is Denjoy integrable for each $x^* \in X^*$, Theorem 3 guarantees that f is Denjoy–Dunford integrable. On the other hand, by the very definition of the Denjoy–Dunford integral, the above equality means that

$$\Big\langle x^*, (\mathrm{DD})\int_a^b f\Big\rangle = \Big\langle x^*, (\mathrm{DD})\int_a^b f\chi_E\Big\rangle + \sum_{k=1}^{\infty} \Big\langle x^*, (\mathrm{DD})\int_{a_k}^{b_k} f\Big\rangle$$

for each $x^* \in X^*$. Of course, this is just our statement. ■

Let us now give our generalizations of Theorems 1 and 2 for Denjoy–Pettis integrals.

THEOREM 8. *Assume that X has no subspace isomorphic to c_0, let $f : [a,b] \to X$ be Denjoy–Pettis integrable, and let P be a closed set in $[a,b]$. Then there exists a portion P_0 of P such that f is Pettis integrable on P_0 and if $((a_k, b_k))_k$ is an enumeration of the intervals contiguous to P_0 then the series $\sum_k (\mathrm{DP})\int_{a_k}^{b_k} f$ is unconditionally convergent, and $\lim_k \omega(\int_{a_k}^t x^*f, [a_k, b_k]) = 0$ for each $x^* \in X^*$.*

Proof. By Gordon's Theorem 5, f is Pettis integrable on a portion of P (and hence on any measurable subset of this portion). If we apply Theorem 6 to f in this portion, we get a smaller portion P_0 such that if $((a_k, b_k))_k$ is an enumeration of the intervals contiguous to P_0, then the series $\sum_k (\mathrm{DP})\int_{a_k}^{b_k} f$ is weakly unconditionally Cauchy, and $\lim_k \omega(\int_{a_k}^t x^*f, [a_k, b_k]) = 0$ for each $x^* \in X^*$. Since $\sum_k (\mathrm{DP})\int_{a_k}^{b_k} f$ is a series in X and X has no subspace isomorphic to c_0, Bessaga and Pełczyński's classical theorem [1, Chapter V, Theorem 8] guarantees that the series is actually unconditionally convergent. ■

THEOREM 9. *Let E be a bounded, closed subset of $\mathbb{R}$ with bounds a and b and let $((a_k, b_k))_k$ be an enumeration of the intervals contiguous to E in (a,b). Suppose that $f : [a,b] \to X$ is Denjoy–Pettis integrable on E and on each $[a_k, b_k]$. If $\lim_k \omega(\int_{a_k}^t x^*f, [a_k, b_k]) = 0$ for each $x^* \in X^*$ and the series $\sum_k (\mathrm{DP})\int_{a_k}^{b_k} f$ is unconditionally convergent, then f is Denjoy–Pettis integrable on $[a,b]$ and*

$$(\mathrm{DP})\int_a^b f = (\mathrm{DP})\int_a^b f\chi_E + \sum_{k=1}^{\infty} (\mathrm{DP})\int_{a_k}^{b_k} f.$$

Proof. Notice first that we can apply Theorem 7, so that f is Denjoy–Dunford integrable on $[a,b]$. To show that f is in fact Denjoy–Pettis integrable we need to show that $(\mathrm{DD})\int_J f$ belongs to X for each (closed) interval J in $[a,b]$.

Take such a J. Define $E_0 = E \cap J$. On the one hand, E_0 is a closed set, and f is Denjoy–Pettis integrable on E_0, because this just says that $f\chi_{E_0}$ is Denjoy–Pettis integrable on J. On the other hand, $((a_k, b_k) \cap J)_k$ is an enumeration of the intervals contiguous to E_0 in J (of course we should omit here the empty intersections). Observe also that $((a_k, b_k) \cap J)_k$ is "almost" a subsequence of $((a_k, b_k))_k$. To be precise, except possibly two intervals (those which meet the endpoints of J), it is clear that $(a_k, b_k) \cap J$ is either empty or (a_k, b_k). Moreover, the two possible exceptional intervals are subintervals of (a_k, b_k) and so f is Denjoy–Pettis integrable on them. Therefore, it is clear that $\sum_k (\mathrm{DP})\int_{[a_k,b_k]\cap J} f$ is an unconditionally convergent series in X, and so $w^*\text{-}\sum_k (\mathrm{DP})\int_{[a_k,b_k]\cap J} f$ is just $\sum_k (\mathrm{DP})\int_{[a_k,b_k]\cap J} f$. Thus, if we apply Theorem 7 to E_0 in J, we get

$$(\mathrm{DD})\int_J f = (\mathrm{DP})\int_J f\chi_{E_0} + \sum_{k=1}^{\infty} (\mathrm{DP}) \int_{[a_k,b_k]\cap J} f,$$

and in particular, $(\mathrm{DD})\int_J f$ belongs to X. Hence, f is Denjoy–Pettis integrable in $[a,b]$. Finally, for $J = [a,b]$, the preceding equality is just the equality in the statement. ■

4. An example and its consequences. A natural question concerning the theorems in the preceding section is whether the hypothesis "X has no subspace isomorphic to c_0" in Theorem 5 (or in Theorem 8) can be removed. It was already posed by Gordon (see [4, paragraph before Theorem 38]). We give an example showing that the answer is negative. As an immediate consequence we deduce that the assertion of Theorem 5 holds precisely in the spaces X not containing c_0.

Of course, here the interest of our example is in connection with the Denjoy–Pettis integral. However, as we have already pointed out, it is actually an example on the classical Dunford and Pettis integrals.

To study the example we first give two lemmas. The first one is a simple exercise on the Dunford integral. We include a proof for completeness. The second one will be the crucial ingredient in the construction of our example.

LEMMA 2. *Let (f_k) be a sequence of X-valued Dunford integrable functions defined on $[a,b]$. Assume that for almost all t the series $\sum_k f_k(t)$ converges and define*

$$f(t)=\sum_k f_k(t).$$

Suppose that

$$\sum_k \sup\Big\{\int_a^b |x^* f_k(t)|\,dt : x^*\in X^*,\ \|x^*\|\le 1\Big\}<\infty.$$

Then f is Dunford integrable, and for each measurable subset A of $[a,b]$ the series

$$\sum_k (\mathrm{D})\int_A f_k$$

is convergent and

$$(\mathrm{D})\int_A f=\sum_k (\mathrm{D})\int_A f_k.$$

Proof. For each x^* in $B(X^*)$, the unit ball of X^*, we have

$$\int_a^b |x^* f(t)|\,dt=\int_a^b\Big|\sum_k x^* f_k(t)\Big|\,dt\le\sum_k\int_a^b |x^* f_k(t)|\,dt<\infty.$$

Therefore, f is Dunford integrable on $[a,b]$. Notice also that the convergence of the series $\sum_k \int_a^b |x^* f_k(t)|\,dt$ implies that for each measurable subset A of $[a,b]$ we have

$$\int_A \sum_k x^* f_k(t)\,dt=\sum_k\int_A x^* f_k(t)\,dt. \tag{4}$$

On the other hand, given a measurable subset A of $[a,b]$, we have

$$\begin{aligned}\sum_k\Big\|(\mathrm{D})\int_A f_k\Big\| &= \sum_k \sup\Big\{\Big\langle x^*,(\mathrm{D})\int_A f_k\Big\rangle : x^*\in B(X^*)\Big\}\\ &= \sum_k \sup\Big\{\int_A x^* f_k : x^*\in B(X^*)\Big\}\\ &\le \sum_k \sup\Big\{\int_a^b |x^* f_k| : x^*\in B(X^*)\Big\}<\infty.\end{aligned}$$

So, the series $\sum_k (\mathrm{D})\int_A f_k$ is convergent. Therefore, for each $x^*\in X^*$, it follows from (4) that

$$\begin{aligned}\Big\langle x^*,(\mathrm{D})\int_A f\Big\rangle &= \int_A x^* f(t)\,dt=\int_A\sum_k x^* f_k(t)\,dt\\ &= \sum_k\int_A x^* f_k(t)\,dt=\sum_k\Big\langle x^*,(\mathrm{D})\int_A f_k\Big\rangle\\ &= \Big\langle x^*,\sum_k (\mathrm{D})\int_A f_k\Big\rangle. \ \blacksquare\end{aligned}$$

LEMMA 3. *Let (J_k) be a sequence of closed nontrivial intervals. Then there exists a double sequence (I_n^k) of nontrivial closed intervals with the following properties*:

(i) *For each $k\in\mathbb{N}$ the I_n^k's are subintervals of J_k.*
(ii) *For each $k\in\mathbb{N}$ we have $\max I_n^k<\min I_{n+1}^k$ for all $n\in\mathbb{N}$.*
(iii) *$\sum_{k,n\in\mathbb{N}}|I_n^k|<\infty$, where $|\ |$ denotes length (Lebesgue measure).*
(iv) *If $1\le j\le k-1$, then one (and only one) of the following two conditions holds*:
1. *$\bigcup_m I_m^k$ and $\bigcup_n I_n^j$ are disjoint.*
2. *There exists $n_0\in\mathbb{N}$ such that $\bigcup_m I_m^k\subset I_{n_0}^j$.*

Proof. We construct the double sequence by induction on k. For $k=1$, take any sequence $(I_n^1)_n$ of nontrivial closed subintervals of J_1 satisfying (ii) and

$$\sum_n |I_n^1|<1/2.$$

Assume we have found $(I_n^1)_n, (I_n^2)_n,\ldots,(I_n^{k-1})_n$ satisfying (i), (ii), (iv) and

$$\sum_n |I_n^j|<1/2^j$$

for $j=1,\ldots,k-1$. Let us find $(I_n^k)_n$. We denote by $\mathrm{int}(I)$ the interior of the interval I. If

$$\mathrm{int}(J_k)\cap\mathrm{int}(I_n^j)=\emptyset$$

for $j=1,\dots,k-1$ and $n\in\mathbb{N}$, we can take as $(I_n^k)_n$ any sequence of nontrivial closed intervals in J_k satisfying (ii) and

$$\sum_n |I_n^k| < 1/2^k. \tag{5}$$

Otherwise, let j_0 be the greatest $j\in\{1,\dots,k-1\}$ such that $\operatorname{int}(J_k)$ meets $\bigcup_n \operatorname{int}(I_n^j)$, and let $n_0\in\mathbb{N}$ be such that

$$\operatorname{int}(J_k)\cap\operatorname{int}(I_{n_0}^{j_0})\neq\emptyset.$$

Then we can take any sequence $(I_n^k)_n$ of closed nontrivial intervals in $\operatorname{int}(J_k)\cap\operatorname{int}(I_{n_0}^{j_0})$ satisfying (ii) and (5). It is straightforward to show that the double sequence so constructed has the required properties. ■

EXAMPLE. *There exists a measurable function $f_0:[a,b]\to c_0$ such that*

1. *f_0 is Dunford integrable,*
2. *$(\mathrm{D})\int_J f_0$ belongs to c_0 for each subinterval J in $[a,b]$, but*
3. *f_0 is not Pettis integrable on any subinterval J in $[a,b]$.*

To construct such a function let us begin with a much easier task. Let us recall a standard way of constructing a measurable Dunford integrable function $f:[a,b]\to c_0$ such that $(\mathrm{D})\int_J f$ belongs to c_0 for each subinterval J in $[a,b]$, while f is not Pettis integrable on $[a,b]$. We take any sequence (I_n) of nontrivial closed subintervals of $[a,b]$ such that $\max I_n<\min I_{n+1}$ for all $n\in\mathbb{N}$, and define $f:[a,b]\to c_0$ by

$$f(t)=\left(\frac{1}{2|I_{2n-1}|}\chi_{I_{2n-1}}(t)-\frac{1}{2|I_{2n}|}\chi_{I_{2n}}(t)\right)_n.$$

It is immediate that it is a well defined measurable function, and for each (α_n) in the unit ball of $c_0^*=\ell_1$ we have

$$\begin{aligned}\int_a^b |\langle f(t),(\alpha_n)\rangle|\,dt &= \int_a^b\left|\sum_n \alpha_n\left(\frac{1}{2|I_{2n-1}|}\chi_{I_{2n-1}}(t)-\frac{1}{2|I_{2n}|}\chi_{I_{2n}}(t)\right)\right|dt\\ &\le \sum_n|\alpha_n|\int_a^b\left(\frac{1}{2|I_{2n-1}|}\chi_{I_{2n-1}}(t)+\frac{1}{2|I_{2n}|}\chi_{I_{2n}}(t)\right)dt\\ &=\sum_n|\alpha_n|\le 1.\end{aligned}$$

So f is Dunford integrable. Given a measurable subset E of $[a,b]$, to determine $(\mathrm{D})\int_E f(t)\,dt$, which is a sequence in $\ell_\infty=c_0^{**}$, it is enough to evaluate it on the canonical basis (e_n) of $\ell_1=c_0^*$. So we get

$$\begin{aligned}\Big\langle e_n,(\mathrm{D})\int_E f(t)\,dt\Big\rangle &= \int_E\left(\frac{1}{2|I_{2n-1}|}\chi_{I_{2n-1}}(t)-\frac{1}{2|I_{2n}|}\chi_{I_{2n}}(t)\right)dt\\ &=\frac{|E\cap I_{2n-1}|}{2|I_{2n-1}|}-\frac{|E\cap I_{2n}|}{2|I_{2n}|}.\end{aligned}$$

Hence, we get

$$(\mathrm{D})\int_E f(t)\,dt=\left(\frac{|E\cap I_{2n-1}|}{2|I_{2n-1}|}-\frac{|E\cap I_{2n}|}{2|I_{2n}|}\right)_n. \tag{6}$$

To see that f is not Pettis integrable we take $A_0=\bigcup_n I_{2n-1}$. Using (6), we get

$$(\mathrm{D})\int_{A_0} f(t)\,dt=\left(\frac12,\frac12,\frac12,\dots\right)\in\ell_\infty\setminus c_0.$$

On the other hand, it follows from (6) that if A is any measurable subset of $[a,b]$ containing all the I_m's, then

$$(\mathrm{D})\int_A f(t)\,dt=0. \tag{7}$$

To see that $(\mathrm{D})\int_J f$ belongs to c_0 for each subinterval J in $[a,b]$, assume first that J meets only a finite number of I_n's, say $I_1,\dots,I_{2n}$. By (6), $(\mathrm{D})\int_J f$ is just

$$\left(\frac{|J\cap I_1|}{2|I_1|}-\frac{|J\cap I_2|}{2|I_2|},\dots,\frac{|J\cap I_{2n-1}|}{2|I_{2n-1}|}-\frac{|J\cap I_{2n}|}{2|I_{2n}|},0,0,0,\dots\right)$$

and so it belongs to c_0. Take now a subinterval $J=[c,d]$ in $[a,b]$ which meets an infinite number of I_n's. Since $\max I_n<\min I_{n+1}$ for all $n\in\mathbb{N}$, it is clear that $[a,c]$ meets only a finite number of the I_n's, and $[a,d]$ contains all of them. Then $[a,c]$ is a measurable set of the type just studied, and we can apply (7) to $[a,d]$. Therefore,

$$0=(\mathrm{D})\int_a^d f=(\mathrm{D})\int_a^c f+(\mathrm{D})\int_c^d f$$

and so

$$(\mathrm{D})\int_c^d f=-(\mathrm{D})\int_a^c f\in c_0.$$

Hence we have shown that $(\mathrm{D})\int_J f\in c_0$ for each subinterval J of $[a,b]$.

Let us now begin the construction of a function with the required properties 1–3.

Consider first an enumeration (J_k) of all intervals in $[a,b]$ with rational endpoints. Our idea is to construct a function f_0 reproducing the preceding function to a small scale in each J_k. To do this we add up many functions of the type constructed before. Of course, the point is that we have to be very careful not to let them interfere with each other. This is why we need the preceding lemma.

Applying the preceding lemma to the sequence (J_k), we get a double sequence (I_n^k) of nontrivial closed intervals satisfying the corresponding conditions (i)–(iv). For each $k \in \mathbb{N}$ denote by f_k the function defined using the sequence $(I_n^k)_n$ according to the preceding procedure. That is, $f_k : [a,b] \to c_0$ is defined by

$$f_k(t) = \left(\frac{1}{2|I_{2n-1}^k|}\chi_{I_{2n-1}^k}(t) - \frac{1}{2|I_{2n}^k|}\chi_{I_{2n}^k}(t)\right)_n.$$

Let us summarize the main properties of f_k:

(a) f_k is a measurable Dunford integrable function.

(b) $\int_a^b |x^* f_k(t)|\,dt \le 1$ for each x^* in the unit ball of $c_0^* = \ell_1$.

(c) (D) $\int_A f_k = 0$ for each measurable subset A of $[a,b]$ containing

$$\operatorname{supp}(f_k) = \{t \in [a,b] : f_k(t) \neq 0\} = \bigcup_n I_n^k.$$

(d) (D) $\int_J f_k$ belongs to c_0 for each subinterval J in $[a,b]$.

(e) (D) $\int_{A_k} f_k \in \ell_\infty \setminus c_0$, where $A_k = \bigcup_n I_{2n-1}^k$.

Notice that the set of all $t \in [a,b]$ such that $f_k(t) \neq 0$ for infinitely many k's is

$$\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}\operatorname{supp}(f_k) = \bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}\bigcup_{n=1}^{\infty} I_n^k.$$

From Lemma 3(iii) we deduce that this is a null set, or, in other words, for almost all $t \in [a,b]$ the sequence $(f_k(t))$ has only finitely many terms different from 0. Therefore,

$$f_0(t) = \sum_{k=1}^{\infty} \frac{1}{2^k} f_k(t)$$

is a measurable function defined almost everywhere. Thanks to (a) and (b), we can apply Lemma 2 to conclude that f_0 is Dunford integrable and for each measurable subset E of $[a,b]$ the series $\sum_k \frac{1}{2^k}$ (D) $\int_E f_k$ is convergent and

(8) $$\text{(D)}\int_E f_0 = \sum_k \frac{1}{2^k}\,\text{(D)}\int_E f_k.$$

In particular, it follows from (d) that (D) $\int_J f_0 \in c_0$ for each subinterval J in $[a,b]$.

To finish our proof we have to show that f_0 is Pettis integrable in no subinterval J in $[a,b]$. Of course, it is enough to see that it is not Pettis integrable in J_k for any $k \in \mathbb{N}$. Given $k_0 \in \mathbb{N}$, by Lemma 3(i), $A_{k_0} = \bigcup_n I_{2n-1}^{k_0}$ is a subset of J_{k_0}. Let us show that (D) $\int_{A_{k_0}} f_0 \notin c_0$. By (8) we have

(9) $$\begin{aligned}\text{(D)}\int_{A_{k_0}} f_0 &= \sum_k \frac{1}{2^k}\,\text{(D)}\int_{A_{k_0}} f_k \\ &= \sum_{k=1}^{k_0-1} \frac{1}{2^k}\,\text{(D)}\int_{A_{k_0}} f_k + \frac{1}{2^{k_0}}\,\text{(D)}\int_{A_{k_0}} f_{k_0} \\ &\quad + \sum_{k=k_0+1}^{\infty} \frac{1}{2^k}\,\text{(D)}\int_{A_{k_0}} f_k.\end{aligned}$$

Now, by (e),

$$\frac{1}{2^{k_0}}\,\text{(D)}\int_{A_{k_0}} f_{k_0} \in \ell_\infty \setminus c_0.$$

So, to complete our proof, it is enough to show that the other two summands in (9),

$$\sum_{k=1}^{k_0-1} \frac{1}{2^k}\,\text{(D)}\int_{A_{k_0}} f_k \quad\text{and}\quad \sum_{k=k_0+1}^{\infty} \frac{1}{2^k}\,\text{(D)}\int_{A_{k_0}} f_k,$$

belong to c_0. Let us begin with the first one. By (iv) of the preceding lemma, if $k < k_0$ then either $\bigcup_n I_n^{k_0}$ and $\operatorname{supp}(f_k) = \bigcup_m I_m^k$ are disjoint or $\bigcup_n I_n^{k_0}$ is contained in a certain $I_{m_0}^k$. Since A_{k_0} is contained in $\bigcup_n I_n^{k_0}$, in the first case (D) $\int_{A_{k_0}} f_k = 0$. In the second case, notice that f_k is constant on A_{k_0} just because it is constant on $I_{m_0}^k$. To be precise, m_0 will have the form $m_0 = 2n_0 - 1$ or $m_0 = 2n_0$, and then at the points of A_{k_0} the function f_k takes the value

$$\Big(\overbrace{0,\ldots,0}^{n_0-1}, (-1)^{m_0+1}\frac{1}{2}\frac{1}{|I_{m_0}^k|}, 0, 0, \ldots\Big).$$

Therefore, we have

$$\text{(D)}\int_{A_{k_0}} f_k = |A_{k_0}|\Big(\overbrace{0,\ldots,0}^{n_0-1}, (-1)^{m_0+1}\frac{1}{2}\frac{1}{|I_{m_0}^k|}, 0, 0, \ldots\Big) \in c_0.$$

So, it is clear that in any case, the first summand belongs to c_0.

Finally, let us consider the other summand. We will see that in fact (D) $\int_{A_{k_0}} f_k$ vanishes for all $k > k_0$. We use again (iv) of the preceding lemma.

If $k > k_0$ then either $\operatorname{supp}(f_k) = \bigcup_m I^k_m$ does not meet any of the $I^{k_0}_{2n-1}$'s or it is contained in one of them. In any of these cases we have $(\mathrm{D})\int_{A_{k_0}} f_k = 0$. In the first case, because A_{k_0} and $\operatorname{supp}(f_k)$ are disjoint, and in the second one, since A_{k_0} contains $\operatorname{supp}(f_k)$, we can apply (c). This completes the proof.

REMARK 2. Recall that if an X-valued Dunford integrable function f is not Pettis integrable, then it is not Pettis integrable even if we "enlarge" the space X. That is, if Y is a Banach space having X as a subspace and we consider f as a function with values in Y, then f is not Pettis integrable either. This is an immediate consequence of the following elementary fact: *if $f : [a,b] \to Y$ is Pettis integrable and $f([a,b])$ lies in the subspace X of Y, then $(\mathrm{P})\int_E f$ lies in X for every measurable subset E of $[a,b]$*. To prove this it is enough to notice that for every measurable subset E of $[a,b]$, if $x^* \in Y^*$ and x^* vanishes on X then $\langle (\mathrm{P})\int_E f, x^*\rangle = \int_E x^* f(t)\, dt = 0$.

From the preceding remark and the example we immediately get the following

PROPOSITION 2. *If X is a Banach space having a subspace isomorphic to c_0 then there exists a measurable function $f_0 : [a,b] \to X$ such that*

(a) *f_0 is Dunford integrable,*
(b) *$(\mathrm{D})\int_J f_0$ belongs to X for each subinterval J in $[a,b]$, but*
(c) *f_0 is not Pettis integrable on any subinterval J in $[a,b]$.*

Finally, we have the following

THEOREM 10. *The following are equivalent*:

(a) *X does not contain c_0.*
(b) *Each X-valued Denjoy–Pettis integrable function defined on $[a,b]$ is Pettis integrable on a portion of every closed set.*
(c) *Each X-valued Dunford integrable function f defined on $[a,b]$ such that $(\mathrm{D})\int_J f$ belongs to X for every subinterval J in $[a,b]$ is Pettis integrable on some subinterval of $[a,b]$.*

Proof. That (a) implies (b) is just Gordon's Theorem 5 (which has been subsumed in our Theorem 8). That (b) implies (c) is immediate because each Dunford integrable function such that $(\mathrm{D})\int_J f$ belongs to X for every subinterval J in $[a,b]$ is clearly Denjoy–Pettis integrable. That (c) implies (a) is just the preceding proposition. ■

REMARK 3. It is well known that X does not contain c_0 if and only if each X-valued Dunford integrable function defined on $[a,b]$ such that $(\mathrm{D})\int_J f$ belongs to X for every subinterval J in $[a,b]$ is Pettis integrable on $[a,b]$ (see for instance [4, Theorem 23]). The preceding theorem may be seen as an improvement of this result.

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