STUDIA MATHEMATICA 130 (2) (1998)

Remarks on the Bergman kernel function of a worm domain

by

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Abstract. We use a recent result of M. Christ to show that the Bergman kernel function of a worm domain cannot be C^∞-smoothly extended to the boundary.

1. Introduction. The present paper is motivated by M. Christ's work [6] devoted to the study of the Kohn–Neumann operator and the Bergman projection on the so-called worm domains in $\mathbb{C}^2$ constructed by Diederich and Fornæss in [7]. M. Christ proved in his important paper that for each worm domain D its Bergman projection does not preserve $C^\infty(\overline{D})$.

An important condition in the study of biholomorphic mappings is the so-called *condition* A: for every $t \in D$, the Bergman kernel function $K(z,t)$ extends to a function from $C^\infty(\overline{D})$ (see [12], [11], [4]). The regularity of the Bergman projection on $C^\infty(\overline{D})$ implies condition A ([11], [4]).

We shall use M. Christ's result to prove that condition A is not valid for worm domains.

2. Preliminaries

DEFINITION 1. A *worm domain* in $\mathbb{C}^2$ is an open bounded domain defined by

$$D = \{z : |z_1 + e^{i\log|z_2|^2}| < 1 - \Phi(\log|z_2|^2)\}$$

where the function Φ vanishes identically on some interval $[-r,r]$ of positive length.

Diederich and Fornæss proved in [7] that Φ can be chosen such that

(*) D is pseudoconvex bounded with C^∞-boundary, and strictly pseudoconvex at every boundary point except those in $E = \{z : z_1 = 0, |\log|z|^2| \le r\}$.

Let $L^2_{(0,k)}(D)$, $k = 0,1,2$, denote the space of square integrable $(0,k)$-forms on D. On each worm domain this space decomposes as the direct

1991 *Mathematics Subject Classification*: Primary 32F20.

orthogonal sum $\bigoplus_{j\in\mathbb{Z}} L^2_{k,j}(D)$, with $w \in L^2_{k,j}(D)$ iff $R_\theta w = e^{ij\theta}w$ where $R_\theta(w) = w(z_1, e^{i\theta}z_2)$ for $k = 0,2$ and $R_\theta(w_1 d\overline{z}_1 + w_2 d\overline{z}_2) = (R_\theta w_1)d\overline{z}_1 + (R_\theta w_2)e^{-i\theta}d\overline{z}_2$ for $k = 1$.

For each $s \le 0$ and each k the Sobolev space $W^s_{2,(0,k)}(D)$ decomposes into the direct sum $\bigoplus_{j\in\mathbb{Z}} W^s_{2,(0,k),j}(D)$ of forms satisfying $R_\theta w = e^{ij\theta}w$.

The above decompositions imply in particular the following decomposition of the Bergman kernel function:

$$K(z,t) = \sum_{j=-\infty}^{\infty} z_2^j k_j(z,t)\overline{t}_2^j$$

where $z_2^j k_j(z,t)\overline{t}_2^j$ is the reproducing kernel of the Hilbert space $L^2_{0,j}(D) \cap \mathrm{Hol}(D)$ (compare Kiselman [9]). The functions $k_j(z,t)$ depend locally only on z_1 and t_1. Note that if $f(z_1, |z|^2)$ is holomorphic on D then $\partial f/\partial z_2 = 0$.

Note that if for some t the function $K(z,t) \in C^\infty(\overline{D})$ then also $k_j(z,t)$ is in $C^\infty(\overline{D})$ for each j.

Similar facts hold for harmonic functions. The Laplace operator maps $W^s_{2,(0,0),j}(D)$ into $W^{s-2}_{2,(0,0),j}(D)$ and therefore we have the decomposition $L^2(D) \cap \mathrm{Harm}(D) = \bigoplus L^2_{0,j}(D) \cap \mathrm{Harm}(D)$.

Thus the reproducing kernel $G(z,t)$ for the space of harmonic functions can be written as

$$G(z,t) = \sum_{j=-\infty}^{\infty} G_j(z,t)$$

where $G_j(z,t)$ is a reproducing kernel of $L^2_{0,j}(D) \cap \mathrm{Harm}(D)$.

If D has a C^∞ smooth boundary then for every $t \in D$, $G(z,t)$ and each $G_j(z,t)$ belong to $C^\infty(\overline{D})$.

DEFINITION 2. A function h harmonic on D is said to be *of polynomial growth* if there exists $m > 0$ such that $h(z)\,\mathrm{dist}(z, \partial D)^m$ is bounded on D. The space $\mathrm{Harm}^{-\infty}(D)$ is dual to $\mathrm{Harm}^\infty(D) = \mathrm{Harm}(D) \cap C^\infty(\overline{D})$ (see [2]).

In the sequel we denote by N_1 the Kohn operator solving the equation $u = (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})N_1 u$ on $(0,1)$-forms ([8]), by N_2 the operator solving the Dirichlet problem and by P_0 the Bergman projection $P_0 f = \int_D K(z,t)f(t)\,dV_t$, $f \in L^2(D)$. We have $P_0 = 1 - \overline{\partial}^* N_1 \overline{\partial}$ ([8]). The operator $P_1 = 1 - \overline{\partial}^* N_2 \overline{\partial}$ is the orthogonal projection onto the space of $\overline{\partial}$-closed $(0,1)$-forms.

The projection P_1 is regular with respect to the Sobolev norms $\|\cdot\|^s_2$. D. Barrett proved in [1] that the operators N_1 and P_0 are not regular with respect to $\|\cdot\|^s_2$ if D is a worm domain satisfying $(*)$ and s is sufficiently large.

The operators P_0, P_1, N_1, N_2 each map an appropriate $L^2_{k,j}(D)$ space onto itself for $k = 0,1,2$.

3. The main theorem

THEOREM. *Let D be a C^∞-smooth pseudoconvex worm domain satisfying $(*)$. Let $K(z,t)$ be its Bergman function. Denote by A the set of all $t \in D$ such that $K(z,t) \in C^\infty(\overline{D})$. Then there exists a function h ($h \not\equiv 0$) harmonic on D with polynomial growth near the boundary such that*

$$A \subset \{z \in D : h(z) = 0\}.$$

Proof. M. Christ proved in [6] the following:

PROPOSITION. *Denote by V^s_j the space of $(0,1)$-forms $w \in C^\infty_{(0,1)}(\overline{D}) \cap W^s_{2,(0,1),j}(\overline{D})$ such that $N_1 w \in C^\infty_{(0,1)}(\overline{D})$. There exists a discrete subset $S \subset \mathbb{R}^+$ (depending on D) such that for every $s \in \mathbb{R}^+ \setminus S$ and $j \in \mathbb{Z}$ there exists a constant C_{sj} depending only on D, s, j such that*

$$\|N_1 w\|^s_2 \le C_{sj}\|w\|^s_2 \quad \textit{for } w \in V^s_j.$$

We shall always assume in the sequel that $s \notin S$.

Let us denote by $C^\infty_{1,j}(\overline{D})$ the space $C^\infty_{(0,1)}(\overline{D}) \cap W^s_{2,(0,1),j}(\overline{D})$. The results of D. Barrett [1] imply that N_1 cannot map $C^\infty_{1,j}(\overline{D})$ into itself for any j.

It follows from the formula (4) of Boas–Straube's paper [5] (see also below) that if $w \in C^\infty_{1,j}(\overline{D})$ and w is orthogonal to all $\overline{\partial}$-closed forms then $N_1 w \in C^\infty_{1,j}(\overline{D})$.

Recall that $P_1 = 1 - \overline{\partial}^* N_2 \overline{\partial}$ maps $C^\infty_{1,j}(\overline{D})$ into itself.

Let A be the set of all $t \in D$ for which

$$K(z,t) \in C^\infty(\overline{D}), \quad K(z,t) = \sum_{j=-\infty}^{\infty} z_2^j k_j(z,t)\overline{t}_2^j.$$

For each j and $t \in A$, $z_2^j k_j(z,t)\overline{t}_2^j \in C^\infty(\overline{D})$.

Let us now consider the reproducing kernel of the space of square integrable harmonic functions on D:

$$G(z,t) = \sum_{j=-\infty}^{\infty} G_j(z,t) \quad \text{(see Preliminaries)}.$$

We have $P_0(G_j(z,t)) = z_2^j k_j(z,t)\overline{t}_2^j$. Suppose now that the set $\{G_j(z,t) : t \in A\}$ is linearly dense in $C^\infty_{1,j}(\overline{D}) \cap \mathrm{Harm}(D)$. Then the set

$$B = \{\overline{\partial}G_j(\cdot,t) : t \in A\} \cup \{\overline{\partial}h : h \in C^\infty_{0,j}(\overline{D}) \text{ and } P_0(h) = 0\} \cup \{w \in C^\infty_{1,j}(\overline{D}) : w \perp \ker\overline{\partial}\}$$

is linearly dense in $C^\infty_{1,j}(\overline{D})$.

This can be proved in the following way:

Let $w \in C^\infty_{1,j}(\overline{D})$. Since P_1 maps $C^\infty_{1,j}(\overline{D})$ into itself we have $w = w_1 + w_2$, where $\overline{\partial} w_1 = 0$ and $w_2 \perp \ker \overline{\partial} \in B$. By the result of J. J. Kohn [10] there exists $u \in C^\infty(\overline{D})$ such that $\overline{\partial} u = w_1$. Since $w_1 \in C^\infty_{1,j}(\overline{D})$ we can take $u \in C^\infty_{1,j}(\overline{D})$. The orthogonal projection Π onto the space of harmonic square integrable functions given by $\Pi u = \int_D G(z,t)u(t)\,dt$ maps $L^2_{1,j}(D)$ into itself and $C^\infty(\overline{D})$ into itself (see [2]). Thus $u = \Pi u + (u - \Pi u)$, $u - \Pi u$ is orthogonal to the holomorphic functions and $\overline{\partial}(u - \Pi u) \in B$. Since $\Pi u \in \overline{\operatorname{span}}\, G_j(z,t)$ we have $\partial \Pi u \in \overline{\operatorname{span}}\, \overline{\partial} G_j(z,t)$. The formula (4) of [5] and the ellipticity of N_2 ensure that N_1 maps B into $C^\infty_{1,j}(\overline{D})$, as follows.

The above-mentioned formula for $(0,1)$-forms says that

$$N_1 = P_1 w_\tau N_{\tau,1}[w_{-\tau} P_1 + \overline{\partial} w_{-\tau} \wedge (I - P_0)\overline{\partial}^*_\tau N_{\tau,1} P_1] + (I - P_1)\overline{\partial}^*_\tau N_{\tau,2} P_2 w_\tau \overline{\partial} N_{\tau,1}[w_{-\tau}(I - P_1)].$$

The symbols $N_{\tau,1}$, $N_{\tau,2}$, $\overline{\partial}^*_\tau$ denote the Neumann operators and the adjoint operator for $\overline{\partial}$, constructed with respect to the weight $w_\tau(z) = e^{-\tau|z|^2}$. J. J. Kohn proved in [10] that for each $s > 0$ there exists τ_0 such that $N_{\tau,1}$ and $N_{\tau,2}$ are regular in the Sobolev norm $\|\cdot\|^s_2$ if $\tau > \tau_0$. Since $D \subset \mathbb{C}^2$, we have $P_2 = I$ and N_2 is regular in Sobolev norms (elliptic). Thus the projection P_1 maps $W^s_{2,(0,1)}(D)$ into itself for each $s > 0$. If $w = \overline{\partial} G_j(z,t)$ then $(I - P_0)\overline{\partial}^*_\tau N_{\tau,1} P_1 w = G_j(z,t) - z_2^j k_j(z,t)\overline{t}_2^j$; if $w = \overline{\partial} h$, $h \in C^\infty(\overline{D})$, $P_0(h) = 0$ then $(I - P_0)\overline{\partial}^*_\tau N_{\tau,1} P_1 w = h$, and if $w \perp \ker \overline{\partial}$ then $P_1 w = 0$. Hence N_1 maps B into $C^\infty_{1,j}(\overline{D})$.

Since B is linearly dense in $C^\infty_{1,j}(\overline{D})$ we have

$$\|N_1 w\|^s_2 \le C_{sj}\|w\|^s_2 \quad \text{on } \overline{\operatorname{span}}\, B = C^\infty_{1,j}(\overline{D}).$$

We got a contradiction with Christ's and Barrett's results.

Hence $\{G(z,t) : t \in A\}$ is not linearly dense in $C^\infty_{1,j}(\overline{D}) \cap \operatorname{Harm}(D)$.

By Bell's result [2] on duality between $\operatorname{Harm}^\infty(D)$ and $\operatorname{Harm}^{-\infty}(D)$ there exists a non-zero $h \in \operatorname{Harm}^{-\infty}(D)$ such that $h = 0$ on A.

It could also be proved that h can be taken from the space of harmonic functions of polynomial growth which depend only on z_1 and $|z_2|$.

4. Open problems

PROBLEM 1. The Theorem implies that $K(z,t) \notin C^\infty(\overline{D})$ for t belonging to an open dense subset of D. However, we do not know how bad the function $K(z,t)$, $t \in D \setminus A$, can be. Results of [9] suggest that $K(z,t)$ may not be in the Hölder class $\Lambda_\alpha(\overline{D})$.

PROBLEM 2. The proof of the Theorem suggests the following conjecture: A C^∞-smooth bounded pseudoconvex domain D has property A if and only if the Bergman projection maps $C^\infty(\overline{D})$ into itself.

As far as we know the best result in this direction is due to Bell–Boas [3]: The Bergman projection maps $C^\infty(\overline{D})$ into itself if for each $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and C_k such that

$$\left|\frac{\partial^\alpha}{\partial z^\alpha} K(z,t)\right| \le C_k \operatorname{dist}(t, \partial D)^{-m} \quad \text{for each } z,t \in D \text{ and } |\alpha| = k.$$

PROBLEM 3. Do Christ's a priori estimates hold for every smooth bounded pseudoconvex domain? Do they imply similar a priori estimates for the Bergman projection?

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Received March 14, 1997
Revised version October 13, 1997
(3858)