A generalized Kahane–Khinchin inequality

by

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Abstract. The inequality

$$\left| \log \left( \sum a_n e^{2\pi i \varphi_n} \right) \right| d\varphi_1 \ldots d\varphi_n \geq C \log \left( \sum |a_n|^2 \right)^{1/2}$$

with an absolute constant $C$, and similar ones, are extended to the case of $a_n$ belonging to an arbitrary normed space $X$ and an arbitrary compact group of unitary operators on $X$ instead of the operators of multiplication by $e^{2\pi i \varphi}$.

The classical Khinchin inequality has the form

$$\left( \sum a_n^2 \right)^{1/2} \leq C E \left( \sum a_n \xi_n \right),$$

where $\{a_n\}$ is a finite (or an infinite, with a bounded sum of squares) sequence of real numbers, $\xi_n$ is a sequence of Rademacher random functions (this means that the $\xi_n$ are independent and $\xi_n = \pm 1$ with probability $1/2$), $E$ denotes the expected value, and $C$ is an absolute constant.

For any $p > 0$ and any random function $S$ we define

$$\|S\|_p = \left( E|S|^p \right)^{1/p}, \quad p > 0.$$ 

Then inequality (1) can be rewritten in the form

$$\|S\|_2 \leq C \|S\|_1$$

for

$$S = \sum a_n \xi_n.$$ 

For $\{a_n\}$ complex the sum

$$S = \sum a_n e^{2\pi i \varphi_n}$$

is usually considered, where the $\varphi_n$ are independent and uniformly distributed on $[0, 1]$. Inequality (3) is also true for (5) (with another constant $C$).

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It follows from the Jensen inequality that (2) is an increasing function of p (in particular, the inequality opposite to (3) with C = 1 is trivial). That is why the inequality

$$\|S\|_p \leq C(p, q)\|S\|_q, \quad 0 < q < p,$$

for the sums (4), (5) or more general ones is also named the Khinchin inequality.

J. P. Kahane proved inequality (6) for the sums (4) and (5) with $a_n$ belonging to an arbitrary Banach space $X$; the constant $C(p, q)$ depends neither on $a_n \in X$, nor on $X$ \(^{(1)}\). This inequality is also valid for the sums

$$S = \sum a_n x_n, \quad x_n \in X,$$

where $a_n$ are elements of a compact (in the strong topology) subgroup $\Gamma$ of the group of unitary linear operators on $X$, and the expected value in (2) is calculated with respect to the direct product of the Haar measures on $\Gamma$. The only additional restriction on $\Gamma$ is the following: the operator

$$e^\pi : x \rightarrow -x$$

belongs to $\Gamma$.

D. Ulrich [9] and the author [1] simultaneously and independently proved a stronger result than inequality (6), namely

$$\|S\|_2 \leq C\|S\|_0,$$

where $S$ is defined in (5) and

$$\|S\|_0 = \exp C(\log |S|)$$

(the opposite inequality with $C = 1$ is obvious). This inequality has a number of applications: in the theory of Bloch functions [9], in the theory of value distribution of holomorphic mappings into Banach space [1–2], in the theory of almost periodic functions [3]. It should be mentioned that inequality (9) with the sum of Rademacher functions (4) is not true.

It is easy to see that the definition (2) also makes sense for $p < 0$, and (2) and (10) define a decreasing function $p \rightarrow \|S\|_p$ for all $p \in \mathbb{R}$. Ye. A. Gorin and the author [4–5] obtained sufficient conditions for sequences of independent random vectors $x_n \in \mathbb{R}^m$ and linear operators $A_n \in L(\mathbb{R}^m, \mathbb{R}^m)$ under which inequality (6) is satisfied for $S = \sum A_n x_n$ with $q \leq 0 < p$. This inequality was named the generalized Khinchin inequality.

In particular, (6) is true for any $p > q > -(m - 1)$ when $x_n$ are equidistributed on the unit sphere in $\mathbb{R}^m$ and $A_n$ are operators of multiplication by $a_n \in \mathbb{R}$. This result for $m = 2$ is slightly stronger than (9).

\(^{(1)}\) For example, this inequality follows from Th. 4, p. 20 of [6]; see also [8].

The proofs of all these results make use of the Fourier transform and seem to be of little use in the infinite-dimensional case. D. Ulrich [10] gave a new method and proved (9) for the sum (5) and $a_n \in X$, where $X$ is an arbitrary Banach space and $C$ is an absolute constant. For a random vector $S$ with log-concave distribution inequality (9) was established by R. Latała [7]. Therefore it is natural to describe groups of unitary operators in $X$ for which inequality (9) (or (6) for $q < 0$) is satisfied for the sums (7). Observe that the group $\Gamma = \{e^{\pi i \varphi} \}$ is not of this type. This situation differs from the case of $q > 0$ (Khinchin–Khinchin inequality).

The purpose of the present paper is to prove the following result:

**Theorem.** Let $X$ be an arbitrary normed space and $\Gamma$ be a compact and connected (in the strong topology) group of unitary linear operators on $X$ with the Haar measure $\mu$ such that

$$\int x \mu(x) dx = 0 \quad \text{for all } x \in X$$

and

$$\exists \delta < \infty \forall \varphi \leq 1/2 \forall x \in X \quad \mu\{x : \|ax - x\| \leq \varepsilon \|x\|\} \geq \varepsilon^\delta.$$

Then the inequality

$$\|S\|_2 \leq C\|S\|_{-\varphi}$$

is true for any $q \in [0, 1)$ and any $x_1, \ldots, x_n \in X$ for the sum (7) with $a_n \in \Gamma$; the constant $C$ depends only on $q$ and $\varphi$. Condition (12) can be omitted if the norm in $X$ is defined by the scalar product.

Inequality (13), together with the Kahane–Khinchin inequality, imply that inequality (6) for the sums (7) is true for all $q, p \in \mathbb{R}, -1 < q \leq p < 0$. Then (13) is also true for an infinite sum if the latter converges almost surely. This follows from the Fatou lemma and the equivalence between almost sure convergence and convergence in $L^p(\Gamma^\infty)$ for all $p \geq 1$ for the sums (7) (see, for example, [6]).

Conditions (11) and (12) are satisfied, for example, when $X$ is a complex normed space and $\Gamma = \{x \rightarrow e^{i\varphi} x, \varphi \in [0, 1]\}$; in this case our theorem gives a stronger inequality than (9). Conditions (11) and (12) are also satisfied when $X$ is the space of functions integrable on the unit circle $T$ and such that

$$\int_T x(t) dt = 0 \quad \text{and} \quad \int_T |x(t + h) - x(t)| dt \leq K|h|^p \|x(t)\|_{L(T)};$$

here $\Gamma$ is the group of shifts $\{x(t) \rightarrow x(t + h)\}$. Inequality (13) for $q = 0$ has
the form
\[
\left( \int_\mathbb{R} \left( \sum_{j=1}^n x_j(t + h_j) \right)^2 \, dh_1 \ldots dh_n \right)^{1/2} \leq C \exp \int_\mathbb{R} \log \left( \sum_{j=1}^n x_j(t + h_j) \right) \, dh_1 \ldots dh_n,
\]
where \(x_j\) are arbitrary functions of \(t\) and the constant \(C\) depends only on \(p\) and \(K\).

We would like to emphasize that the connectedness of \(\Gamma\) separates just the cases \(\Gamma = \{e, e^{-\frac{\pi}{n}}\}\) and \(\Gamma = \{ x \to e^{i\pi/n} x \}, \varphi \in [0, 1].\) Condition (11) is weaker than condition (8) that was used for the Kahane-Khinchin inequality (however, Kahane's proof needs only a little change when we replace (8) by (11)). Condition (12) means that the orbits of any element of \(X\) are "not very long". Perhaps this condition appears because of the method of the proof; the author could neither omit condition (12) nor prove its necessity.

**Proof of (the Theorem).** First of all we prove (13) for sums \(a_1 x_1 + a_2 x_2\) with a constant \(C_2\). Further we assume that (13) is true for any sum (7) of \(n\) terms with a constant \(C_{n-1}\) and prove (13) for a sum (7) of \(n\) terms with a constant \(C_n\). The theorem will be proved if the product \(\prod_{n=3}^{\infty} C_n / C_{n-1}\) converges (14). Of course we may assume \(q \neq 0\).

It is easy to see that for any \(x, x' \in X\),
\[
\left\| x' \right\| = \left\| \int_\Gamma (\alpha x - x') \, d\mu(\alpha) \right\| \leq \int_\Gamma \left\| \alpha x - x' \right\| \, d\mu(\alpha).
\]
Therefore for any \(x' \in X\) with \(\left\| x' \right\| = 1\), there exists an element \(x''\) belonging to the orbit \(\Gamma(x')\) such that \(\left\| x - x'' \right\| \geq 1\). Consequently, the connected image of \(\Gamma(x')\) under the map \(\tau : x \to \left\| x - x'' \right\|\) contains the segment \([0, 1]\).

Let \(t \in [0, 1/4), N = \lfloor 1/(2t) \rfloor\) and \(x_1, \ldots, x_N\) be the preimages of the points \(j/N\) under the map \(\tau\). The balls \(\{ x \in X : \left\| x - x_j \right\| < t \}\) are pairwise disjoint, so
\[
\sum_{j=1}^N \mu(\alpha : \left\| \alpha x' - x_j \right\| < t) \leq 1.
\]
The properties of the Haar measure imply that \(\mu(\alpha : \left\| \alpha x' - z \right\| < t)\) does not depend on \(x' \in \Gamma(x')\), therefore
\[
\mu(\alpha : \left\| \alpha x' - x \right\| < t) \leq 1/N < 4t
\]
(14) Earlier this method permitted M. I. Kadets to obtain an extremely simple and elementary proof of (8) with \(a_n \in C\) (private communication).

For all \(x \in \Gamma(x')\) and \(t \in (0, 1/4)\), so
\[
\int_0^t \left\| \alpha x' - x\right\|^{-q} \, d\mu(\alpha) \leq \frac{2}{q} \int_0^t \left\| \alpha x' - x\right\| \, d\mu(\alpha : \left\| \alpha x' - x\right\| < t) \leq 4t/(1 - q).
\]

Let \(n = 2\). We may assume \(\left\| x_1 \right\| = 1\) and \(\left\| x_2 \right\| \leq 1\). If a point \(x' \in \Gamma(x_1)\) is nearest to \(-x_2\), then for any \(\alpha \in \Gamma\)
\[
\left\| \alpha x_1 - x' \right\| \leq 2 \left\| \alpha x_1 + x_2 \right\|
\]
and
\[
\left( \int_\Gamma \left\| \alpha x_1 + a_2 x_2 \right\|^{-q} \, d\mu(\alpha) \right)^{-1/q} \geq C(q).
\]
This inequality implies (13) for \(n = 2\) because \(\left\| a_1 x_1 + a_2 x_2 \right\| \leq 2\).

Let now \(n \geq 3\). For \(k \leq n\) define
\[
S_k = a_1 x_1 + \ldots + a_k x_k.
\]
It is easy to see that
\[
\left\| S_n \right\| \leq (n - 1)^{-1} \left( (n - 2)^{-1} \sum_{j \neq k} \left\| S_n - a_j x_j - a_k x_k \right\| \right)
\]
Therefore
\[
\left\| S_n \right\| \leq (n - 1)^{-1} \left( (n - 2)^{-1} \sum_{j \neq k} \left\| S_n - a_j x_j - a_k x_k \right\| \right)
\]
(15)
\[
\left\| S_n \right\| \leq \frac{n - 2}{n} \left\| S_n \right\|
\]
under a suitable numbering of \(x_{n-1}, x_n\). On the other hand, inequality (14) implies
\[
\left\| S_{n-2} \right\| \leq \left\| S_{n-1} \right\| \leq \left\| S_n \right\|
\]
and
\[
\left\| x_n \right\| \leq \left\| S_n \right\|.
\]
Consider for a fixed \(\beta \in \Gamma\) the expression
\[
\tilde{S}(\beta) = a_1 x_1 + \ldots + a_{n-2} x_{n-2} + a_{n-1} (x_{n-1} - \beta x_n)
\]
and define
\[
r(\beta) = \left\| \tilde{S}(\beta) \right\| \left\| S_n \right\|^{-1} \leq 1, \quad M_n = \max_{\beta} r(\beta).
\]
Inequalities (16)-(17) and
\begin{equation}
\|\hat{S}(\beta)\|_2 \leq \|S_{n-1}\|_2 + \|x_n\| \tag{18}
\end{equation}
imply $M_n \leq 3$; the inequalities (15) and $\|\hat{S}(\beta)\|_2 \geq \|S_{n-2}\|_2$ imply the estimate
\begin{equation}
r(\beta) \geq \max\{-4/n, -8/9\}. \tag{19}
\end{equation}
It is not difficult to see that
\begin{equation}
\int r(\beta)\,d\mu(\beta) = \|S_n\|_2^{-2} \int \|\hat{S}(\beta)\|_2^{-2} \,d\mu(\beta) - 1 = 0. \tag{20}
\end{equation}
Suppose that the inductive hypothesis is true. Replace $S_{n-1}$ by $\hat{S}(\beta)$ in (13). Then
\begin{equation}
\|S_n\|_{-q}^{-2} = \int \|\hat{S}(\beta)\|_{-q}^{-2} \,d\mu(\beta) \leq C_{n-1}^2 \int \|\hat{S}(\beta)\|_2^{-2} \,d\mu(\beta) \tag{21}
= C_{n-1}^2 \|S_n\|_2^{-2} \int (1 + r(\beta))^{-q/2} \,d\mu(\beta).
\end{equation}
Since $(1+t)^{-q/2} \leq 1 - (q/2)t + A(q)t^2$ for $t \geq -8/9$ and $q \in (0, 1)$, (19)-(21) imply
\begin{equation}
\|S_n\|_2 \leq C_n \|S_{n-1}\|_{-q}^{-2} \left[1 + A(q) \int r(\beta)^2 \,d\mu(\beta)\right]^{1/q}
\end{equation}
and
\begin{equation}
\int r(\beta)^2 \,d\mu(\beta) \leq \left(M_n + \frac{8}{3n}\right) \int r(\beta) \,d\mu(\beta) \leq \left(M_n + \frac{8}{3n}\right) \frac{16}{3n}.
\end{equation}
If the norm in $X$ is generated by a scalar product, then by (11) we obtain
\begin{equation}
\|S_k\|_2^2 = \sum_{j \leq h} |x_j|^2.
\end{equation}
Therefore
\begin{equation}
\|\hat{S}(\beta)\|_2^2 = \|S_{n-2}\|_2^2 + \|x_{n-1} + \beta x_n\|^2 \leq 2\|S_n\|_2^2 - \|S_{n-2}\|_2^2.
\end{equation}
Now inequality (15) implies the estimate $M_n \leq 4/n$, and we may take $C_n = (1 + 36A(q)n^{-2})^{1/q}C_{n-1}$. In the general case we calculate the measure of the set
\begin{equation}
D = \{\beta \in \Gamma : |r(\beta) - r(\beta_0)| \leq M_n/2\},
\end{equation}
where $\beta_0$ is a point of $\Gamma$ such that $r(\beta_0) = M_n$. It follows from (16)-(18) that for any $\beta, \beta' \in D$ that
\begin{equation}
|r(\beta) - r(\beta')| \leq \frac{4\|\hat{S}(\beta) - \hat{S}(\beta')\|_2}{\|S_n\|_2} \leq 4\|\beta x_n - \beta' x_n\| \cdot \|x_n\|^{-1}.
\end{equation}
Then condition (12) and the estimate $M_n \leq 3$ imply
\begin{equation}
\mu(D) \geq \mu(\{\beta : \|\beta x_n - \beta_0 x_n\| \leq M_n\|x_n\|/8\}) \geq (M_n/8)^q.
\end{equation}
By (19) and (20) we obtain
\begin{equation}
\mu(D) \leq \frac{1}{2} \int r(\beta) \,d\mu(\beta) \leq \int_{\beta | r(\beta) \leq 0} r(\beta) \,d\mu(\beta) \leq \frac{8}{3n}.
\end{equation}
and
\begin{equation}
M_n \leq 8(23^{-1}n^{-1})^{1/(q+1)}. \quad \text{Therefore we can take}
\end{equation}
\begin{equation}
C_n = C_{n-1}(1 + 86A(q)n^{-(q+2)/(q+1)})^{1/q}.
\end{equation}
The theorem is proved.

References


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