Contents of Volume 130, Number 2

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (1998): Vols. 127–131 (15 issues); $32 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by

drukarnia
herman & herman
02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

PRINTED IN POLAND

ISSN 0039-3223

A generalized Kahane–Khinchin inequality

by

S. Yu. FAVOROV (Kharkov)

Abstract. The inequality

$$\int \log\Big|\sum a_n e^{2\pi i\varphi_n}\Big|\, d\varphi_1\ldots d\varphi_n \ge C\log\Big(\sum |a_n|^2\Big)^{1/2}$$

with an absolute constant C, and similar ones, are extended to the case of a_n belonging to an arbitrary normed space X and an arbitrary compact group of unitary operators on X instead of the operators of multiplication by $e^{2\pi i\varphi}$.

The classical Khinchin inequality has the form

$$\Big(\sum a_n^2\Big)^{1/2} \le C\mathcal{E}\Big(\sum a_n\xi_n\Big), \tag{1}$$

where $\{a_n\}$ is a finite (or an infinite, with a bounded sum of squares) sequence of real numbers, ξ_n is a sequence of Rademacher random functions (this means that the ξ_n are independent and $\xi_n = \pm 1$ with probability $1/2$), $\mathcal{E}$ denotes the expected value, and C is an absolute constant.

For any $p > 0$ and any random function S we define

$$\|S\|_p = (\mathcal{E}|S|^p)^{1/p}, \quad p > 0. \tag{2}$$

Then inequality (1) can be rewritten in the form

$$\|S\|_2 \le C\|S\|_1 \tag{3}$$

for

$$S = \sum a_n\xi_n. \tag{4}$$

For $\{a_n\}$ complex the sum

$$S = \sum a_n e^{2\pi i\varphi_n} \tag{5}$$

is usually considered, where the φ_n are independent and uniformly distributed on $[0,1]$. Inequality (3) is also true for (5) (with another constant C).

1991 *Mathematics Subject Classification*: 60E15, 46B09.

It follows from the Jensen inequality that (2) is an increasing function of p (in particular, the inequality opposite to (3) with $C = 1$ is trivial). That is why the inequality

$$\|S\|_p \leq C(p,q)\|S\|_q, \quad 0 < q < p, \tag{6}$$

for the sums (4), (5) or more general ones is also named the Khinchin inequality.

J. P. Kahane proved inequality (6) for the sums (4) and (5) with a_n belonging to an arbitrary Banach space X; the constant $C(p,q)$ depends neither on $a_n \in X$, nor on X ([1]). This inequality is also valid for the sums

$$S = \sum \alpha_n x_n, \quad x_n \in X, \tag{7}$$

where α_n are elements of a compact (in the strong topology) subgroup Γ of the group of unitary linear operators on X, and the expected value in (2) is calculated with respect to the direct product of the Haar measures on Γ. The only additional restriction on Γ is the following: the operator

$$e^- : x \to -x \quad \text{belongs to } \Gamma. \tag{8}$$

D. Ullrich [9] and the author [1] simultaneously and independently proved a stronger result than inequality (6), namely

$$\|S\|_2 \leq C\|S\|_0, \tag{9}$$

where S is defined in (5) and

$$\|S\|_0 = \exp \mathcal{E}(\log |S|) \tag{10}$$

(the opposite inequality with $C = 1$ is obvious). This inequality has a number of applications: in the theory of Bloch functions [9], in the theory of value distribution of holomorphic mappings into Banach space [1–2], in the theory of almost periodic functions [3]. It should be mentioned that inequality (9) with the sum of Rademacher functions (4) is not true.

It is easy to see that the definition (2) also makes sense for $p < 0$, and (2) and (10) define a decreasing function $p \to \|S\|_p$ for all $p \in \mathbb{R}$. Ye. A. Gorin and the author [4–5] obtained sufficient conditions for sequences of independent random vectors $x_n \in \mathbb{R}^m$ and linear operators $A_n \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ under which inequality (6) is satisfied for $S = \sum A_n x_n$ with $q \leq 0 < p$. This inequality was named the *generalized Khinchin inequality*.

In particular, (6) is true for any $p > q > -(m-1)$ when x_n are equidistributed on the unit sphere in $\mathbb{R}^m$ and A_n are operators of multiplication by $a_n \in \mathbb{R}$. This result for $m = 2$ is slightly stronger than (9).

([1]) For example, this inequality follows from Th. 4, p. 20 of [6]; see also [8].

The proofs of all these results make use of the Fourier transform and seem to be of little use in the infinite-dimensional case. D. Ullrich [10] gave a new method and proved (9) for the sum (5) and $a_n \in X$, where X is an arbitrary Banach space and C is an absolute constant. For a random vector S with log-concave distribution inequality (9) was established by R. Latała [7]. Therefore it is natural to describe groups of unitary operators in X for which inequality (9) (or (6) for $q < 0$) is satisfied for the sums (7). Observe that the group $\Gamma = \{e, e^-\}$ is not of this type. This situation differs from the case of $q > 0$ (Kahane–Khinchin inequality).

The purpose of the present paper is to prove the following result:

THEOREM. *Let X be an arbitrary normed space and Γ be a compact and connected (in the strong topology) group of unitary linear operators on X with the Haar measure μ such that*

$$\int_\Gamma \alpha(x)\, d\mu(\alpha) = 0 \quad \textit{for all } x \in X \tag{11}$$

and

$$\exists \varrho < \infty\ \forall \varepsilon \leq 1/2\ \forall x \in X \quad \mu\{\alpha : \|\alpha x - x\| \leq \varepsilon\|x\|\} \geq \varepsilon^\varrho. \tag{12}$$

Then the inequality

$$\|S\|_2 \leq C\|S\|_{-q} \tag{13}$$

is true for any $q \in [0,1)$ and any $x_1, \ldots, x_n \in X$ for the sum (7) *with $\alpha_j \in \Gamma$; the constant C depends only on q and ϱ. Condition* (12) *can be omitted if the norm in X is defined by the scalar product.*

Inequality (13), together with the Kahane–Khinchin inequality, imply that inequality (6) for the sums (7) is true for all $q, p \in \mathbb{R}, -1 < q \leq 0 < p$. Then (13) is also true for an infinite sum if the latter converges almost surely. This follows from the Fatou lemma and the equivalence between almost sure convergence and convergence in $L^p(\Gamma^\infty)$ for all $p \geq 1$ for the sums (7) (see, for example, [6]).

Conditions (11) and (12) are satisfied, for example, when X is a complex normed space and $\Gamma = \{x \to e^{2\pi i\varphi}x\}$, $\varphi \in [0,1]$; in this case our theorem gives a stronger inequality than (9). Conditions (11) and (12) are also satisfied when X is the space of functions integrable on the unit circle $\mathbb{T}$ and such that

$$\int_\mathbb{T} x(t)\, dt = 0 \quad \text{and} \quad \int_\mathbb{T} |x(t+h) - x(t)|\, dt \leq K|h|^\varrho \|x(t)\|_{L(\mathbb{T})};$$

here Γ is the group of shifts $\{x(t) \to x(t+h)\}$. Inequality (13) for $q = 0$ has the form

$$\Big(\int\limits_{\mathbb{T}^n}\Big\|\sum_{j=1}^n x_j(t+h_j)\Big\|^2_{L(\mathbb{T})}dh_1\dots dh_n\Big)^{1/2}$$
$$\le C\exp\int\limits_{\mathbb{T}^n}\log\Big\|\sum_{j=1}^n x_j(t+h_j)\Big\|_{L(\mathbb{T})}dh_1\dots dh_n,$$

where x_j are arbitrary functions of t and the constant C depends only on ϱ and K.

We would like to emphasize that the connectedness of Γ separates just the cases $\Gamma=\{e,e^-\}$ and $\Gamma=\{x\to e^{2\pi i\varphi}x\}$, $\varphi\in[0,1]$. Condition (11) is weaker than condition (8) that was used for the Kahane–Khinchin inequality (however, Kahane's proof needs only a little change when we replace (8) by (11)). Condition (12) means that the orbits of any element of X are "not very long". Perhaps this condition appears because of the method of the proof; the author could neither omit condition (12) nor prove its necessity.

Proof (of the Theorem). First of all we prove (13) for sums $\alpha_1x_1+\alpha_2x_2$ with a constant C_2. Further we assume that (13) is true for any sum (7) of $n-1$ terms with a constant C_{n-1} and prove (13) for a sum (7) of n terms with a constant C_n. The theorem will be proved if the product $\prod_{n=3}^\infty C_n/C_{n-1}$ converges ([2]). Of course we may assume $q\ne 0$.

It is easy to see that for any $x,x'\in X$,

$$\|x'\|=\Big\|\int\limits_\Gamma(\alpha x-x')\,d\mu(\alpha)\Big\|\le\int\limits_\Gamma\|\alpha x-x'\|\,d\mu(\alpha). \tag{14}$$

Therefore for any $x'\in X$ with $\|x'\|=1$, there exists an element x'' belonging to the orbit $\Gamma(x')$ such that $\|x'-x''\|\ge 1$. Consequently, the connected image of $\Gamma(x')$ under the map $\tau:x\to\|x-x''\|$ contains the segment $[0,1]$. Let $t\in(0,1/4)$, $N=[1/(2t)]$ and $x_1,\dots,x_N$ be the preimages of the points j/N under the map τ. The balls $\{x\in X:\|x-x_j\|<t\}$ are pairwise disjoint, so

$$\sum_{j=1}^N\mu\{\alpha:\|\alpha x'-x_j\|<t\}\le 1.$$

The properties of the Haar measure imply that $\mu\{\alpha:\|\alpha x'-x\|<t\}$ does not depend on $x\in\Gamma(x')$, therefore

$$\mu\{\alpha:\|\alpha x'-x\|<t\}\le 1/N<4t$$

([2]) Earlier this method permitted M. I. Kadets to obtain an extremely simple and elementary proof of (9) with $a_n\in\mathbb{C}$ (private communication).

for all $x\in\Gamma(x')$ and $t\in(0,1/4)$, so

$$\begin{aligned}\int\limits_\Gamma\|\alpha x'-x\|^{-q}\,d\mu(\alpha)&\le\int\limits_0^2 t^{-q}\,d\mu\{\alpha:\|\alpha x'-x\|<t\}\\&=2^{-q}+q\int\limits_0^2 t^{-q-1}\mu\{\alpha:\|\alpha x'-x\|<t\}\,dt\\&\le 4^q/(1-q).\end{aligned}$$

Let $n=2$. We may assume $\|x_1\|=1$ and $\|x_2\|\le 1$. If a point $x'\in\Gamma(x_1)$ is nearest to $-x_2$, then for any $\alpha\in\Gamma$,

$$\|\alpha x_1-x'\|\le 2\|\alpha x_1+x_2\|$$

and

$$\Big(\int\|\alpha_1x_1+\alpha_2x_2\|^{-q}\,d\mu(\alpha_1)\,d\mu(\alpha_2)\Big)^{-1/q}$$
$$=\Big(\int\|\alpha x_1+x_2\|^{-q}\,d\mu(\alpha)\Big)^{-1/q}\ge C(q).$$

This inequality implies (13) for $n=2$ because $\|\alpha_1x_1+\alpha_2x_2\|_2\le 2$.

Let now $n\ge 3$. For $k\le n$ define

$$S_k=\alpha_1x_1+\dots+\alpha_kx_k.$$

It is easy to see that

$$\|S_n\|_2\le(n-1)^{-1}(n-2)^{-1}\sum_{j\ne k}\|S_n-\alpha_jx_j-\alpha_kx_k\|_2.$$

Therefore

$$\|S_{n-2}\|_2\ge\frac{n-2}{n}\|S_n\|_2 \tag{15}$$

under a suitable numbering of x_{n-1},x_n. On the other hand, inequality (14) implies

$$\|S_{n-2}\|_2\le\|S_{n-1}\|_2\le\|S_n\|_2 \tag{16}$$

and

$$\|x_n\|\le\|S_n\|_2. \tag{17}$$

Consider for a fixed $\beta\in\Gamma$ the expression

$$\widehat S(\beta)=\alpha_1x_1+\dots+\alpha_{n-2}x_{n-2}+\alpha_{n-1}(x_{n-1}+\beta x_n)$$

and define

$$r(\beta)=\|\widehat S(\beta)\|_2^2\|S_n\|_2^{-2}-1,\qquad M_n=\max_\Gamma r(\beta).$$

Inequalities (16)–(17) and

$$\|\widehat{S}(\beta)\|_2 \le \|S_{n-1}\|_2 + \|x_n\| \tag{18}$$

imply $M_n \le 3$; the inequalities (15) and $\|\widehat{S}(\beta)\|_2 \ge \|S_{n-2}\|_2$ imply the estimate

$$r(\beta) \ge \max\{-4/n, -8/9\}. \tag{19}$$

It is not difficult to see that

$$\int_\Gamma r(\beta)\,d\mu(\beta) = \|S_n\|_2^{-2} \int_\Gamma \|\widehat{S}(\beta)\|_2^2\,d\mu(\beta) - 1 = 0. \tag{20}$$

Suppose that the inductive hypothesis is true. Replace S_{n-1} by $\widehat{S}(\beta)$ in (13). Then

$$\begin{aligned} \|S_n\|_{-q}^{-q} &= \int_\Gamma \|\widehat{S}(\beta)\|_{-q}^{-q}\,d\mu(\beta) \le C_{n-1}^q \int_\Gamma \|\widehat{S}(\beta)\|_2^{-q}\,d\mu(\beta) \\ &= C_{n-1}^q \|S_n\|_2^{-q} \int_\Gamma (1+r(\beta))^{-q/2}\,d\mu(\beta). \end{aligned} \tag{21}$$

Since $(1+t)^{-q/2} \le 1-(q/2)t + A(q)t^2$ for $t \ge -8/9$ and $q \in (0,1)$, (19)–(21) imply

$$\|S_n\|_2 \le C_{n-1}\|S_n\|_{-q}\Big[1 + A(q)\int_\Gamma r(\beta)^2\,d\mu(\beta)\Big]^{1/q}$$

and

$$\int_\Gamma r(\beta)^2\,d\mu(\beta) \le \left(M_n + \frac{8}{3n}\right)\int_\Gamma |r(\beta)|\,d\mu(\beta) \le \left(M_n + \frac{8}{3n}\right)\frac{16}{3n}.$$

If the norm in X is generated by a scalar product, then by (11) we obtain

$$\|S_k\|_2^2 = \sum_{j\le k} \|x_j\|^2.$$

Therefore

$$\|\widehat{S}(\beta)\|_2^2 = \|S_{n-2}\|_2^2 + \|x_{n-1} + \beta x_n\|^2 \le 2\|S_n\|_2^2 - \|S_{n-2}\|_2^2.$$

Now inequality (15) implies the estimate $M_n \le 4/n$, and we may take $C_n = (1+36A(q)n^{-2})^{1/q}C_{n-1}$. In the general case we calculate the measure of the set

$$D = \{\beta \in \Gamma : |r(\beta) - r(\beta_0)| \le M_n/2\},$$

where β_0 is a point of Γ such that $r(\beta_0) = M_n$. It follows from (16)–(18) that for any $\beta, \beta' \in \Gamma$,

$$|r(\beta) - r(\beta')| \le \frac{4\|\widehat{S}(\beta) - \widehat{S}(\beta')\|_2}{\|S_n\|_2} \le 4\|\beta x_n - \beta' x_n\| \cdot \|x_n\|^{-1}.$$

Then condition (12) and the estimate $M_n \le 3$ imply

$$\mu(D) \ge \mu\{\beta : \|\beta x_n - \beta_0 x_n\| \le M_n\|x_n\|/8\} \ge (M_n/8)^\varrho.$$

By (19) and (20) we obtain

$$\mu(D)\cdot\frac{1}{2}M_n \le \int_D r(\beta)\,d\mu(\beta) \le \int_{\beta : r(\beta)\le 0} |r(\beta)|\,d\mu(\beta) \le \frac{8}{3n}$$

and $M_n \le 8(23^{-1}n^{-1})^{1/(\varrho+1)}$. Therefore we can take

$$C_n = C_{n-1}(1 + 86A(q)n^{-(\varrho+2)/(\varrho+1)})^{1/q}.$$

The theorem is proved.

References

[1] S. Yu. Favorov, *The distribution of values of holomorphic mappings of $\mathbb{C}^n$ into Banach space*, Funktsional. Anal. i Prilozhen. 21 (1987), no. 3, 91–92 (in Russian); English transl.: Functional Anal. Appl. 21 (1987), 251–252.

[2] —, *Growth and distribution of values of holomorphic mappings of a finite-dimensional space into a Banach space*, Sibirsk. Mat. Zh. 31 (1990), no. 1, 161–171 (in Russian); English transl.: Siberian Math. J. 31 (1990), 137–146.

[3] —, *Estimates for asymptotic measures and Jessen functions for almost periodic functions*, Dopov. Nats. Akad. Ukraïni 1996 (10), 27–30 (in Russian).

[4] Ye. A. Gorin and S. Yu. Favorov, *Generalizations of Khinchin's inequality*, Teor. Veroyatnost. i Primenen. 35 (1990), 763–767 (in Russian); English transl.: Theory Probab. Appl. 35 (1990), 766–771.

[5] —, *Variants of the Khinchin inequality*, in: Studies in the Theory of Functions of Several Real Variables, Yaroslavl', 1990, 52–63 (in Russian).

[6] J. P. Kahane, *Some Random Series of Functions*, Cambridge Univ. Press, New York, 1985.

[7] R. Latała, *On the equivalence between geometric and arithmetic means for log-concave measures*, in: Proceedings of Convex Geometry Seminar, MSRI, Berkeley, 1996, to appear.

[8] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Math. 1200, Springer, 1986.

[9] D. C. Ullrich, *Khinchine's inequality and the zeros of Bloch functions*, Duke Math. J. 57 (1988), 519–535.

[10] —, *An extension of the Kahane–Khinchine inequality in a Banach space*, Israel J. Math. 62 (1988), 56–62.

Department of Mathematics
Kharkov State University
Swobodi sq. 4
Kharkov, 310077, Ukraine
E-mail: favorov@ilt.Kharkov.ua

Received March 13, 1997
Revised version November 6, 1997
(3857)