Hull-minimal ideals in the Schwartz algebra
of the Heisenberg group

by

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Abstract. For every closed subset $C$ in the dual space $H_n$ of the Heisenberg group $H_n$ we describe via the Fourier transform the elements of the hull-minimal ideal $j(C)$ of the Schwartz algebra $S(H_n)$ and we show that in general for two closed subsets $C_1$, $C_2$ of $H_n$, the product of $j(C_1)$ and $j(C_2)$ is different from $j(C_1 \cap C_2)$.

0. Introduction. Let $A$ be an algebra. We are interested in the structure of some special ideals of $A$. In this paper an ideal of $A$ is always a two-sided ideal. Denote by $\text{Prim}(A)$ the primitive ideal space of $A$, i.e. the space of all the ideals $J$ of $A$ of the form $J = \ker(T)$ where $(T, V)$ denotes an algebraically irreducible (or simple) representation $T$ of $A$ on a vector space $V$. We provide $\text{Prim}(A)$ with the Jacobson topology. In this topology a subset $C$ of $\text{Prim}(A)$ is closed if it is the hull $h(I)$ of some ideal $I$ of $A$, i.e.

$$C = h(I) = \{ J \in \text{Prim}(A) : J \supset I \}.$$ 

For a subset $C \subset \text{Prim}(A)$ let

$$\ker(C) = \bigcap_{J \in C} J \subset A \quad \text{and} \quad I(C) = \bigcup_{h(I) = C} I.$$ 

The hull of $I(C)$ contains of course $C$.

For certain algebras $A$, we have $h(I(C)) = C$, i.e. there exists a minimal ideal $j(C)$ with hull $C$. That means that there exists an ideal $j(C)$ of $A$ such that the hull of $j(C)$ is equal to $C$ and $j(C) \subset I$ for every ideal $I$ of $A$ whose hull is contained in $C$. It has been shown in [LRS] and in [Lui] that $j(C)$ exists for every closed subset $C$ in the primitive ideal space of the Schwartz algebra of a nilpotent Lie group.

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[77]
In Section 1, we repeat the arguments used in these two papers and we show the existence of \( j(C) \) for any semisimple symmetric polynomially bounded Fréchet algebra \( A \) (see Proposition 1.9). Given now two closed subsets \( C_1, C_2 \) in \( \text{Prim}(A) \), what can be said about \( j(C_1) \cdot j(C_2) \) or \( j(C_1) \cap j(C_2) \)? Under what conditions do we have \( j(C_1) \cdot j(C_2) = j(C_1 \cup C_2) \)? In easy cases, for instance if \( A \) is abelian or if \( C_1 \) and \( C_2 \) are separated, the equality does hold (see 1.12, 1.13 below).

In Section 2, we describe the ideals \( j(C) \) for the Heisenberg algebra \( \mathcal{S}(H_n) \), where \( H_n \) denotes the \((2n + 1)\)-dimensional Heisenberg group. Although this description is not very precise, it suffices to show that in many cases \( j(C_1) \cdot j(C_2) \neq j(C_1 \cup C_2) \) (see 2.9 below).

The paper finishes with open questions on the nature of \( j(C) \) in the Heisenberg case; for instance, what is \( j(C_1) \ast j(C_2) \) in the general case?

### 1. Hull-minimal ideals in Fréchet algebras

1.1. As an example consider a completely regular semisimple commutative Banach algebra \( A \). By Gelfand’s theory, \( A \) is isomorphic to an algebra of continuous functions vanishing at infinity on the dual space \( A^\vee \). “Regular” means that for every closed subset \( C \) of \( A^\vee \) and every point \( \chi \in A^\vee \setminus C \), there exists \( a \in A \) such that \( \varepsilon(a) \) vanishes on \( C \), but not at \( \chi \). Then, given a closed subset \( C \) of \( A^\vee \), the ideal consisting of all the \( a \in A \) such that the support of their Fourier–Gelfand transform \( \tilde{a} \) is compact and disjoint from \( C \) is the minimal ideal of \( A \) with hull \( C \) (see [BD], [S3], and [Wadi, 1.4(iii)]).

As a second example, let \( H \) be a Hilbert space and let \( A \) be the algebra of all compact operators on \( H \). The identity representation of \( A \) on \( H \) is up to equivalence the only algebraically irreducible representation of \( A \) and so \( \text{Prim}(A) \) consists of only one point. The subset \( j(A) \) of all operators with finite rank is a minimal dense ideal of \( A \). It is well known that every \( C^* \)-algebra has such a minimal dense ideal, the so-called Pedersen ideal.

1.2. Lemma. Let \( C \) be a closed subset of \( \text{Prim}(A) \). Suppose that there exist \( a, b \in A \) such that \( b \in \ker(C) \) and \( b \cdot a = a \). Then every ideal \( I \) of \( A \) with \( h(I) \subseteq C \) contains \( a \).

1.3. Hull-kernel regularity

1.3.1. Definition. We say that a semisimple algebra \( A \) is hull-kernel regular (or h.k. regular) if for any closed subset \( C \) of \( \text{Prim}(A) \) and for every \( J \in \text{Prim}(A) \setminus C \) there exist \( b_J, a_J \in A \) such that \( b_J \in \ker(C) \), \( a_J \notin J \) and \( b_J \cdot a_J = a_J \).

1.3.2. Proposition. Let \( A \) be a h.k. regular algebra. For any closed subset \( C \) of \( \text{Prim}(A) \), the minimal ideal \( j(C) \) exists and is generated by the elements \( a_J, J \notin C \).

Proof. The hull of the ideal \( I \) generated by the \( a_J, J \notin C \), is equal to \( C \), since for every \( J \notin C \), \( a_J = b_J \cdot a_J \in \ker(C) \) and \( a_J \notin J \). By Lemma 1.2, \( I \subseteq \bigcap_{h(I) = C} I = I(C) \). Hence \( I = I(C) \) since \( h(I) = C \), we see that \( j(C) = I \).

1.3.3. Remark. If \( A \) is h.k. regular, then the minimal ideal \( j(C) \) of a closed set \( C \subseteq \text{Prim}(A) \) can be described in the following way:

\[
\begin{align*}
\left\{ \sum_{J \in \mathbb{F}} b_J \cdot a_J \cdot y_J : b_J, y_J \in \tilde{A} = C_l \oplus A, \right. \\
F \text{ a finite subset of } \text{Prim}(A) \setminus C \}
\end{align*}
\]

1.3.4. Example. In many algebras it is impossible to find elements \( a, b \) such that \( b \cdot a = a \) and \( a \neq 0 \). For instance, let \( A \) be the convolution algebra \( L^1(\mathbb{R}, w) \), where the weight \( w \) is the function \( w(t) = e^{2\pi|t|}, t \in \mathbb{R} \), and where

\[
L^1(\mathbb{R}, w) = \left\{ f \in L^1(\mathbb{R}) : \|f\|_w = \int_{\mathbb{R}} w(t)|f(t)| dt < \infty \right\}.
\]

The primitive ideal space of this algebra is easily seen to be homeomorphic to the subset \( \mathbb{R} + i[-1, 1] \) of the complex numbers and \( A \) is isomorphic to the subalgebra of continuous bounded functions on \( \mathbb{R} + i[-1, 1] \) which are holomorphic on \( \mathbb{R} + i[-1, 1] \). This isomorphism is given by the Fourier transform \( f \mapsto \hat{f} \), where

\[
\hat{f}(a + ib) = \int_{\mathbb{R}} f(t) e^{-2\pi i (a+ib) t} dt, \quad f \in A, \ a + bi \in \mathbb{R} + i[-1, 1].
\]

Hence, if \( g \cdot f = f \) in \( A \), then \( \tilde{g} \cdot \hat{f} = \hat{f} \), which forces \( f \) to be 0, since otherwise \( \tilde{g} = 1 \) constant, \( g \) being holomorphic.

1.4. Definition. We say that an algebra \( A \) is a Fréchet algebra if there exists a family \( \{p_k\}_{k \in \mathbb{N}} \) of norms on \( A \) such that \( A \) is complete for the topology defined by these norms and \( p_k(a \cdot b) \leq p_k(a)p_k(b) \) for all \( k \in \mathbb{N} \) and \( a, b \in A \).

We say that the Fréchet algebra \( A \) is involutive if it is equipped with an involution *.

1.5. Definition. An element \( a \) in an involutive Fréchet algebra \( (A, \{p_k\}) \) is called polynomially bounded if for every \( k \) there exists a constant \( c_k = c_{a,k} > 0 \) such that

\[
p_k(e(i\lambda a)) \leq c_k(1 + |\lambda|)^k, \quad \forall \lambda \in \mathbb{R}, k \in \mathbb{N}.
\]
Here $e(b)$, $b \in A$, means

$$e(b) = \sum_{k=1}^{\infty} \frac{b^k}{k!} \in A.$$ 

1.6. The functional calculus

1.6.1. To a polynomially bounded $a$ we can apply the functional calculus of $C^\infty$ functions which has been developed in [Di2]. Let $C^\infty_0(\mathbb{R})$ denote the space of all complex-valued $C^\infty$ functions $\varphi$ on $\mathbb{R}$ with compact support such that $\varphi(0) = 0$. The integral

\begin{equation}
\varphi(a) = \frac{1}{2\pi i} \int_{\mathbb{R}} e(i\lambda) \varphi(a) \, d\lambda
\end{equation}

converges in $A$ for any polynomially bounded $a$. This functional calculus has the following property. For every character $\chi$ on a maximal abelian closed subalgebra $A(a)$ containing $a \in A$ we have

$$\chi(\varphi(a)) = \varphi(\chi(a)).$$

In particular, for $\varphi, \psi \in C^\infty_0(\mathbb{R})$,

$$\chi((\psi \cdot \varphi)a) = \chi(\varphi(a)) \cdot \chi(\psi(a)).$$

Suppose now that $A(a)$ is semisimple. Then

\begin{equation}
\psi(a) \cdot \varphi(a) = (\psi \cdot \varphi)(a), \quad \varphi, \psi \in C^\infty_0(\mathbb{R}).
\end{equation}

Take $\psi$ and $\varphi$ such that $\psi \cdot \varphi = \varphi$. We see that

\begin{equation}
\psi(a) \cdot \varphi(a) = \varphi(a).
\end{equation}

1.6.2. Remark. A polynomially bounded element $a$ in a Banach algebra $(A, \| \cdot \|)$ must have real spectrum. Indeed, if the spectrum of $a$ contains a nonreal number $\mu = \alpha + i\beta$, then there exists a character $\chi$ on $A(a)$ such that $\chi(a) = \mu$ and so

$$e^{-\lambda \beta} = |e^{i\lambda \beta}| = |1 + \chi(e^{i\lambda a})| \leq 1 + |\chi(e^{i\lambda a})|$$

and so $|\chi(i\lambda a)|$ grows exponentially in $\lambda$. In order to find polynomially growing elements we must look for symmetric algebras, i.e., involutive algebras for which the spectrum of every selfadjoint element is real.

1.7. Definition. We say that a Fréchet algebra $A$ is symmetric if $A$ has a continuous involution and if there exists a continuous $^*$ homomorphism $\sigma$ from $A$ into a $C^*$-algebra $C$ such that for any $a \in A$, $\text{spec}_A(a) = \text{spec}_C(\sigma(a))$. Here $\text{spec}_A(x)$ denotes the spectrum of an element $x$ in an algebra $B$.

Let $A$ be a Fréchet algebra. We denote by $\hat{A}$ the space of all topologically irreducible unitary representations $(\pi, \mathcal{H})$ of $A$ on a Hilbert space $\mathcal{H}$.

1.8. Proposition. Let $A$ be a symmetric Fréchet algebra. For every algebraically irreducible representation $(T, V)$ of $A$, there exists $(\pi, \mathcal{H}) \in \hat{A}$ such that $(T, V)$ is equivalent to a submodule of $(\pi, \mathcal{H})$.

Proof. Since $A$ is symmetric, so is $\hat{A} = C_1 \oplus A$ and we may assume that $A$ and $C$ have identities.

For any $x \in \ker(\sigma)$ and $y \in A$, the spectrum of $yx$ in $A$ is reduced to $0$. Hence if for some $x \in \ker(\sigma)$, $T(x) \neq 0$, then there exists $v \in V$ such that $T'(x)v \neq 0$ and since $T'$ is simple, we can find an element $y \in A$ such that $T(y)xv = v$, i.e., 1 is in the spectrum of $yx$. This contradiction tells us that $\ker(\sigma) \subset \ker(T)$. The simple module $(T, V)$ is equivalent to the left regular representation of $A$ on $A/M$, where $M$ denotes a proper maximal left ideal of $A$. The sum of $C_1$ and $\sigma(M)$ is direct in $C$, since otherwise $1 \in M \mod \ker(\sigma)$, which implies that $1 \in M$, since $\ker(\sigma) \subset \ker(T) \subset M$. Hence we can define a linear functional $\varphi$ on $\hat{M} = \sigma(C_1 + M) = C_1 + \sigma(M) \subset C$ by setting

$$\varphi(\lambda_1 + \sigma(m)) = \lambda_1, \quad \lambda_1 \in C, \quad m \in M.$$ 

For $x = \lambda_1 + m \in M$, $x - \lambda_1 \in M$ and so $x - \lambda_1$ is not invertible in $A$. Hence $\lambda \in \text{spec}_C(x) = \text{spec}_C(\sigma(x))$ and so

$$\|\varphi(x)\| = |\lambda| \leq \sup\{|\mu| : \mu \in \text{spec}_C(\sigma(x))\} \leq \|\sigma(x)\|.$$ 

Hence by Hahn Banach, there exists a continuous extension $\tilde{\varphi}$ of $\varphi$ to $C$ of norm $\leq 1$. Since $\varphi(1) = 1$ and $\|\tilde{\varphi}\| \leq 1$, $\varphi$ is a positive functional (which annihilates $M$) and so, since $M$ is maximal and $\varphi(\sigma(M)) = 0$, we have $M = \{y \in A : \tilde{\varphi}(\sigma(y^*y)) = 0\}$. In particular, $M$ is closed. Therefore, we can define a Hilbert-space structure on $A/M$ by setting

$$(x + M, y + M) = \tilde{\varphi}(\sigma(y^*x)).$$

The left regular representation of $A$ on $A/M$ extends to a unitary representation $\pi$ of $A$ on the completion $\mathcal{H}$ of $A/M$ (see [Di1], 2.4.4). Since $\pi$ may always assume that $\tilde{\varphi}$ is a pure state, we even know that $\pi$ is irreducible (see [Di1], 2.5).

1.9. Definition. We say that an involutive Fréchet algebra $A$ is polynomially bounded if the set $A_0$ of selfadjoint polynomially bounded elements of $A$ is dense in the real subspace $A_0$ of hermitian elements of $A$.

1.10. Proposition. Let $A$ be a semisimple symmetric polynomials boundy Fréchet algebra. Then $A$ is h.k. regular. In particular, for every closed subset $C$ in $\text{Prim}(A)$, the minimal ideal $j(C)$ exists and is generated by the elements $a_J, J \notin C$.

Proof. Since $A$ is symmetric, for any $J \in \text{Prim}(A)$ we may choose a topologically irreducible unitary representation $(\pi_J, \mathcal{H}_J)$ such that $\ker(\pi_J)$
Let $G = \text{exp } g$ denote a simply connected, connected nilpotent Lie group with Lie algebra $g$. For such a group, the exponential mapping $\exp$ is a diffeomorphism, which allows us to identify the group $G$ with the vector space $g$ as a manifold, and if we equip $g$ with the Baker–Campbell–Hausdorff product

$$X \cdot Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \ldots$$

for all $X, Y \in g$, then $\exp : (g, B, C, H) \rightarrow (G, \cdot)$ is even a group isomorphism. The Haar measure of $G$ is just Lebesgue measure on $g$, the vector space $g$.

The Schwartz algebra $A = S(G)$ of $G$ is by definition the space of rapidly decreasing $C^\infty$ functions on $G$ and is in fact a Fréchet algebra under convolution. It has been shown in [Lu2] that $A$ is symmetric and in [Hu] that $A$ is polynomially bounded. For more details see [LM].

The spaces $\text{Prim}(S(G)), \text{Prim}(L^\infty(G)), \text{Prim}(C^\ast(G))$ and $\widehat{G}$ are homeomorphic (see [Lu2]) and we shall identify them. Furthermore, for any closed subset $C$ of $\widehat{G}$ the minimal ideal $j(C)_{S(G)}$ in $S(G)$ associated with $C$ is of course contained in $j(C)_{L^1(G)} \subseteq L^1(G)$ and in $j(C)_{C^\ast(G)} \subset C^\ast(G)$. Hence

$$j(C)_{L^1(G)} = j(C)_{S(G)} + L^1(G) \ast j(C)_{S(G)} \ast L^1(G)$$

and

$$j(C)_{C^\ast(G)} = j(C)_{S(G)} + C^\ast(G) \ast j(C)_{S(G)} \ast C^\ast(G).$$

4) It follows from 1.6.2 that in the algebra $A = L^1(\mathbb{R}, u)$ no element $f$ is polynomially bounded. Indeed, since $\hat{f}$ is a complex analytic function there always exists $a + ib$ such that $\mu = \hat{f}(a + ib) \notin \mathbb{R}$. Also, this algebra does not admit minimal ideals. Let $C$ be a nonempty closed subset of $\text{Prim}(A)$ such that $\ker(C) \neq (0)$. Then for any $n \in \mathbb{N}$, $\ker(C)^n$ is an ideal of $A$ with hull $C$. But

$$I_\infty = \bigcap_{n \in \mathbb{N}} \ker(C)^n$$

is $(0)$, since the Fourier transform of any element of $I_\infty$ vanishes to infinite order on $C$.

1.12. Let $A$ be a h.k. regular semisimple Fréchet algebra and let $C_1, C_2$ be two closed subsets in $\text{Prim}(A)$. We may ask what happens to the product of $j(C_1)$ with $j(C_2)$. It is clear that

$$j(C_1) \cdot j(C_2) = j(C_1).$$

Obviously we always have

$$h(j(C_1) \cdot j(C_2)) = C_1 \cup C_2,$$
since any \( J \in \text{Prim}(A) \) is a prime ideal. Hence
\[
j(C_1) \cdot j(C_2) \supset j(C_1 \cup C_2) \quad \text{and} \quad \ker(C_1) \cdot \ker(C_2) \supset \ker(C_1 \cup C_2).
\]

1.13. PROPOSITION. Let \( A \) be an abelian h.k. regular algebra. Then for any closed subsets \( C_1, C_2 \) of \( \text{Prim}(A) \), we have
\[
j(C_1) \cdot j(C_2) = j(C_1 \cup C_2).
\]

Proof. Since for any \( J_i \in \text{Prim}(A) \setminus \{C_i\} \), we can find \( b_{J_i}, a_{J_i} \), such that \( b_{J_i} \in \ker(C_i), a_{J_i} \notin \ker(C_i) \) and \( b_{J_i} \cdot a_{J_i} = a_{J_i}, i = 1, 2 \), we see that for \( b = b_{J_1} \cdot b_{J_2} \) and \( a = a_{J_1} \cdot a_{J_2} \), we have \( b \in \ker(C_1 \cup C_2) \) and since \( A \) is abelian,
\[
b \cdot a = b_{J_1} \cdot b_{J_2} \cdot a_{J_1} \cdot a_{J_2} = b_{J_1} a_{J_1} b_{J_2} a_{J_2} = a_{J_1} a_{J_2} = a.
\]
Hence \( a \in j(C_1 \cup C_2) \) and so \( j(C_1) \cdot j(C_2) \subset j(C_1 \cup C_2) \), whence the assertion follows.

We shall see in 2.9 that the situation is much more complicated if \( A \) is not longer abelian. However, in the case where \( C_1, C_2 \) are separated in \( \text{Prim}(A) \), i.e. if there exist two open subsets \( U_1, U_2 \) in \( \text{Prim}(A) \) such that \( C_i \subset U_i, i = 1, 2 \), and \( U_1 \cap U_2 = \emptyset \), we can control \( j(C_1) \cdot j(C_2) \).

1.14. PROPOSITION. Let \( A \) be a h.k. regular algebra. Then for any closed separated subsets \( C_1, C_2 \) of \( \text{Prim}(A) \), we have
\[
j(C_1) \cdot j(C_2) = j(C_1 \cup C_2).
\]

Proof. Let \( K_i \supset \text{Prim}(A) \setminus U_i, i = 1, 2 \). Then \( C_1 \subset K_2 \), \( C_2 \subset K_1 \) and \( \text{Prim}(A) \) is the union of the two closed subsets \( K_1, K_2 \). For any \( J \notin C_1 \cup C_2 \) choose \( b_J \in \ker(C_1 \cup C_2) \) and \( a_J \notin J \) such that \( b_J \cdot a_J = a_J \) (i.e. \( a_J \in j(C_1 \cup C_2) \)); for any \( J \in C_1 \) choose \( b_J \in \ker(K_1) \) and \( a_J \notin J \) with \( b_J \cdot a_J = a_J \) (i.e. \( a_J \in j(K_1) \subset j(C_2) \)); and for \( J \in C_2 \) choose \( b_J \in \ker(K_2) \) and \( a_J \notin J \) with \( b_J \cdot a_J = a_J \) (i.e. \( a_J \in j(K_2) \subset j(C_1) \)). Then, for any \( a \in A, J_1 \notin C_1, J_2 \notin C_2, a \cdot a_{J_1} a_{J_2} \in j(C_1 \cup C_2) \) if \( J_1 \) or \( J_2 \notin C_1 \cup C_2 \),
by the choice of \( a_{J_1}, a_{J_2} \). If \( J_1, J_2 \subset C_1 \cup C_2 \) then \( a_{J_1} \cdot a_{J_2} \in \ker(K_1 \cup K_2) = \ker(\text{Prim}(A)) = \{0\} \). Since \( j(C_i), i = 1, 2 \), is generated by elements of the form
\[
b \cdot a, \quad a, b \in A, J \notin C_i,
\]
we see again that \( j(C_1) \cdot j(C_2) \subset j(C_1 \cup C_2) \).

1.15. PROPOSITION. Let \( A \) be a h.k. regular algebra. Then for any closed subsets \( C_1, C_2 \subset \text{Prim}(A) \), we have
\[
j(C_1 \cap C_2) = j(C_1) + j(C_2).
\]
If \( C_1 \subset C_2 \), then \( j(C_2) \subset j(C_1) \).

Proof. The hull of \( j(C_1) + j(C_2) \) is obviously equal to \( C_1 \cap C_2 \) and so \( j(C_1 \cap C_2) \subset j(C_1) + j(C_2) \). Hence it suffices to show the opposite inclusion. But for closed subsets \( C \subset B \) of \( \text{Prim}(A) \), and for \( J \notin B \), there exist \( b_J, a_J \in j(B) \) such that \( b_J \in \ker(B) \subset \ker(C), a_J \notin J \) and \( b_J a_J = a_J \). Hence also \( a_J \in j(C) \) by 1.3.2, and since the \( a_J, J \notin B \), generate \( j(B) \) we see that
\[
(1.15.1) \quad j(B) \subset j(C)
\]
and so \( j(C_1) \subset j(C_1 \cap C_2) \) and \( j(C_2) \subset j(C_1 \cap C_2) \).

2. The minimal ideals in the Schwartz algebra of the Heisenberg group

2.1. We shall determine the minimal ideals \( j(C) \) in the Schwartz algebra of the Heisenberg group \( H_n \) by describing the Fourier transforms of the elements of \( j(C) \). This section is based on [Fe] and uses its notations.

As a manifold, \( H_n \) is the space \( \mathbb{R}^{2n+1} \). We write \( (p, q, t) \) for the elements of \( H_n \), where \( p, q \in \mathbb{R}^n, t \in \mathbb{R} \). The group law on \( H_n \) is defined by
\[
(p, q, t) \cdot (p', q', t') = (p + p', q + q', t + t' + \frac{1}{2}(p \cdot q' - p' \cdot q)).
\]
Here \( p \cdot q \) means the ordinary euclidean product on \( \mathbb{R}^n \), i.e.
\[
p \cdot q = p_1 q_1 + \ldots + p_n q_n.
\]
The center \( Z \) of \( H_n \) is given by the last coordinate, i.e.
\[
Z = \{0\} \times \{0\} \times \mathbb{R},
\]
and we observe that \( Z \) is also the first commutator \([H_n, H_n] \) of \( H_n \).

The Lie algebra \( h_n \) of \( H_n \) can also be identified with \( \mathbb{R}^{2n+1} \) and the exponential mapping is then the identity mapping. For \( j \in \{1, \ldots, n\} \) we define the vectors
\[
X_j = (\delta_{i,j})_{i=1, \ldots, 2n+1}, \quad Y_j = (\delta_{i,j+n})_{i=1, \ldots, 2n+1}, \quad Z = (\delta_{i,2n+1})_{i=1, \ldots, 2n+1}.
\]
We obtain the classical commutator relations
\[
[X_i, X_j] = \delta_{i,j} Z, \quad 1 \leq i, j \leq n,
\]
and \( Z \) spans the center \( \mathfrak{z} \) of \( h_n \). We also identify the dual space \( h_n^* \) of \( h_n \) with \( \mathbb{R}^{2n+1} \). An element \( (a, b, \lambda) \in h_n^* \) acts on \( (p, q, t) \) by
\[
\langle (a, b, \lambda), (p, q, t) \rangle = a \cdot p + b \cdot q + \lambda t.
\]

2.2. The dual space \( \hat{H}_n \) is the union of the set \( CH\mathcal{A} \) of one-dimensional representations and the set \( \hat{H}_{\infty} \) of infinite-dimensional ones. The characters \( \chi \in CH\mathcal{A} \) are defined through the elements \( \phi = (a, b) \in \mathbb{R}^{2n} \approx \mathfrak{h}_n \),
\[
\chi_{\phi}(p, q, t) = e^{-2\pi i (a \cdot p + b \cdot q)}, \quad (p, q, t) \in H_n.
\]
The infinite-dimensional representations can be parametrized by $\mathbb{R}^* \times \mathbb{R}$, for $\lambda \in \mathbb{R}^*$. We take the linear form $\lambda_\lambda \in \mathfrak{h}_n^*$ for which

$$l_\lambda(X_j) = l_\lambda(Y_j) = 0, \quad j = 1, \ldots, n, \quad l_\lambda(Z) = \lambda.$$ 

The subalgebra $b = \text{span}\{Y_j, Z : j = 1, \ldots, n\} = \{(0, q, t) : q \in \mathbb{R}^n, \ t \in \mathbb{R}\}$ is a polarization at $l_\lambda$ for any $\lambda$ and so if $\chi_\lambda$ denotes the character

$$\chi_\lambda(0, q, t) = e^{-2i\pi \lambda t}, \quad (0, q, t) \in b = \exp(b) = B,$$

then by the Stone-von Neumann theorem,

$$\pi_\lambda = \text{ind}_{H_n}^{\hat{H}_n} \chi_\lambda$$

is irreducible and every $\pi \in \hat{H}_n^\infty$ is of this form. We can identify the Hilbert space $\mathcal{H}_\pi$ of $\pi_\lambda$ with $L^2(\mathbb{R}^n)$ and we obtain the following relations:

$$\pi_\lambda(p, q, t) \xi(v) = e^{-i2\pi \lambda t + i\pi \psi(v + q - (v - p))} \xi(v - p),$$

$$\xi \in L^2(\mathbb{R}^n), \quad v \in \mathbb{R}^n, \ (p, q, t) \in H_n.$$

Hence $\hat{H}_n$ can be identified with $\mathbb{R}^{2n} \cup \mathbb{R}^*$. By Kirillov's theory, $\hat{H}_n$ is homeomorphic to the space of the coadjoint orbits $\mathfrak{h}_n^*/H_n$. The character $\chi_\lambda(a, b)$ corresponds to the linear functional $(a, b, 0) \in \mathbb{R}^{2n+1}$ and the representation $\pi_\lambda$, $\lambda \in \mathbb{R}^*$, to the functional $l_\lambda = (0, 0, 0, \lambda) \in \mathbb{R}^{2n+1}$. The coadjoint orbit $\Omega_\lambda$ of $l_\lambda$ is the affine subspace $\mathbb{R}^n \times \{\lambda\}$. Hence in the orbit space,

$$\lim_{\lambda \to 0} \Omega_\lambda = \mathbb{R}^n \times \{0\} = \bigcup_{a, b \in \mathbb{R}^n} \{(a, b)\},$$

and so in $\hat{H}_n$,

$$\lim_{\lambda \to 0} \pi_\lambda = \bigcup_{a, b \in \mathbb{R}^n} \chi_{(a, b)} = \mathcal{CH}_A.$$

Hence $\hat{H}_n$ is not a Hausdorff space.

We now consider the Schwartz algebra $S(H_n)$ of $H_n$. The elements of $S(H_n)$ are just ordinary Schwartz functions on $\mathbb{R}^{2n+1}$ and $S(H_n)$ is an algebra for the convolution $*$ and the involution $^*$:

$$f \ast g(x) = \int_{H_n} f(y)g(y^{-1}x) \, dy, \quad f^*(x) = \overline{f(x')}.$$

For every $\pi \in \hat{H}_n$ and $f \in S(H_n)$, we can define the operator $\pi(f)$ on $H_n$ by

$$\pi(f) = \int_{H_n} f(x)\pi(x) \, dx.$$

Hence for $(a, b) \in \mathcal{CH}_A$,

$$\chi_{(a, b)}(f) = \hat{f}(a, b, 0) = \int_{H_n} f(p, q, t) e^{-i2\pi \lambda(p+q-a)} \, dp \, dq \, dt.$$
2.4. Closed sets in \(\hat{H}_n\) containing the characters

2.4.1. We now describe the minimal ideals of \(S(H_n)\) associated with the closed subsets \(C\) containing the characters of \(H_n\). This is a situation which can also be understood in the general nilpotent Lie group case (for details see [CG]). Let therefore \(G\) denote any nilpotent simply connected, connected Lie group with Lie algebra \(\mathfrak{g}\). We know that \(\hat{G}\) is homeomorphic to the space \(\mathfrak{g}^*/G\) of coadjoint orbits of \(G\) and that \(\hat{G} \cong \text{Prim}(L^1(G)) \cong \text{Prim}(S(G))\).

We can describe \(\mathfrak{g}^*/G\) and \(\hat{G}\) explicitly in the following way. Let \(\mathcal{B} = \{X_1, \ldots, X_1\}\) be a Jordan–Hölder basis of \(\mathfrak{g}\) and \(\mathcal{B}^* = \{l_1, \ldots, l_n\}\) be its dual basis in \(\mathfrak{g}^*\). For every \(l \in \mathfrak{g}^*\) there exists an index set \(I(l) \subset \{1, \ldots, n\}\) such that if \(V(l)\) denotes the span of \(\{l_i : i \notin I(l)\} \subset \mathfrak{g}^*\), then the coadjoint orbit \(\Omega_l\) of \(l\) meets \(V(l)\) in a single point. Let us take the Vergne polarization \(p(l) = \mathfrak{g}^{2l}(l)\) at \(l\) associated with \(\mathcal{B}\) and let \(\pi_l = \text{ind}_\mathcal{B}^{\mathfrak{g}}(\chi_l)\) be the representation induced from the character \(\chi_l\) of the subgroup \(F(l) = \exp(p(l))\) of \(G\). Then \(\pi_l\) is irreducible and every irreducible representation \(\pi\) of \(G\) is equivalent to some \(\pi_l\). For two elements \(l, p \in \mathfrak{g}^*\), the representations \(\pi_l\) and \(\pi_p\) are equivalent if and only if \(\Omega_l = \Omega_p\).

It has been shown in [LZ] that there exists an index set \(I \subset \{1, \ldots, n\}\) and a Zariski open \(G\)-invariant subset denoted by \(\mathfrak{g}_{gen}^\mathfrak{g}\) in \(\mathfrak{g}^*\), the set of elements in general position, such that \(I(l) = I\) for any \(l \in \mathfrak{g}_{gen}^\mathfrak{g}\). Furthermore, for any \(l \in \mathfrak{g}_{gen}^\mathfrak{g}\), there exists a Mal’tsev basis \(X(l) = \{X_1(l), \ldots, X_p(l)\}\) of \(\mathfrak{g}\) relative to \(p(l)\) (i.e.,

\[ \mathfrak{g} = \mathfrak{R}X_1(l) \oplus \cdots \oplus \mathfrak{R}X_p(l) \oplus p(l) \]

and \(\sum_{i=1}^p \mathfrak{R}X_i(l) + p(l)\) is a subalgebra for any \(j\), a Mal’tsev basis \(Y(l) = \{Y_1(l), \ldots, Y_p(l)\}\) of \(p(l)\) relative to the stabilizer \(\mathfrak{g}(l)\) of \(l\) in \(\mathfrak{g}\) and a Mal’tsev basis \(Z(l) = \{Z_1(l), \ldots\}\) of \(\mathfrak{g}(l)\) such that \(l \mapsto X_j(l), l \mapsto Y_j(l)\) and \(l \mapsto Z_j(l)\) are polynomial mappings. In particular, if \(U\) denotes the Zariski open subset \(\mathfrak{g}_{gen}^\mathfrak{g} \cap \mathcal{V}\) of \(\mathfrak{g}\), where \(\mathcal{V} = \text{span}(l_i : j \notin I)\), then the mapping

\[ (l, T, S) \mapsto \left( \prod_{i=1}^p \text{Ad}^*(\exp(t_iX_i(l))) \prod_{i=1}^p \text{Ad}^*(\exp(s_iY_i(l))) \right) \]

is a diffeomorphism.

If \(G = H_n\), then \(\mathfrak{g}_{gen}^\mathfrak{g}\) corresponds to \(\hat{H}_n\) (in fact, \(p = n\) and for the Jordan–Hölder basis \(\mathcal{B} = \{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}\) we have \(\mathfrak{V} = \mathfrak{R}Z^*\) and \(X_j(l) = X_j, Y_j(l) = Y_j\) for all \(l \in \mathfrak{h}_{gen}^\mathfrak{g}\).

Coming back to our general \(G\), we can use for \(l \in U\) the Mal’tsev basis \(\{Y(l), Z(l)\}\) of \(p(l)\) to write down the kernel \(f_{i}(l)\) of the operator \(\pi_l(f), f \in S(G)\). Define first for \(l \in U\) the polynomial diffeomorphisms

\[ X_i : \mathbb{R}^p \rightarrow G/P(l), \quad T \mapsto \prod_{j=1}^p \exp(t_jX_j(p)), \]

and

\[ E_i : \mathbb{R}^p \times \mathbb{R}^m \rightarrow P(l), \quad (S, U) \mapsto \prod_{j=1}^p \exp(s_jY_j(l)) \prod_{j=1}^m \exp(s_jZ_j(l)). \]

The operator \(\pi_l(f) = \int_G f(x)\pi_l(x)dx\) is a trace class operator whose kernel \(f_l \in C^\infty(G \times G)\) is given by the expression

\[ f_l(x, y) = \int_{\mathbb{R}^p \times \mathbb{R}^m} \frac{f(xp^{-1})e^{-i(l, \log(p))}}{p^{|l|}} dp \]

for all \(x, y \in G, x, y \in P(l),\) and the function \((T, T') \mapsto \tilde{F}(T, T') = F(X(T), X(T'))\) is in \(C^\infty(\mathbb{R}^p \times \mathbb{R}^m)\) (resp. \(T \mapsto \xi(T)\) is in \(C^\infty(\mathbb{R}^p)\)).

Then for \(f \in S(G)\) and for a fixed \(l \in \mathfrak{g}_{gen}^\mathfrak{g}\), the function \(f_{i,l}\) in \(S(G \times G, \mathfrak{g})\) and the mapping \(U \ni l \mapsto f_{i,l}\) from \(U\) into \(S(\mathbb{R}^p \times \mathbb{R}^m)\) where

\[ f_{i,l}(T, T') = f_{i,l}(X(T), X(T')), \quad T, T' \in \mathbb{R}^p, \]

is in \(C^\infty(U, \mathfrak{g}_{gen}^\mathfrak{g})\) (see [LZ]).

We obtain in this fashion a Fourier transform \(f \mapsto \hat{f}\) on \(S(G)\) by setting (2.4.1.1)

\[ \hat{f}(l) = f_{i,l}, \quad l \in U. \]

2.4.2. It has been shown in [LZ] that every mapping

\[ F : U \ni l \mapsto F(l) \in S(G \times G, \mathfrak{g}) \]

with compact support such that \(\hat{F} \in C^\infty(U, \mathfrak{g}_{gen}^\mathfrak{g})\) is in the image of the Fourier transform. In particular, if we choose two smooth functions \(\xi, \eta U \rightarrow S(\mathbb{R}^p)\) with compact support, then there exists a unique \(f_{\xi,\eta} \in S(G)\) such that \(f_{\xi,\eta} = F_{\xi,\eta}\), where

\[ F_{\xi,\eta}(l, x, y) = \xi(l, x)\eta(\eta(l, y), x, y \in G, l \in \mathcal{V}, \xi(l, X(T)(S, U))(T) = \xi(T)e^{-i(l, \log(p))(S, U))}. \]

and similarly for \(\eta\).

2.4.3. DEFINITION. We say that the element \(f_{\xi,\eta} \in S(G)\) is elementary.
For \( g, h \in \mathcal{S}(G) \) we see that
\[
g \ast f_{\xi, \eta} \ast h = f_{\tilde{g}(\xi), \tilde{h}(\eta)},
\]
where
\[
\tilde{g}(\xi)(l, x) = \int_{G/P(l)} g_l(x, y) \xi(l, y) \, dy,
\]
\[
\tilde{h}(\eta)(l, y) = \int_{G/P(l)} h_l(x, y) \eta(l, x) \, dx.
\]
Therefore \( g \ast f_{\xi, \eta} \ast h \) is also elementary.

Let \( \tilde{G}_{\text{gen}} \) be the (open dense) subset of \( \tilde{G} \) corresponding to \( a_{\text{gen}} \). Let \( \tilde{G}_{\text{sing}} \) be its complement. \( \tilde{G}_{\text{gen}} \) is homeomorphic to \( \tilde{U} \) and in particular every open subset of \( \tilde{G} \) disjoint from \( \tilde{G}_{\text{sing}} \) corresponds to an open subset of \( \tilde{U} \).

2.5. Theorem. Let \( C \) be a closed subset of \( \tilde{G} \) containing \( \tilde{G}_{\text{sing}} \). Let \( U_C = \tilde{G} \setminus C \subset \tilde{U} \). The minimal ideal \( j(C) \) in \( \mathcal{S}(G) \) is the vector space spanned by all the elementary \( f_{\xi, \eta} \)'s with support of \( \xi \) and \( \eta \) contained in \( U_C \) and compact.

Proof. Let \( \pi \in \tilde{G} \setminus \tilde{C} \). There exists \( l \in U_C \) such that \( \pi \simeq \pi_l \). Choose \( \tilde{\xi}_0 = \tilde{\xi}(\mathbb{R}P^d) \) and \( \varphi \in C^\infty(U_C) \) such that \( \|\tilde{\xi}_0\|_L = 1, \varphi(l) = 1 \) and \( \text{supp} \varphi \subset U_C \). Choose also \( \psi \in C^\infty(U_C) \) such that \( \psi \cdot \varphi = \psi \). Let
\[
\tilde{\sigma} = \varphi \otimes \tilde{\xi}_0, \quad \tilde{\tau} = \psi \otimes \tilde{\xi}_0 \in C^\infty(\tilde{U}, S_p).
\]
Then \( \pi_l(f_{\tau, \sigma}) = 0 \) for every \( q \notin \text{supp} \psi \), hence \( \pi'(f_{\tau, \sigma}) = 0 \) for every \( \pi' \in C \). Furthermore,
\[
(f_{\tau, \sigma} * f_{\sigma, \tau})(q) = |\varphi|^2(q) |\varphi|^2(q) \tilde{\xi}_0(q) \tilde{\psi}(q) \tilde{\xi}_0(q) = \tilde{f}_{\tau, \sigma}(q)
\]
for every \( q \in U \). Hence
\[
f_{\tau, \sigma} * f_{\sigma, \tau} = f_{\sigma, \sigma}
\]
and so by 1.2, \( f_{\sigma, \sigma} \in j(C) \) and since \( \tilde{f}_{\sigma, \sigma}(l) = \tilde{\xi}_0 \otimes \tilde{\xi}_0 \neq 0 \), we see that the ideal \( I \) generated by the \( \tilde{f}_{\sigma, \sigma} \) admits the set \( C \) as null and hence by the minimality of \( j(C) \), \( I = j(C) \). By 2.4.3, we see that all the elements of \( I \) are finite sums of elementary ones.

Let now \( \tilde{\xi}, \tilde{\eta} \in C^\infty(U_C, S_p) \), with compact support. Take a function \( \varphi \in C^\infty(U_C) \) with compact support in \( U \) such that
\[
\varphi \tilde{\xi} = \tilde{\xi}, \quad \varphi \tilde{\eta} = \tilde{\eta}.
\]
By 2.4.2 there exist \( f, f' \) in \( \mathcal{S}(G) \) such that
\[
f_1(x, y) = \varphi((l) \xi(l, x) \xi_0(l, y)),
\]
\[
f'_1(x, y) = \varphi((l) \xi_0(l, x) \eta(l, y)), \quad l \in U, \ x, y \in G.
\]
Hence for \( f_{\sigma, \sigma} \) as above,
\[
(f * f_{\sigma, \sigma} * f')^\langle q \rangle = \varphi^2(q) \tilde{\xi}_0 \xi_0(\xi_0(q) \eta(q) q = \xi(q) \eta(q) q
\]
for all \( q \in U \). Hence \( f_{\xi, \eta} = f * f_{\sigma, \sigma} * f' \in j(C) \) and so \( j(C) = \text{span} \{ f_{\xi, \eta} \} \).

2.5.1. COROLLARY. Let \( C_1, C_2 \) be closed subsets of \( \tilde{G} \) containing \( \tilde{G}_{\text{sing}} \).

Then
\[
j(C_1) * j(C_2) = j(C_1 \cup C_2).
\]

2.6. Closed sets in the dual of \( H_n \) not containing \( C'H \). A

2.6.1. We now come to the case where the closed subset \( C \) of \( \tilde{H}_n \) does not contain all characters. That means that there exists \( \delta > 0 \) such that \( \delta \notin \mathbb{R}^+ \) is not contained in \( C \), since otherwise we can find a sequence \( \{ \lambda_k \} \subset \mathbb{R}^+ \cap C \) which converges to 0, and so all the limit points of this sequence, i.e. the characters of \( \tilde{H}_n \), belong to \( C \). Hence
\[
(2.6.1.1) \quad \delta = \min \{ |\lambda| : \lambda \in C \cap \mathbb{R}^+ \} > 0
\]
and \( C = C_0 \cup C_\infty \) is the disjoint union of the closed set \( C_0 = C \cap \mathbb{C}'A = C \cap \mathbb{R}^+ \) and \( C_\infty = C \cap \tilde{H}_n^\infty = C \cap \mathbb{R}^+ \).

In order to construct elements in \( j(C) \), we first consider the 3-dimensional Heisenberg group \( H_1 = \mathbb{R}^3 \) whose Lie algebra \( h_1 \) is spanned by the vectors \( X, Y \) and \( Z \) with the nontrivial bracket \( [X, Y] = Z \). We use the heat kernel \( \{ q_t \}_{t \in \mathbb{R}^+} \), associated with the homogeneous operator \( L = X^2 + Y^2 \) on \( H_1 \) (see [PS], 1.68–1.74). The functions \( q_t \) are Schwartz functions of \( L \)-norm 1 such that \( \partial_t q_t = L(q_t) \) for every \( t \in \mathbb{R}^+ \) and formally \( q_t = \exp(tL)q_1 \). This means that for any unitary representation \( \pi \) of \( H_1 \), we have
\[
d \pi(q_t) = \frac{\partial}{\partial t} \pi(q_t)
\]
for any \( t > 0 \) and any \( C^\infty \)-vector \( \xi \) of \( \mathcal{H}_x \) and so
\[
(2.6.1.2) \quad \exp(t\pi(L)) = \pi(q_t), \quad t \in \mathbb{R}^+,
\]
in the sense of functional calculus. Now for \( \lambda \in \mathbb{R}^+ \),
\[
d \pi(\lambda) = d \pi(\lambda X)^2 + d \pi(\lambda Y)^2 = \left( \frac{d}{d\lambda} \right)^2 - 4\pi^2 \lambda^2 M_v^2,
\]
where \( M_v \) denotes multiplication with the function \( v \mapsto v \) in \( L^2(\mathbb{R}) \). The
Hermite functions $h_j$, $j \in \mathbb{N}$,
$$h_j(v) = \frac{g^{1/4}}{\sqrt{2^j j!}} e^{v^2} \frac{d^j}{dv^j}(e^{-2\pi v^2}), \quad v \in \mathbb{R},$$
form an orthonormal basis of $L^2(\mathbb{R})$ consisting of eigenvectors for $d\pi_1(L) = (d/dv)^2 - 4\pi^2 M^2_L$ (see [Fo]). In fact, for $\lambda = 1$,
$$d\pi_1(L)h_j = -2\pi(2j + 1)h_j, \quad j \in \mathbb{N},$$
(see [Fo]). Hence an easy calculation shows that
$$d\pi_1(L)h_{j,\lambda} = -2\pi(2j + 1)|\lambda| h_{j,\lambda}, \quad j \in \mathbb{N},$$
where
$$h_{j,\lambda}(v) = |\lambda|^{j/4} h_j(v/|\lambda|), \quad v \in \mathbb{R}, \quad j \in \mathbb{N}.$$ We can now write the kernel $q_\lambda$ of the operator $\pi_\lambda(g) = \exp(d\pi_1(L))$ where $g = q_1$. From 2.6.1 it follows that
$$2.6.1.3 \quad \pi_\lambda(q_t)h_{j,\lambda} = e^{-2\pi |\lambda|(2j+1)t}h_{j,\lambda}, \quad \forall \lambda, j, t > 0.$$ 2.6.2. Going back to $H_n$, it suffices to consider the Schwartz function $q = q_1 \ast \cdots \ast q_n$ on $H_n$, where $q_i$, $i = 1, \ldots, n$, is the smooth measure defined on $H_n$ by
$$\langle q_i, f \rangle = \int \left. f(\exp(s_iX_i + t_iY_i + u_iZ_i)q_1(s_i, t_i, u_i) \right| ds_i dt_i du_i, \quad f \in \mathcal{S}(H_n).$$ For $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$, let
$$h_{j,\lambda}(v) = |\lambda|^{j/4} h_j(v/\sqrt{|\lambda|}) \cdots h_{j_n}(v_n/\sqrt{|\lambda|}), \quad v \in \mathbb{R}^n,$$
and
$$|j| = j_1 + \cdots + j_n.$$ It follows from (2.6.1.3) that the kernel $q_\lambda$ of $\pi_\lambda(g)$ can be written as
$$q_\lambda(v, p) = \sum_{j \in \mathbb{N}^n} e^{-2\pi |\lambda|(2|j|+n)} h_{j,\lambda}(v)h_{j,\lambda}(p), \quad v, p \in \mathbb{R}^n,$$
and
$$\pi_\lambda(g) = \sum_{j \in \mathbb{N}^n} e^{-2\pi |\lambda|(2|j|+n)} q_j.$$ is the sum of $e^{-2\pi |\lambda|(2|j|+n)}$-times the one-dimensional orthogonal projections $q_j$ onto $\mathcal{C}_{j,\lambda}$. Since $q$ is selfadjoint, we may apply the functional calculus of $C^\infty$ functions to $q$. Therefore if $\varphi \in C^\infty(\mathbb{R})$ and $\varphi(0) = 0$, then
$$\pi_\lambda(\varphi(g)) = \varphi(\pi_\lambda(g)) = \sum_{j \in \mathbb{N}^n} \varphi(e^{-2\pi |\lambda|(2|j|+n)})q_j.$$ and so
$$2.6.2.1 \quad \varphi(g)_\lambda(v, p) = \sum_{j \in \mathbb{N}^n} \varphi(e^{-2\pi |\lambda|(2|j|+n)})h_{j,\lambda}(v)h_{j,\lambda}(p), \quad v, p \in \mathbb{R}^n.$$ Let now $\chi = \chi_{\alpha,\lambda}$ be a unitary character of $H_n$. Multiplication with $\chi$ defines an automorphism of $\mathcal{S}(H_n)$ and we have
$$\varphi(\chi f) = \chi(\varphi(f))$$ for every selfadjoint $f \in \mathcal{S}(H_n)$ and $\varphi \in C^\infty(\mathbb{R})$ with $\varphi(0) = 0$. Furthermore, it follows from (2.2.2) that
$$2.6.2.2 \quad (\chi f)_\lambda(v, p) = e^{-2i\pi \alpha(v-p)} f_j^{2,\lambda}(v-p, p+v/\lambda/2 - b/\lambda) = e^{-2i\pi \alpha(v-p)} f_j(v-b/\lambda, p-b/\lambda), \quad v, p \in \mathbb{R}^n,$$
for every $f = f^* \in \mathcal{S}(H_1)$. In particular,
$$2.6.2.3 \quad (\chi g)_\lambda(v, p) = \sum_{j \in \mathbb{N}^n} e^{-2\pi |\lambda|(2|j|+n)} h_j^{\chi}(v)h_j^{\chi}(p), \quad v, p \in \mathbb{R}^n,$$
where
$$2.7.1. \text{DEFINITION. Let } C \text{ be a closed subset of } \mathbb{R}_n \text{ not containing } CH_\lambda.$$ For every $\chi = (a, b) \in \mathbb{R}_n \setminus C$, let
$$d(\chi, C) = \min(\text{distance of } \chi \text{ to } C_0, \delta),$$
where $\delta$ is as in (2.6.1.1). 2.7.2. DEFINITION. We say that a function $\varphi \in C^\infty(\mathbb{R})$ is adopted to $\chi \in CH_\lambda$ if $\varphi(1) = 1$ and $\varphi(t) = 0$ whenever $|t| < e^{-2\pi d(\chi, C)}$.
2.7.3. DEFINITION. We say that a function $f \in \mathcal{S}(H_n)$ is elementary for $\chi$ if for every $\lambda \in \mathbb{R}^n$,
$$f_\lambda = \sum_{j \in \mathbb{N}^n} \varphi(e^{-2\pi |\lambda|(2|j|+n)}) \xi_{j,\lambda} \otimes \eta_{j,\lambda},$$
where $\varphi$ is adapted to $\chi$ and $\xi_{j,\lambda}, \eta_{j,\lambda}$ are Schwartz functions such that the functions $F, G$ defined on $\mathbb{R}^n \times \mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R}^*$ by
$$\left. F(\lambda) = \sum_{j \in \mathbb{N}^n} \xi_{j,\lambda} \otimes h_j^{\chi}, \quad G(\lambda) = \sum_{j \in \mathbb{N}^n} h_j^{\chi} \otimes \eta_j^{\chi}, \quad \lambda \in \mathbb{R}^n, \right.$$ are in $\mathcal{S}(H_n)$.
2.8. PROPOSITION. Let $C$ be a closed subset of $\mathbb{H}_n$ not containing all characters of $H_n$. The minimal ideal $j(C)$ is the span of all the $f$'s in $\mathcal{S}(H_n)$.
which are elementary for some $\chi \in \CH\setminus C$ or which are elementary in $j(C \cup j(A))$.

Proof. Let $\chi = \chi_{a,b} \in \CH\setminus C$ and let $\varphi$ be adapted to $\chi$. It follows from the definition of $d(\chi, C)$ that for the Schwartz function $g$ of 2.6,\

\[ \varphi(\chi g)^{\chi}(u,v) = \varphi(g)^{\chi}(u-a,v-b) = 0, \quad \text{for } f(x^2) = d(\chi, C)^2 \text{ and } f(x) > d(\chi, C)^2. \]

Hence, if we take $\psi$ adapted to $\chi$ such that $\varphi \cdot \psi = \varphi$, then we see that $\psi(\chi g) \in \ker(C \cup R\lambda)$ for all $\lambda \in \ker(C \cup R\lambda)$ where $R\lambda = \{ \lambda \in \ker(C \cup R\lambda) \mid \lambda \geq 0 \}$, $\varphi(\chi g)^{\chi}(\lambda) = \varphi(\chi g, \lambda) = 0$ and $\psi(\chi g) * \varphi(\chi g) = \varphi(\chi g)$. Hence, $\varphi(\chi g) \in j(C \cup R\lambda)$ and so $\varphi(\chi g) \in j(C \cup R\lambda)$ generate $j(C \cup R\lambda)$. The function $\varphi(t) = \varphi(t)/t$, $t \in \mathbb{R}^+$, extends to an element of $C_{c,0}^{\infty}(\mathbb{R})$, since $\varphi$ vanishes in a neighbourhood of 0. Hence, we can write\

\[ \varphi(\chi g)^{\chi}(u,v) = \sum_{\lambda \in \ker(C \cup R\lambda)} \varphi(\chi g)^{\chi}(u-v, \lambda) = \sum_{\lambda \in \ker(C \cup R\lambda)} \varphi(\chi g)^{\chi}(u,v) = \varphi(\chi g)^{\chi}(u,v), \]

and so

\[ \xi_{i,\lambda} = \varphi(\chi g)^{\chi}(u,v), \quad \eta_{i,\lambda} = \varphi(\chi g)^{\chi}(u,v). \]

i.e.

\[ F = \tilde{F} = \tilde{G}. \]

If $f, g$ are in $S(H\lambda)$ then

\[ (f * \varphi(\chi g) * g)^{\lambda}(\lambda) = f_{\chi} \circ \varphi(\chi g) \circ g_{\lambda}, \]

where

\[ \eta_{i,\lambda} = \int_{\mathbb{R}^n} \varphi(\chi g)^{\chi}(u,v) \, du, \quad \xi_{i,\lambda} = \int_{\mathbb{R}^n} \varphi(\chi g)^{\chi}(u,v) \, du. \]

On the other hand,

\[ f_{\lambda}(u,v) = \sum_{\lambda \in \ker(C \cup R\lambda)} f_{\chi,\lambda}(u) h_{\lambda,\chi}(v), \quad g_{\lambda}(u,v) = \sum_{\lambda \in \ker(C \cup R\lambda)} g_{\lambda,\chi}(u) h_{\lambda,\chi}(v). \]

Since $h_{\lambda,\chi}(\lambda)$ is not 0, we see that $\langle h_{\lambda,\chi}(\lambda), h_{\lambda,\chi}(\lambda) \rangle_{L^2} = 0$ for some $\lambda \neq 0$ and so the analytic function $\lambda \mapsto \langle h_{\lambda,\chi}(\lambda), h_{\lambda,\chi}(\lambda) \rangle_{L^2}$, $\lambda \in \mathbb{R}^+$, is not 0. Hence the mapping

\[ \lambda \mapsto \varphi(\lambda, \lambda) \in \ker(C \cup R\lambda), \lambda \in \mathbb{R}^+, \]

is not 0 in any neighbourhood of 0. Furthermore, the functions $h_{\lambda,\chi}(\lambda), h_{\lambda,\chi}(\lambda)$ being linearly independent, we see that also the mapping $\varphi(\chi g, \lambda) \circ \varphi(\chi g, \lambda)$ is not identically 0 in any neighbourhood of 0. Hence $\varphi(\chi g) \circ \varphi(\chi g)$ is in $\mathcal{S}(C \cup j(C'))$ but not in $\mathcal{S}(C \cup j(C'))$, since for $g \in \mathcal{S}(C \cup j(C'), \lambda)$ vanishes in a neighbourhood of 0 by 2.5.

2.10. Questions and remarks

2.10.1. According to Mehler’s formula the function $q_{\lambda}$ on $H\lambda$ is equal to

\[ q_{\lambda}(u,v) = \left( \frac{e^{2|\lambda|} - e^{-2|\lambda|}}{|\lambda|} \right)^{1/2} \times \exp \left( -\pi \left( e^{2|\lambda|} + e^{-2|\lambda|} \right) \right) \left( u^2 + v^2 + 4\pi |\lambda| |\lambda| \right) \]

\[ \rightarrow \sqrt{1/2} e^{-\pi(u-v)^2/2} \quad \text{as } \lambda \rightarrow 0, \]

and so
\[ q_{\lambda} \left( \frac{x}{\lambda} + \frac{y}{\lambda} - \frac{z}{\lambda} \right) \]
\[ = \left( \frac{2|\lambda|}{e^{2|\lambda|} - e^{-2|\lambda|}} \right)^{1/2} \times \exp \left( -\frac{2\pi(e^{2|\lambda|} + e^{-2|\lambda|})(x^2|\lambda| + y^2|\lambda|)}{e^{2|\lambda|} - e^{-2|\lambda|}} + 2(-x^2|\lambda| + y^2|\lambda|) \right) \]
\[ \quad \times \exp \left( -\frac{\pi}{2} \left( \frac{(e^{2|\lambda|} + e^{-2|\lambda|})^2}{e^{2|\lambda|} - e^{-2|\lambda|}} |\lambda|^2 \right) + 4 \frac{(e|\lambda| - e^{-|\lambda|})^2}{(e^{2|\lambda|} - e^{-2|\lambda|})|\lambda|^2} \right) \]
\[ \rightarrow \sqrt{1/2 e^{-\pi x^2/2 - 4y^2}} \quad \text{as} \quad \lambda \rightarrow 0. \]

2.10.5. Let \( C \subset CHA \) be a closed subset not containing 0. The description of \( j(C) \) given in 2.8 is not very precise. Choose a real function \( \xi \in S(\mathbb{R}) \) such that \( \text{supp}(\xi) \subset [-1/4, 1/4] \). Choose \( \varphi \in S'(\mathbb{R}) \) such that \( \varphi \) is compactly supported and vanishes on \( C \) and \( \varphi(0) = 1 \). For any \( \lambda \neq 0 \) let
\[ g_{\lambda}(u, p) = \sum_{j \in \mathbb{Z}} \varphi(\lambda j) e^{-i2\pi \lambda j(u - p)} |\lambda|^j \xi(\lambda u) \xi(\lambda p) \]
\[ = \sum_{j \in \mathbb{Z}} (\lambda j \varphi(\lambda j) e^{-i2\pi \lambda j(u - p)}) \xi(\lambda u) \xi(\lambda p) \]
\[ = \hat{\varphi}(v - p, \lambda) \xi(\lambda u) \xi(\lambda p), \]
where
\[ \varphi(x, \lambda) = \sum_{j \in \mathbb{Z}} \varphi(x + j/\lambda). \]

Hence
\[ g_{\lambda}(u/\lambda + r/2, u/\lambda - r/2) = \hat{\varphi}(r, \lambda) \xi(u + \lambda r/2) \xi(u - \lambda r/2). \]

Therefore
\[ \lim_{\lambda \to 0} g_{\lambda}(u/\lambda + r/2, u/\lambda - r/2) = \hat{\varphi}(r) \xi^2(u) \]
and since \( \xi(u + \lambda r/2) \xi(u - \lambda r/2) = 0 \) for any \( u \in \mathbb{R} \) whenever \( |\lambda r| > 1/2 \) we see that the function
\[ \tilde{f}(r, u, \lambda) = \hat{\varphi}(r, \lambda) \xi(u + \lambda r/2) \xi(u - \lambda r/2) \varphi(\lambda), \quad r, u, \lambda \in \mathbb{R}, \]
is a Schwartz function. Hence there exists \( f \in S(H_1) \) such that \( f_{\lambda} = g_{\lambda} \) for any \( \lambda \in \mathbb{R}^* \). We see that for any \( \lambda \neq 0 \), \( \pi_{\lambda}(f) \) has finite rank and that
\[ \text{rank}(\pi_{\lambda}(f)) \leq C/|\lambda| \]
for any \( \lambda \neq 0 \) and some constant independent of \( \lambda \).

Is \( f \) contained in \( j(C) \)?

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References

Corrigenda to: “Generalizations of theorems of Fejér and Zygmund on convergence and boundedness of conjugate series”

(Studia Math. 57 (1976), 241–249)

by

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Abstract. Proposition 4.1(i) of [1] is incorrect, i.e., the sequence of Cesàro-segments \( \sigma_n \) of a sequence \( x \) in a translation invariant BK-space is not necessarily bounded. Theorem 4.2(ii) of [1] and the proof of Proposition 4.3 of [1] are corrected. All other statements of [1], including Proposition 4.3 itself, are correct.

1. Notations and definitions. We use the notations and definitions of [1]. Two more definitions:

\[
\tilde{E}^2 := \{ x \in \Omega : \| x \|_2 := \left( \sum_{k = -\infty}^{\infty} |x_k|^2 \right)^{1/2} < \infty \},
\]

\[
\tilde{M}^d := \{ x \in \tilde{M} : x = \hat{\mu}, \mu \in M_{2n} \text{ is discrete} \}.
\]

2. The error in Proposition 4.1(i) of [1]. The error in the proof of this proposition consists in the assumption of the existence of the \( E \)-valued Riemann integral \( \int_0^{2\pi} K_n(t)x \cdot e(-t) \, dt \), where \( x \in E \) and \( K_n \) is the \( n \)th Fejér-kernel. This is pointed out in detail in [2].

A counterexample to 4.1(i) of [1] is \( E = \tilde{M}^d \). In fact, since \( \tilde{E} \cap \tilde{M}^d = \{ 0 \} \), evidently \( \tilde{M}^d \not\subset (\tilde{M}^d)_{EB} = \{ 0 \} \).

A less trivial counterexample is \( \hat{E} = \tilde{M}^d + \tilde{E}^2 = \{ x \in \Omega : x = a + b, a \in \tilde{M}^d, b \in \tilde{E}^2 \} \) with \( \| x \|_E := \| a \|_{\tilde{M}^d} + \| b \|_{\tilde{E}^2} \). Through this example Ulf Boettcher brought the incorrectness of 4.1(i) in [1] to our attention. Evidently \( \hat{E} \) is a translation invariant BK-space. If \( \sigma_n x \in \tilde{L}^2 \) for all \( n = 0, 1, 2, \ldots \) then \( \sigma_n x \in \tilde{E}^2 \) for all \( n = 0, 1, 2, \ldots \). Hence \( E \not\subset \hat{E} \). Hence \( \hat{E} \not\subset \tilde{E} \).

3. Corrected version of Theorem 4.2(ii) of [1]. If \( E \) is a translation invariant BK-space with \( E \subset E_{EB} \), then \( E_{AB} \cap \tilde{E} \subset \tilde{E}_{AB} \).

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