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Hull-minimal ideals in the Schwartz algebra of the Heisenberg group

by

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Abstract. For every closed subset C in the dual space $\widehat{H}_n$ of the Heisenberg group H_n we describe via the Fourier transform the elements of the hull-minimal ideal $j(C)$ of the Schwartz algebra $\mathcal{S}(H_n)$ and we show that in general for two closed subsets C_1, C_2 of $\widehat{H}_n$, the product of $j(C_1)$ and $j(C_2)$ is different from $j(C_1 \cap C_2)$.

0. Introduction. Let A be an algebra. We are interested in the structure of some special ideals of A. In this paper an *ideal* of A is always a two-sided ideal. Denote by $\mathrm{Prim}(A)$ the *primitive ideal space* of A, i.e. the space of all the ideals J of A of the form $J = \ker(T)$ where (T, V) denotes an algebraically irreducible (or simple) representation T of A on a vector space V. We provide $\mathrm{Prim}(A)$ with the Jacobson topology. In this topology a subset C of $\mathrm{Prim}(A)$ is closed if it is the hull $h(I)$ of some ideal I of A, i.e. if

$$C = h(I) = \{J \in \mathrm{Prim}(A) : J \supset I\}.$$

For a subset $C \subset \mathrm{Prim}(A)$ let

$$\ker(C) = \bigcap_{J \in C} J \subset A \quad \text{and} \quad I(C) = \bigcap_{h(I)=C} I.$$

The hull of $I(C)$ contains of course C.

For certain algebras A, we have $h(I(C)) = C$, i.e. there exists a minimal ideal $j(C)$ with hull C. That means that there exists an ideal $j(C)$ of A such that the hull of $j(C)$ is equal to C and $j(C) \subset I$ for every ideal I of A whose hull is contained in C. It has been shown in [LRS] and in [Lu1] that $j(C)$ exists for every closed subset C in the primitive ideal space of the Schwartz algebra of a nilpotent Lie group.

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In Section 1 we repeat the arguments used in these two papers and we show the existence of $j(C)$ for any semisimple symmetric polynomially bounded Fréchet algebra A (see Proposition 1.9). Given now two closed subsets C_1, C_2 in $\mathrm{Prim}(A)$, what can be said about $j(C_1)\cdot j(C_2)$ or $j(C_1)\cap j(C_2)$? Under what conditions do we have $j(C_1)\cdot j(C_2) = j(C_1\cup C_2)$? In easy cases, for instance if A is abelian or if C_1 and C_2 are separated, the equality does hold (see 1.12, 1.13 below).

In Section 2, we describe the ideals $j(C)$ for the Heisenberg algebra $\mathcal{S}(H_n)$, where H_n denotes the $(2n+1)$-dimensional Heisenberg group. Although this description is not very precise, it suffices to show that in many cases $j(C_1)\cdot j(C_2) \neq j(C_1 \cup C_2)$ (see 2.9 below).

The paper finishes with open questions on the nature of $j(C)$ in the Heisenberg case; for instance, what is $j(C_1) * j(C_2)$ in the general case?

1. Hull-minimal ideals in Fréchet algebras

1.1. As an example consider a completely regular semisimple commutative Banach algebra A. By Gelfand's theory, A is isomorphic to an algebra of continuous functions vanishing at infinity on the dual space $\widehat{A}$. "Regular" means that for every closed subset C of $\widehat{A}$ and every point $\chi \in \widehat{A}\setminus C$, there exists $a \in A$ such that $\widehat{a}$ vanishes on C, but not at χ. Then, given a closed subset C of $\widehat{A}$, the ideal consisting of all the $a\in A$ such that the support of their Fourier–Gelfand transform $\widehat{a}$ is compact and disjoint from C is the minimal ideal of A with hull C (see [BD], §23 and [Rei], 1.4(iii)).

As a second example, let $\mathcal{H}$ be a Hilbert space and let A be the algebra of all compact operators on $\mathcal{H}$. The identity representation of A on $\mathcal{H}$ is up to equivalence the only algebraically irreducible representation of A and so $\mathrm{Prim}(A)$ consists of only one point. The subset j of A of all operators with finite rank is a minimal dense ideal of A. It is well known that every C^*-algebra has such a minimal dense ideal, the so-called Pedersen ideal.

Let now A be any complex algebra and let C be a closed subset of $\mathrm{Prim}(A)$. We recall a general condition for an element a of A to be contained in every ideal I with $h(I)\subset C$ (see [Lu1], 2.7).

1.2. Lemma. *Let C be a closed subset of* $\mathrm{Prim}(A)$. *Suppose that there exist $a, b\in A$ such that $b\in \ker(C)$ and $b\cdot a = a$. Then every ideal I of A with $h(I)\subset C$ contains a.*

1.3. *Hull-kernel regularity*

1.3.1. Definition. We say that a semisimple algebra A is *hull-kernel regular* (or h.k. regular) if for any closed subset C of $\mathrm{Prim}(A)$ and for every $J\in \mathrm{Prim}(A)\setminus C$ there exist $b_J, a_J \in A$ such that $b_J\in\ker(C)$, $a_J\notin J$ and $b_J\cdot a_J = a_J$.

1.3.2. Proposition. *Let A be a h.k. regular algebra. For any closed subset C of* $\mathrm{Prim}(A)$, *the minimal ideal $j(C)$ exists and is generated by the elements a_J, $J\notin C$.*

Proof. The hull of the ideal I generated by the a_J, $J\notin C$, is equal to C, since for every $J'\notin C$, $a_{J'} = b_{J'}\cdot a_{J'} \in \ker(C)\cdot a_{J'} \subset \ker(C)$ and $a_{J'}\notin J$. By Lemma 1.2, $I\subset \bigcap_{h(I')=C} I' = I(C)$. Hence $I = I(C)$ and since $h(I) = C$, we see that $j(C) = I$. ∎

1.3.3. Remark. If A is h.k. regular, then the minimal ideal $j(C)$ of a closed set $C\subset \mathrm{Prim}(A)$ can be described in the following way:

$$j(C) = \Big\{ \sum_{J\in F} x_J\cdot a_J\cdot y_J : x_J, y_J\in \widetilde{A} = \mathbb{C}1\oplus A,\quad F \text{ a finite subset of } \mathrm{Prim}(A)\setminus C\Big\}.$$

1.3.4. Example. In many algebras it is impossible to find elements a, b such that $b\cdot a = a$ and $a\neq 0$. For instance, let A be the convolution algebra $L^1(\mathbb{R}, w)$, where the weight w is the function $w(t) = e^{2\pi|t|}$, $t\in\mathbb{R}$, and where

$$L^1(\mathbb{R}, w) = \Big\{ f\in L^1(\mathbb{R}) : \|f\|_w = \int_{\mathbb{R}} w(t)|f(t)|\,dt < \infty\Big\}.$$

The primitive ideal space of this algebra is easily seen to be homeomorphic to the subset $\mathbb{R}+i[-1,1]$ of the complex numbers and A is isomorphic to the subalgebra of continuous bounded functions on $\mathbb{R}+i[-1,1]$ which are holomorphic on $\mathbb{R}+i]-1,1[$. This isomorphism is given by the Fourier transform $f\mapsto \widehat{f}$, where

$$\widehat{f}(a+ib) = \int_{\mathbb{R}} f(t)e^{-2\pi i t(a+ib)}\,dt, \qquad f\in A,\ a+ib\in \mathbb{R}+i[-1,1].$$

Hence, if $g*f = f$ in A, then $\widehat{g}\cdot\widehat{f} = \widehat{f}$, which forces f to be 0, since otherwise $\widehat{g} =$ constant 1, $\widehat{g}$ being holomorphic.

1.4. Definition. We say that an algebra A is a *Fréchet algebra* if there exists a family $\{p_k\}_{k\in\mathbb{N}}$ of norms on A such that A is complete for the topology defined by these norms and $p_k(a\cdot b)\le p_k(a)p_k(b)$ for all $k\in\mathbb{N}$, and $a, b\in A$.

We say that the Fréchet algebra A is *involutive* if it is equipped with an involution $*$.

1.5. Definition. An element a in an involutive Fréchet algebra $(A, \{p_k\})$ is called *polynomially bounded* if for every k there exists a constant $c_k = c_{a,k} > 0$ such that

$$p_k(e(i\lambda a)) \le c_k(1+|\lambda|)^{c_k}, \qquad \forall\lambda\in\mathbb{R},\ k\in\mathbb{N}.$$

Here $e(b)$, $b \in A$, means

$$e(b) = \sum_{k=1}^{\infty} \frac{b^k}{k!} \in A.$$

1.6. *The functional calculus*

1.6.1. To a polynomially bounded a we can apply the functional calculus of C^∞ functions which has been developed in [Di2]. Let $C^\infty_{c,0}(\mathbb{R})$ denote the space of all complex-valued C^∞ functions φ on $\mathbb{R}$ with compact support such that $\varphi(0) = 0$. The integral

$$\varphi(a) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\lambda) e(i\lambda a)\, d\lambda \tag{1.6.1.1}$$

converges in A for any polynomially bounded a. This functional calculus has the following property. For every character χ on a maximal abelian closed subalgebra $A(a)$ containing $a \in A$ we have

$$\chi(\varphi(a)) = \varphi(\chi(a)).$$

In particular, for $\varphi, \psi \in C^\infty_{c,0}(\mathbb{R})$,

$$\chi((\psi \cdot \varphi)a) = \chi(\psi(a)) \cdot \chi(\varphi(a)).$$

Suppose now that $A(a)$ is semisimple. Then

$$\psi(a) \cdot \varphi(a) = (\psi \cdot \varphi)(a), \qquad \varphi, \psi \in C^\infty_{c,0}(\mathbb{R}). \tag{1.6.1.2}$$

Take ψ and φ such that $\psi \cdot \varphi = \varphi$. We see that

$$\psi(a) \cdot \varphi(a) = \varphi(a). \tag{1.6.1.3}$$

1.6.2. Remark. A polynomially bounded element a in a Banach algebra $(A, \|\ \|)$ must have real spectrum. Indeed, if the spectrum of a contains a nonreal number $\mu = \alpha + i\beta$, then there exists a character χ on $A(a)$ such that $\chi(a) = \mu$ and so

$$e^{-\lambda\beta} = |e^{i\lambda\mu}| = |1 + \chi(e(i\lambda a))| \le 1 + \|e(i\lambda a)\|$$

and so $\|e(i\lambda a)\|$ grows exponentially in λ. In order to find polynomially growing elements we must look for symmetric algebras, i.e. involutive algebras for which the spectrum of every selfadjoint element is real.

1.7. Definition. We say that a Fréchet algebra A is *symmetric* if A has a continuous involution and if there exists a continuous * homomorphism σ from A into a C^*-algebra $\mathcal{C}$ such that for any $a \in A$, $\mathrm{spec}_A(a) = \mathrm{spec}_{\mathcal{C}}(\sigma(a))$. Here $\mathrm{spec}_B(x)$ denotes the spectrum of an element x in an algebra B.

Let A be a Fréchet algebra. We denote by $\widehat{A}$ the space of all topologically irreducible unitary representations $(\pi, \mathcal{H})$ of A on a Hilbert space $\mathcal{H}$.

1.8. Proposition. *Let A be a symmetric Fréchet algebra. For every algebraically irreducible representation (T, V) of A, there exists $(\pi, \mathcal{H}) \in \widehat{A}$ such that (T, V) is equivalent to a submodule of $(\pi, \mathcal{H})$.*

Proof. Since A is symmetric, so is $\widetilde{A} = \mathbb{C}1 \oplus A$ and we may assume that A and $\mathcal{C}$ have identities.

For any $x \in \ker(\sigma)$ and $y \in A$, the spectrum of yx in A is reduced to (0). Hence if for some $x \in \ker(\sigma)$, $T(x) \neq 0$, then there exists $v \in V$ such that $T(x)v \neq 0$ and since T is simple, we can find an element $y \in A$ such that $T(y)T(x)v = v$, i.e. 1 is in the spectrum of yx. This contradiction tells us that $\ker(\sigma) \subset \ker(T)$. The simple module (T, V) is equivalent to the left regular representation of A on A/M, where M denotes a proper maximal left ideal of A. The sum of $\mathbb{C}1$ and $\sigma(M)$ is direct in $\mathcal{C}$, since otherwise $1 \in M \bmod \ker(\sigma)$, which implies that $1 \in M$, since $\ker(\sigma) \subset \ker(T) \subset M$. Hence we can define a linear functional φ on $\widetilde{M} = \sigma(\mathbb{C}1 + M) = \mathbb{C}1 + \sigma(M) \subset \mathcal{C}$ by setting

$$\varphi(\lambda 1 + \sigma(m)) = \lambda, \qquad \lambda \in \mathbb{C},\ m \in M.$$

For $x = \lambda 1 + m \in M$, $x - \lambda 1 \in M$ and so $x - \lambda 1$ is not invertible in A. Hence $\lambda \in \mathrm{spec}_A(x) = \mathrm{spec}_{\mathcal{C}}(\sigma(x))$ and so

$$|\varphi(\sigma(x))| = |\lambda| \le \sup\{|\mu| : \mu \in \mathrm{spec}_{\mathcal{C}}(\sigma(x))\} \le \|\sigma(x)\|_{\mathcal{C}}.$$

Hence by Hahn–Banach, there exists a continuous extension $\widetilde{\varphi}$ of φ to $\mathcal{C}$ of norm ≤ 1. Since $\varphi(1) = 1$ and $\|\widetilde{\varphi}\|_{\mathrm{op}} \le 1$, $\widetilde{\varphi}$ is a positive functional (which annihilates $\widetilde{M}$) and so, since M is maximal and $\varphi(\sigma(M)) = (0)$, we have $M = \{y \in A : \widetilde{\varphi}(\sigma(y^* y)) = 0\}$. In particular, M is closed. Therefore, we can define a Hilbert-space structure on A/M by setting

$$\langle x + M, y + M \rangle = \widetilde{\varphi}(\sigma(y^* x)).$$

The left regular representation of A on A/M extends to a unitary representation π of A on the completion $\mathcal{H}$ of A/M (see [Di1], 2.4.4). Since we may always assume that $\widetilde{\varphi}$ is a pure state, we even know that π is irreducible (see [Di1], 2.5). ■

1.9. Definition. We say that an involutive Fréchet algebra A is *polynomially bounded* if the set A_0 of selfadjoint polynomially bounded elements of A is dense in the real subspace A_{h} of hermitian elements of A.

1.10. Proposition. *Let A be a semisimple symmetric polynomially bounded Fréchet algebra. Then A is h.k. regular. In particular, for every closed subset C in* Prim(A), *the minimal ideal $j(C)$ exists and is generated by the elements a_J, $J \notin C$.*

Proof. Since A is symmetric, for any $J \in \mathrm{Prim}(A)$ we may choose a topologically irreducible unitary representation $(\pi_J, \mathcal{H}_J)$ such that $\ker(\pi_J)$

$= J$. We fix $J \in \mathrm{Prim}(A) \setminus C$. There exists $u \in A$ such that $u \in \ker(C) = \bigcap_{J' \in C} J'$ and $u \notin J$, since C is closed. Hence $\pi_{J'}(u^*u) = 0$ for all $J' \in C$ and $\pi_J(u^*u) \neq 0$. We may suppose, after having multiplied $v = u^*u$ with a positive scalar, that $\|\pi_J(v)\|_{\mathrm{op}} = 1$. Since σ is continuous, there exists a submultiplicative norm p on A such that

$$\|\sigma(a)\|_C \le p(a), \quad a \in A.$$

Then for any unitary representation π of A, we have

$$\|\pi(a)\|_{\mathrm{op}} \le \|\sigma(a)\|_C \le p(a), \quad a \in A.$$

We choose a_0 in A_0 such that $p(a_0 - v) \le 1/10$ and real C^∞ functions φ, ψ such that ψ vanishes in a neighbourhood of $[-2/10, 2/10]$, $\varphi = 1$ on $[9/10, 11/10]$ and $\psi \cdot \varphi = \varphi$. Hence, for $b_J = \psi(a_0)$ and $a_J = \varphi(a_0)$, we see by (1.6.1.3) that $b_J \cdot a_J = a_J$, since A is semisimple. Furthermore, for $J' \in C$,

$$\pi_{J'}(b_J) = \psi(\pi_{J'}(a_0)) = 0,$$

since

$$\|\pi_{J'}(a_0)\|_{\mathrm{op}} = \|\pi_{J'}(a_0 - v)\|_{\mathrm{op}} \le p(a_0 - v) \le 1/10,$$

and

$$\pi_J(a_J) = \varphi(\pi_J(a_0)) \neq 0,$$

because

$$\begin{aligned} |\|\pi_J(a_0)\|_{\mathrm{op}} - 1| &= |\|\pi_J(a_0)\|_{\mathrm{op}} - \|\pi_J(v)\|_{\mathrm{op}}| \le \|\pi_J(a_0 - v)\|_{\mathrm{op}} \\ &\le p(a_0 - v) \le 1/10. \ \blacksquare \end{aligned}$$

1.11. EXAMPLES. 1) If A is a C^*-algebra, then for any selfadjoint element $a \in \widetilde{A} = \mathbb{C}1 \oplus A$, $u(\lambda) = \exp(i\lambda a) = \sum_{k=0}^\infty (i\lambda a)^k/k!$ is unitary and so $\|u(\lambda)\|_{C^*} = 1$ for any $\lambda \in \mathbb{R}$ (see [Di1], 1.3.9). Hence A is a symmetric polynomially bounded Banach algebra.

2) Let now G be a nilpotent locally compact group or more generally a locally compact group of polynomial growth. That means that for any compact neighbourhood U of the identity element e of G, the Haar measure $|U^k|$ of the powers $U^k = \{u_1 \ldots u_k : u_i \in U, i = 1, \ldots, k\}$, $k \in \mathbb{N}$, grows at most polynomially in k. In [Di2] it is shown that for every $f = f^*$ in $L^1(G) \cap L^2(G)$ with compact support or more generally of exponential decrease, we have

$$\|e(i\lambda f)\|_1 \le c(1 + |\lambda|)^c$$

for some positive constant c depending on f. Furthermore, for nilpotent or connected groups of polynomial growth, we know that $L^1(G)$ is symmetric (see [Lu2]). Hence $L^1(G)$, for G nilpotent, is a symmetric polynomially bounded Banach algebra.

3) Let now $G = \exp \mathfrak{g}$ denote a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. For such a group, the exponential mapping exp is a diffeomorphism, which allows us to identify the group G with the vector space $\mathfrak{g}$ as a manifold, and if we equip $\mathfrak{g}$ with the Baker–Campbell–Hausdorff product

$$X \cdot Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \ldots$$

for all $X, Y \in \mathfrak{g}$, then $\exp : (\mathfrak{g}, \mathrm{B.C.H.}) \to (G, \cdot)$ is even a group isomorphism. The Haar measure of G is just Lebesgue measure dX on the vector space $\mathfrak{g}$.

The Schwartz algebra $A = \mathcal{S}(G)$ of G is by definition the space of rapidly decreasing C^∞ functions on G and is in fact a Fréchet algebra under convolution. It has been shown in [Lu2] that A is symmetric and in [Hu] that A is polynomially bounded. For more details see [LM].

The spaces $\mathrm{Prim}(\mathcal{S}(G))$, $\mathrm{Prim}(L^1(G))$, $\mathrm{Prim}(C^*(G))$ and $\widehat{G}$ are homeomorphic (see [Lu2]) and we shall identify them. Furthermore, for any closed subset C of $\widehat{G}$ the minimal ideal $j(C)_{\mathcal{S}(G)}$ in $\mathcal{S}(G)$ associated with C is of course contained in $j(C)_{L^1(G)} \subset L^1(G)$ and in $j(C)_{C^*(G)} \subset C^*(G)$. Hence

$$j(C)_{L^1(G)} = j(C)_{\mathcal{S}(G)} + L^1(G) * j(C)_{\mathcal{S}(G)} * L^1(G)$$

and

$$j(C)_{C^*(G)} = j(C)_{\mathcal{S}(G)} + C^*(G) * j(C)_{\mathcal{S}(G)} * C^*(G).$$

4) It follows from 1.6.2 that in the algebra $A = L^1(\mathbb{R}, w)$ no element f is polynomially bounded. Indeed, since $\widehat{f}$ is a complex analytic function there always exists $a + ib$ such that $\mu = \widehat{f}(a + ib) \notin \mathbb{R}$. Also, this algebra does not admit minimal ideals. Let C be a nonempty closed subset of $\mathrm{Prim}(A)$ such that $\ker(C) \neq (0)$. Then for any $n \in \mathbb{N}$, $\ker(C)^n$ is an ideal of A with hull C. But

$$I_\infty = \bigcap_{n \in \mathbb{N}} \ker(C)^n$$

is (0), since the Fourier transform of any element of I_∞ vanishes to infinite order on C.

1.12. Let A be a h.k. regular semisimple Fréchet algebra and let C_1, C_2 be two closed subsets in $\mathrm{Prim}(A)$. We may ask what happens to the product of $j(C_1)$ with $j(C_2)$. It is clear that

$$(1.12.1) \qquad j(C_1) \cdot j(C_1) = j(C_1).$$

Obviously we always have

$$(1.12.2) \qquad h(j(C_1) \cdot j(C_2)) = C_1 \cup C_2,$$

since any $J \in \mathrm{Prim}(A)$ is a prime ideal. Hence

$$j(C_1)\cdot j(C_2) \supset j(C_1\cup C_2) \quad \text{and} \quad \ker(C_1)\cdot\ker(C_2)\subset \ker(C_1\cup C_2).$$

1.13. PROPOSITION. *Let A be an abelian h.k. regular algebra. Then for any closed subsets C_1, C_2 of* $\mathrm{Prim}(A)$, *we have*

$$j(C_1)\cdot j(C_2) = j(C_1\cup C_2).$$

Proof. Since for any $J_i \in \mathrm{Prim}(A)\setminus(C_i)$, we can find b_{J_i}, a_{J_i} such that $b_{J_i}\in\ker C_i$, $a_{J_i}\notin\ker(C_i)$ and $b_{J_i}\cdot a_{J_i} = a_{J_i}$, $i=1,2$, we see that for $b = b_{J_1}\cdot b_{J_2}$ and $a = a_{J_1}\cdot a_{J_2}$, we have $b\in\ker(C_1\cup C_2)$ and since A is abelian,

$$b\cdot a = b_{J_1}\cdot b_{J_2}\cdot a_{J_1}\cdot a_{J_2} = b_{J_1}a_{J_1}b_{J_2}a_{J_2} = a_{J_1}a_{J_2} = a.$$

Hence $a\in j(C_1\cup C_2)$ and so $j(C_1)\cdot j(C_2)\subset j(C_1\cup C_2)$, whence the assertion follows. ■

We shall see in 2.9 that the situation is much more complicated if A is no longer abelian. However, in the case where C_1, C_2 are separated in $\mathrm{Prim}(A)$, i.e. if there exist two open subsets U_1, U_2 in $\mathrm{Prim}(A)$ such that $C_i\subset U_i$, $i=1,2$, and $U_1\cap U_2=\emptyset$, we can control $j(C_1)\cdot j(C_2)$.

1.14. PROPOSITION. *Let A be a h.k. regular algebra. Then for any closed separated subsets C_1, C_2 of* $\mathrm{Prim}(A)$, *we have*

$$j(C_1)\cdot j(C_2) = j(C_1\cup C_2).$$

Proof. Let $K_i = \mathrm{Prim}(A)\setminus U_i$, $i=1,2$. Then $C_1\subset K_2$, $C_2\subset K_1$ and $\mathrm{Prim}(A)$ is the union of the two closed subsets K_1, K_2. For any $J\notin C_1\cup C_2$ choose $b_J\in\ker(C_1\cup C_2)$ and $a_J\notin J$ such that $b_J\cdot a_J = a_J$ (i.e. $a_J\in j(C_1\cup C_2)$); for any $J\in C_1$ choose $b_J\in\ker(K_1)$ and $a_J\notin J$ with $b_J\cdot a_J = a_J$ (i.e. $a_J\in j(K_1)\subset j(C_2)$); and for $J\in C_2$ choose $b_J\in\ker(K_2)$ and $a_J\notin J$ with $b_J\cdot a_J = a_J$ (i.e. $a_J\in j(K_2)\subset j(C_1)$). Then, for any $a\in A$, $J_1\notin C_1$, $J_2\notin C_2$,

$$a_{J_1}\cdot a\cdot a_{J_2}\in j(C_1\cup C_2) \quad \text{if } J_1 \text{ or } J_2\notin C_1\cup C_2,$$

by the choice of a_{J_1} or a_{J_2}. If $J_1, J_2\in C_1\cup C_2$ then $a_{J_1}\cdot a\cdot a_{J_2}\in\ker(K_1\cup K_2) = \ker(\mathrm{Prim}(A)) = (0)$. Since $j(C_i)$, $i=1,2$, is generated by elements of the form

$$b\cdot a_J\cdot a, \qquad a,b\in A,\ J\notin C_i,$$

we see again that $j(C_1)\cdot j(C_2)\subset j(C_1\cup C_2)$. ■

1.15. PROPOSITION. *Let A be a h.k. regular algebra. Then for any closed subsets $C_1, C_2\subset\mathrm{Prim}(A)$, we have*

$$j(C_1\cap C_2) = j(C_1)+j(C_2).$$

If $C_1\subset C_2$, then $j(C_2)\subset j(C_1)$.

Proof. The hull of $j(C_1)+j(C_2)$ is obviously equal to $C_1\cap C_2$ and so $j(C_1\cap C_2)\subset j(C_1)+j(C_2)$. Hence it suffices to show the opposite inclusion. But for closed subsets $C\subset B$ of $\mathrm{Prim}(A)$, and for $J\notin B$, there exist $b_J, a_J\in j(B)$ such that $b_J\in\ker(B)\subset\ker(C)$, $a_j\notin J$ and $b_Ja_J = a_J$. Hence also $a_J\in j(C)$ by 1.3.2, and since the a_J, $J\notin B$, generate $j(B)$ we see that

$$j(B)\subset j(C) \tag{1.15.1}$$

and so $j(C_1)\subset j(C_1\cap C_2)$ and $j(C_2)\subset j(C_1\cap C_2)$. ■

2. The minimal ideals in the Schwartz algebra of the Heisenberg group

2.1. We shall determine the minimal ideals $j(C)$ in the Schwartz algebra of the Heisenberg group H_n by describing the Fourier transforms of the elements of $j(C)$. This section is based on [Fo] and uses its notations.

As a manifold, H_n is the space $\mathbb{R}^{2n+1}$. We write (p,q,t) for the elements of H_n, where $p,q\in\mathbb{R}^n$, $t\in\mathbb{R}$. The group law on H_n is defined by

$$(p,q,t)\cdot(p',q',t') = \big(p+p', q+q', t+t'+\tfrac{1}{2}(p\cdot q' - p'\cdot q)\big).$$

Here $p\cdot q$ means the ordinary euclidean product on $\mathbb{R}^n$, i.e.

$$p\cdot q = p_1q_1+\ldots+p_nq_n.$$

The center $\mathcal{Z}$ of H_n is given by the last coordinate, i.e.

$$\mathcal{Z} = \{0\}\times\{0\}\times\mathbb{R},$$

and we observe that $\mathcal{Z}$ is also the first commutator $[H_n, H_n]$ of H_n.

The Lie algebra $\mathfrak{h}_n$ of H_n can also be identified with $\mathbb{R}^{2n+1}$ and the exponential mapping exp is then the identity mapping. For $j\in\{1,\ldots,n\}$ we define the vectors

$$X_j = (\delta_{i,j})_{i=1,\ldots,2n+1}, \quad Y_j = (\delta_{i,j+n})_{i=1,\ldots,2n+1}, \quad Z = (\delta_{i,2n+1})_{i=1,\ldots,2n+1}.$$

We obtain the classical commutator relations

$$[X_i, Y_j] = \delta_{i,j}Z, \qquad 1\le i,j\le n,$$

and Z spans the center $\mathfrak{z}$ of $\mathfrak{h}_n$. We also identify the dual space $\mathfrak{h}_n^*$ of $\mathfrak{h}_n$ with $\mathbb{R}^{2n+1}$. An element $(a,b,\lambda)\in\mathfrak{h}_n^*$ acts on (p,q,t) by

$$\langle(a,b,\lambda),(p,q,t)\rangle = a\cdot p + b\cdot q + \lambda t.$$

2.2. The dual space $\widehat{H}_n$ is the union of the set $\mathcal{CHA}$ of one-dimensional representations and the set $\widehat{H}_n^\infty$ of infinite-dimensional ones. The characters $\chi\in\mathcal{CHA}$ are defined through the elements $\phi = (a,b)\in\mathbb{R}^{2n}\simeq\mathfrak{z}^\perp\subset\mathfrak{h}_n^*$,

$$\chi_\phi(p,q,t) = e^{-2i\pi(a\cdot p+b\cdot q)}, \qquad (p,q,t)\in H_n.$$

The infinite-dimensional representations can be parametrized by $\mathbb{R}^* = \mathbb{R}\setminus 0$. For $\lambda \in \mathbb{R}^*$, we take the linear form $l_\lambda \in \mathfrak{h}_n^*$ for which

$$l_\lambda(X_j) = l_\lambda(Y_j) = 0, \quad j = 1,\dots,n, \quad l_\lambda(Z) = \lambda.$$

The subalgebra $\mathfrak{b} = \operatorname{span}\{Y_j, Z : j = 1,\dots,n\} = \{(0,q,t) : q \in \mathbb{R}^n,\ t \in \mathbb{R}\}$ is a polarization at l_λ for any λ and so if χ_λ denotes the character

$$\chi_\lambda(0,q,t) = e^{-2i\pi\lambda t}, \qquad (0,q,t) \in \mathfrak{b} = \exp(\mathfrak{b}) = B,$$

then by the Stone–von Neumann theorem,

$$\pi_\lambda = \operatorname{ind}_B^{H_n}\chi_\lambda$$

is irreducible and every $\pi \in \widehat{H}_n^\infty$ is of this form. We can identify the Hilbert space $\mathcal{H}_\lambda$ of π_λ with $L^2(\mathbb{R}^n)$ and we obtain the following relations:

$$\pi_\lambda(p,q,t)\xi(v) = e^{-i2\pi\lambda t + i\pi(v\cdot q + q\cdot(v-p))}\xi(v-p),$$
$$\xi \in L^2(\mathbb{R}^n),\ v \in \mathbb{R}^n,\ (p,q,t) \in H_n.$$

Hence $\widehat{H}_n$ can be identified with $\mathbb{R}^{2n} \cup \mathbb{R}^*$. By Kirillov's theory, $\widehat{H}_n$ is homeomorphic to the space of the coadjoint orbits $\mathfrak{h}_n/H_n$. The character $\chi_{(a,b)}$ corresponds to the linear functional $(a,b,0) \in \mathbb{R}^{2n+1} \simeq \mathfrak{h}_n^*$, and the representation π_λ, $\lambda \in \mathbb{R}^*$, to the functional $l_\lambda = (0,0,\lambda) \in \mathbb{R}^{2n+1}$. The coadjoint orbit Ω_λ of l_λ is the affine subspace $\mathbb{R}^{2n} \times \{\lambda\}$. Hence in the orbit space,

$$\lim_{\lambda\to 0}\Omega_\lambda = \mathbb{R}^{2n} \times \{0\} = \bigcup_{a,b\in\mathbb{R}^n}\{(a,b)\},$$

and so in $\widehat{H}_n$,

$$\lim_{\lambda\to 0}\pi_\lambda = \bigcup_{a,b\in\mathbb{R}^n}\chi_{(a,b)} = \mathcal{CHA}. \tag{2.2.1}$$

Hence $\widehat{H}_n$ is not a Hausdorff space.

We now consider the Schwartz algebra $\mathcal{S}(H_n)$ of H_n. The elements of $\mathcal{S}(H_n)$ are just ordinary Schwartz functions on $\mathbb{R}^{2n+1}$ and $\mathcal{S}(H_n)$ is an algebra for the convolution $*$ and the involution *:

$$f * g(x) = \int_{H_n} f(y)g(y^{-1}x)\,dy, \qquad f^*(x) = \overline{f}(x^{-1}).$$

For every $\pi \in \widehat{H}_n$ and $f \in \mathcal{S}(H_n)$, we can define the operator $\pi(f)$ on $\mathcal{H}_\pi$ by

$$\pi(f) = \int_{H_n} f(x)\pi(x)\,dx.$$

Hence for $(a,b) \in \mathcal{CHA}$,

$$\chi_{(a,b)}(f) = \widehat{f}(a,b,0) = \int_{H_n} f(p,q,t)e^{-i2\pi(a\cdot p + b\cdot q)}\,dp\,dq\,dt.$$

For $\lambda \in \mathbb{R}^*$, $\pi_\lambda(f)$ is a kernel operator with Schwartz kernel f_λ, i.e. for $\xi \in L^2(\mathbb{R}^n)$,

$$\pi_\lambda(f)\xi(v) = \int_{\mathbb{R}^n} f_\lambda(v,p)\xi(p)\,dp,$$

where

$$f_\lambda(v,p) = f^{2,3}(v-p, \tfrac{1}{2}\lambda(p+v), \lambda), \qquad v,p \in \mathbb{R}^n,\ \lambda \in \mathbb{R}, \tag{2.2.2}$$

and

$$f^{2,3}(p,r,\lambda) = \int_{\mathbb{R}^n\times\mathbb{R}} f(p,q,t)e^{2i\pi(q\cdot r - t\lambda)}\,dq\,dt, \qquad p,r \in \mathbb{R}^n,\ \lambda \in \mathbb{R}.$$

We can thus define the group Fourier transform $\widehat{f}$ of $f \in \mathcal{S}(H_n)$ on $\widehat{H}_n^\infty$ by letting

$$\widehat{f}(\lambda) = f_\lambda, \qquad \lambda \in \mathbb{R}.$$

In particular,

$$\widehat{f}(0)(p,v) = f_0(p,v) = f^{2,3}(v-p,0,0) = \lim_{\lambda\to 0} f_\lambda(v,p).$$

We see that for $\lambda \neq 0$, $\widehat{f}(\lambda) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and $v \mapsto f_0(v,0)$ is a Schwartz function on $\mathbb{R}^n$.

For $f, g \in \mathcal{S}(H_n)$ we have

$$(f*g)_\lambda = f_\lambda \circ g_\lambda$$

where for two kernels F, G,

$$F \circ G(v,p) = \int_{\mathbb{R}^n} F(v,u)G(u,p)\,du.$$

It is well known (and easy to verify) that the mapping $f \mapsto \widehat{f}$ is injective and it is easy to see that a mapping $F : \mathbb{R} \to C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^*)$ is in the range of the Fourier transform if and only if the function

$$\widetilde{F}(r,u,\lambda) = F\left(\frac{u}{\lambda} + \frac{r}{2}, \frac{u}{\lambda} - \frac{r}{2}, \lambda\right), \qquad (r,u,\lambda) \in \mathbb{R}^{2n+1},\ \lambda \neq 0, \tag{2.2.3}$$

is the restriction to $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^*$ of a Schwartz function, since then the function $f(p,q,t) = \int_{\mathbb{R}^{n+1}} \widetilde{F}(p,u,\lambda)e^{-i2\pi(u\cdot q - t\lambda)}\,d\lambda\,du$ is in $\mathcal{S}(H_n)$ and $f_\lambda(v,p) = F(v,p,\lambda)$ for all $(v,p,\lambda) \in \mathbb{R}^{2n+1}$.

2.3. Definition. We denote by $\widehat{\mathcal{S}}(H_n)$ the range of the Fourier transform of $\mathcal{S}(H_n)$. Hence

$$\widehat{\mathcal{S}}(H_n) = \{F \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^*) : \widetilde{F} \in \mathcal{S}(\mathbb{R}^{2n+1})\}.$$

2.4. *Closed sets in $\widehat{H}_n$ containing the characters*

2.4.1. We now describe the minimal ideals of $\mathcal{S}(H_n)$ associated with the closed subsets C containing the characters of H_n. This is a situation which can also be understood in the general nilpotent Lie group case (for details see [CG]). Let therefore G denote any nilpotent simply connected, connected Lie group with Lie algebra $\mathfrak{g}$. We know that $\widehat{G}$ is homeomorphic to the space $\mathfrak{g}^*/G$ of coadjoint orbits of G and that $\widehat{G} \simeq \mathrm{Prim}(L^1(G)) \simeq \mathrm{Prim}(\mathcal{S}(G))$.

We can describe $\mathfrak{g}^*/G$ and $\widehat{G}$ explicitly in the following way. Let $\mathcal{B} = \{X_n, \ldots, X_1\}$ be a Jordan–Hölder basis of $\mathfrak{g}$ and $\mathcal{B}^* = \{l_1, \ldots, l_n\}$ be its dual basis in $\mathfrak{g}^*$. For every $l \in \mathfrak{g}^*$ there exists an index set $I(l) \subset \{1, \ldots, n\}$ such that if $\mathcal{V}(l)$ denotes the span of $\{l_i : i \notin I(l)\} \subset \mathfrak{g}^*$, then the coadjoint orbit Ω_l of l meets $\mathcal{V}(l)$ in a single point. Let us take the Vergne polarization $\mathfrak{p}(l) = \mathfrak{p}^{\mathcal{B}}(l)$ at l associated with $\mathcal{B}$ and let $\pi_l = \mathrm{ind}_{P(l)}^G \chi_l$ be the representation induced from the character χ_l of the subgroup $P(l) = \exp(\mathfrak{p}(l))$ of G. Then π_l is irreducible and every irreducible representation π of G is equivalent to some π_l. For two elements $l, p \in \mathfrak{g}^*$, the representations π_l and π_p are equivalent if and only if $\Omega_l = \Omega_p$.

It has been shown in [LZ] that there exists an index set $I \subset \{1, \ldots, n\}$ and a Zariski open G-invariant subset denoted by $\mathfrak{g}^*_{\mathrm{gen}}$ in $\mathfrak{g}^*$, the set of elements in general position, such that $I(l) = I$ for any $l \in \mathfrak{g}^*_{\mathrm{gen}}$. Furthermore, for any $l \in \mathfrak{g}^*_{\mathrm{gen}}$, there exists a Mal'tsev basis $\mathfrak{X}(l) = \{X_1(l), \ldots, X_p(l)\}$ of $\mathfrak{g}$ relative to $\mathfrak{p}(l)$ (i.e.

$$\mathfrak{g} = \mathbb{R}X_1(l) \oplus \ldots \oplus \mathbb{R}X_p(l) \oplus \mathfrak{p}(l)$$

and $\sum_{i=p}^{j} \mathbb{R}X_i(l) + \mathfrak{p}(l)$ is a subalgebra for any j), a Mal'tsev basis $\mathcal{Y}(l) = \{Y_1(l), \ldots, Y_p(l)\}$ of $\mathfrak{p}(l)$ relative to the stabilizer $\mathfrak{g}(l)$ of l in $\mathfrak{g}$ and a Mal'tsev basis $\mathcal{Z}(l) = \{Z_1(l), \ldots, Z_m(l)\}$ of $\mathfrak{g}(l)$ such that $l \mapsto X_j(l)$, $l \mapsto Y_j(l)$ and $l \mapsto Z_j(l)$ are polynomial mappings. In particular, if $\mathfrak{U}$ denotes the Zariski open subset $\mathfrak{g}^*_{\mathrm{gen}} \cap \mathcal{V}$ of $\mathcal{V}$, where $\mathcal{V} = \mathrm{span}(l_j : j \notin I)$, then the mapping

$$\mathfrak{U} \times \mathbb{R}^p \times \mathbb{R}^p \to \mathfrak{g}^*_{\mathrm{gen}},$$
$$(l, T, S) \mapsto \Big(\prod_{i=1}^{p} \mathrm{Ad}^*(\exp(t_i X_i(l))) \prod_{i=1}^{p} \mathrm{Ad}^*(\exp(s_i Y_i(l)))\Big)(l),$$

is a diffeomorphism.

If $G = H_n$, then $\mathfrak{g}^*_{\mathrm{gen}}$ corresponds to $\widehat{H}_n^\infty$, $p = n$ and for the Jordan–Hölder basis $\mathcal{B} = \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ we have $\mathcal{V} = \mathbb{R}Z^*$ and $X_j(l) = X_j$, $Y_j(l) = Y_j$ for all $l \in (\mathfrak{h}_n)^*_{\mathrm{gen}}$.

Coming back to our general G, we can use for $l \in \mathfrak{U}$ the Mal'tsev basis $\{\mathcal{Y}(l), \mathcal{Z}(l)\}$ of $\mathfrak{p}(l)$ to write down the kernel f_l of the operator $\pi_l(f)$, $f \in \mathcal{S}(G)$. Define first for $l \in \mathfrak{U}$ the polynomial diffeomorphisms

$$X_l : \mathbb{R}^p \to G/P(l), \qquad T \mapsto \prod_{j=1}^{p} \exp(t_j X_j(p)),$$

and

$$E_l : \mathbb{R}^p \times \mathbb{R}^m \to P(l), \qquad (S, U) \mapsto \prod_{j=1}^{p} \exp(s_j Y_j(l)) \prod_{j=1}^{m} \exp(u_j Z_j(l)).$$

The operator $\pi_l(f) = \int_G f(x)\pi_l(x)\,dx$ is a trace class operator whose kernel $f_l \in C^\infty(G \times G)$ is given by the expression

$$f_l(x, y) = \int_{P(l)} f(xpy^{-1}) e^{-i\langle l, \log(p)\rangle}\,dp$$
$$= \int_{\mathbb{R}^p \times \mathbb{R}^m} f(xE_l(S, U)y^{-1}) e^{-i\langle l, \log(E_l(S,U))\rangle}\,dS\,dU, \qquad l \in \mathfrak{g}^*_{\mathrm{gen}},\ x, y \in G.$$

Let $C^\infty(\mathfrak{U}, \mathcal{S}_{2p})$ and $C^\infty(\mathfrak{U}, \mathcal{S}_p)$ be the spaces of all smooth mappings from $\mathfrak{U}$ into $\mathcal{S}(\mathbb{R}^p \times \mathbb{R}^p)$ and $\mathcal{S}(\mathbb{R}^p)$, respectively.

Let $\mathcal{S}(G \times G, l)$ (resp. $\mathcal{S}(G, l)$) be the space of all smooth functions $F : G \times G \to \mathbb{C}$ (resp. $\xi : G \to \mathbb{C}$) such that

$$F(xq, yq') = \overline{\chi_l(q)}\chi_l(q')F(x, y) \qquad (\text{resp. } \xi(xq) = \overline{\chi_l(q)}\xi(x))$$

for all $x, y \in G$, $q, q' \in P(l)$, and the function $(T, T') \mapsto \widetilde{F}(T, T') = F(X_l(T), X_l(T'))$ is in $\mathcal{S}(\mathbb{R}^p \times \mathbb{R}^p)$ (resp. $T \mapsto \xi(X_l(T))$ is in $\mathcal{S}(\mathbb{R}^p)$).

Then for $f \in \mathcal{S}(G)$ and for a fixed $l \in \mathfrak{g}^*_{\mathrm{gen}}$, the function f_l is in $\mathcal{S}(G \times G, l)$ and the mapping $\mathfrak{U} \ni l \mapsto \widetilde{f}_l$ from $\mathfrak{U}$ into $\mathcal{S}(\mathbb{R}^p \times \mathbb{R}^p)$ where

$$\widetilde{f}_l(T, T') = f_l(X_l(T), X_l(T')), \qquad T, T' \in \mathbb{R}^p,$$

is in $C^\infty(\mathfrak{U}, \mathcal{S}_{2p})$ (see [LZ]).

We obtain in this fashion a Fourier transform $f \mapsto \widehat{f}$ on $\mathcal{S}(G)$ by setting

$$\widehat{f}(l) = f_l, \qquad l \in \mathfrak{U}. \tag{2.4.1.1}$$

2.4.2. It has been shown in [LZ] that every mapping

$$F : \mathfrak{U} \ni l \mapsto F(l) \in \mathcal{S}(G \times G, l)$$

with compact support such that $\widetilde{F} \in C^\infty(\mathfrak{U}, \mathcal{S}_{2p})$ is in the image of the Fourier transform. In particular, if we choose two smooth functions $\widetilde{\xi}, \widetilde{\eta} : \mathfrak{U} \to \mathcal{S}(\mathbb{R}^p)$ with compact support, then there exists a unique $f_{\xi,\eta} \in \mathcal{S}(G)$ such that $\widehat{f_{\xi,\eta}} = F_{\xi,\eta}$, where

$$F_{\xi,\eta}(l, x, y) = \xi(l, x)\overline{\eta(l, y)}, \qquad x, y \in G,\ l \in \mathcal{V},$$
$$\xi(l, X_l(T)E_l(S, U)) = \widetilde{\xi}(l, T)e^{-i\langle l, \log(E_l(S,U))\rangle},$$

and similarly for η.

2.4.3. Definition. We say that the element $f_{\xi,\eta}$ of $\mathcal{S}(G)$ is *elementary*.

For $g, h \in \mathcal{S}(G)$ we see that

$$g * f_{\xi,\eta} * h = f_{\hat{g}(\xi),\hat{h}(\eta)},$$

where

$$\widehat{g}(\xi)(l,x) = \int_{G/P(l)} g_l(x,y)\xi(l,y)\,dy,$$
$$\widehat{h}(\eta)(l,y) = \int_{G/P(l)} h_l(x,y)\eta(l,x)\,dx.$$

Therefore $g * f_{\xi,\eta} * h$ is also elementary.

Let $\widehat{G}_{\text{gen}}$ be the (open dense) subset of $\widehat{G}$ corresponding to $\mathfrak{g}^*_{\text{gen}}$. Let $\widehat{G}_{\text{sing}}$ be its complement. $\widehat{G}_{\text{gen}}$ is homeomorphic to $\mathfrak{U}$ and in particular every open subset of $\widehat{G}$ disjoint from $\widehat{G}_{\text{sing}}$ corresponds to an open subset of $\mathfrak{U}$.

2.5. THEOREM. *Let C be a closed subset of $\widehat{G}$ containing $\widehat{G}_{\text{sing}}$. Let $\mathfrak{U}_C = \widehat{G} \setminus C \subset \mathfrak{U}$. The minimal ideal $j(C)$ in $\mathcal{S}(G)$ is the vector space spanned by all the elementary $f_{\xi,\eta}$'s with support of $\widetilde{\xi}$ and of $\widetilde{\eta}$ contained in $\mathfrak{U}_C$ and compact.*

Proof. Let $\pi \in \widehat{G} \setminus C$. There exists $l \in \mathfrak{U}_C$ such that $\pi \simeq \pi_l$. Choose $\widetilde{\xi}_0 \in \mathcal{S}(\mathbb{R}^p)$ and $\varphi \in C_c^\infty(\mathfrak{U}_C)$ such that $\|\widetilde{\xi}_0\|_{L^2} = 1$, $\varphi(l) = 1$ and $\operatorname{supp}(\varphi) \subset \mathfrak{U}_C$. Choose also $\psi \in C_c^\infty(\mathfrak{U}_C)$ such that $\psi \cdot \varphi = \varphi$. Let

$$\widetilde{\sigma} = \varphi \otimes \widetilde{\xi}_0, \qquad \widetilde{\tau} = \psi \otimes \widetilde{\xi}_0 \in C^\infty(\mathfrak{U}, \mathcal{S}_p).$$

Then $\pi_q(f_{\tau,\tau}) = 0$ for every $q \notin \operatorname{supp}(\psi)$, hence $\pi'(f_{\tau,\tau}) = 0$ for every $\pi' \in C$. Furthermore,

$$\begin{aligned}(f_{\tau,\tau} * f_{\sigma,\sigma})^\wedge(q) &= |\psi|^2 \cdot |\varphi|^2(q)\langle \widetilde{\xi}_0, \widetilde{\xi}_0\rangle_{L^2} \xi_0(q) \otimes \overline{\xi_0(q)} \\ &= |\varphi|^2(q)\xi_0(q) \otimes \overline{\xi_0(q)} = \widehat{f}_{\sigma,\sigma}(q)\end{aligned}$$

for every $q \in \mathfrak{U}$. Hence

$$f_{\tau,\tau} * f_{\sigma,\sigma} = f_{\sigma,\sigma}$$

and so by 1.2, $f_{\sigma,\sigma} \in j(C)$ and since $\widehat{f}_{\sigma,\sigma}(l) = \xi_0 \otimes \overline{\xi}_0 \neq 0$, we see that the ideal I generated by the $f_{\sigma,\sigma}$ admits the set C as hull and hence by the minimality of $j(C)$, $I = j(C)$. By 2.4.3, we see that all the elements of I are finite sums of elementary ones.

Let now $\widetilde{\xi}, \widetilde{\eta} \in C^\infty(\mathfrak{U}_C, \mathcal{S}_p)$, with compact support. Take a function $\varphi \in C^\infty(\mathfrak{U}_C)$ with compact support in U such that

$$\varphi\widetilde{\xi} = \widetilde{\xi}, \qquad \varphi\widetilde{\eta} = \widetilde{\eta}.$$

By 2.4.2 there exist f, f' in $\mathcal{S}(G)$ such that

$$f_l(x,y) = \varphi(l)\xi(l,x)\overline{\xi_0(l,y)},$$
$$f'_l(x,y) = \varphi(l)\xi_0(l,x)\overline{\eta(l,y)}, \qquad l \in \mathcal{U},\ x, y \in G.$$

Hence for $f_{\sigma,\sigma}$ as above,

$$(f * f_{\sigma,\sigma} * f')^\wedge(q) = \varphi^2(q)\langle \widetilde{\xi}_0, \widetilde{\xi}_0\rangle_{L^2}\xi(v,q)\overline{\eta(p,q)} = \xi(v,q)\overline{\eta(p,q)}$$

for all $q \in \mathfrak{U}$. Hence $f_{\xi,\eta} = f * f_{\sigma,\sigma} * f' \in j(C)$ and so $j(C) = \operatorname{span}(\{f_{\xi,\eta}\})$. ∎

2.5.1. COROLLARY. *Let C_1, C_2 be closed subsets of $\widehat{G}$ containing $\widehat{G}_{\text{sing}}$. Then*

$$j(C_1) * j(C_2) = j(C_1 \cup C_2).$$

2.6. *Closed sets in the dual of H_n not containing $\mathcal{CHA}$*

2.6.1. We now come to the case where the closed subset C of $\widehat{H}_n$ does not contain all characters. That means that there exists $\delta > 0$ such that $]-\delta, \delta[\cap \mathbb{R}^*$ is not contained in C, since otherwise we can find a sequence $\{\lambda_k\} \subset \mathbb{R}^* \cap C$ which converges to 0, and so all the limit points of this sequence, i.e. the characters of H_n, belong to C. Hence

(2.6.1.1) $$\delta = \min\{|\lambda| : \lambda \in C \cap \mathbb{R}^*\} > 0$$

and $C = C_0 \cup C_\infty$ is the disjoint union of the closed set $C_0 = C \cap \mathcal{CHA} = C \cap \mathbb{R}^{2n}$ and $C_\infty = C \cap \widehat{H}_n^\infty = C \cap \mathbb{R}^*$.

In order to construct elements in $j(C)$, we first consider the 3-dimensional Heisenberg group $H_1 = \mathbb{R}^3$ whose Lie algebra $\mathfrak{h}_1$ is spanned by the vectors X, Y and Z with the nontrivial bracket $[X, Y] = Z$. We use the heat kernel $\{q_t\}_{t\in\mathbb{R}_+}$ associated with the homogeneous operator $L = X^2 + Y^2$ on H_1 (see [FS], 1.68–1.74). The functions q_t are Schwartz functions of L^1-norm 1 such that $\partial_t(q_t) = L(q_t)$ for every $t \in \mathbb{R}_+$ and formally $q_t = \exp(tL)\delta_0$. This means that for any unitary representation π of H_1, we have

$$\frac{d}{dt}\pi(q_t)\xi = d\pi(L)\pi(q_t)\xi$$

for any $t > 0$ and any C^∞-vector ξ of $\mathcal{H}_\pi$ and so

(2.6.1.2) $$\exp(t\pi(L)) = \pi(q_t), \qquad t \in \mathbb{R}_+,$$

in the sense of functional calculus. Now for $\lambda \in \mathbb{R}^*$,

$$d\pi_\lambda(L) = d\pi_\lambda(X)^2 + d\pi_\lambda(Y)^2 = \left(\frac{d}{dv}\right)^2 - 4\pi^2\lambda^2 M_v^2,$$

where M_v denotes multiplication with the function $v \mapsto v$ in $L^2(\mathbb{R})$. The

Hermite functions h_j, $j \in \mathbb{N}$,

$$h_j(v) = \frac{2^{1/4}}{\sqrt{j!}}\left(\frac{-1}{2\sqrt{\pi}}\right)^j e^{\pi v^2}\frac{d^j}{dv^j}(e^{-2\pi v^2}), \qquad v \in \mathbb{R},$$

form an orthonormal basis of $L^2(\mathbb{R})$ consisting of eigenvectors for $d\pi_1(L) = (d/dv)^2 - 4\pi^2 M_v^2$ (see [Fo]). In fact, for $\lambda = 1$,

$$d\pi_1(L)h_j = -2\pi(2j+1)h_j, \qquad j \in \mathbb{N}$$

(see [Fo]). Hence an easy calculation shows that

$$d\pi_\lambda(L)h_{j,\lambda} = -2\pi(2j+1)|\lambda|h_{j,\lambda}, \qquad j \in \mathbb{N},$$

where

$$h_{j,\lambda}(v) = |\lambda|^{1/4}h_j(v\sqrt{|\lambda|}), \qquad v \in \mathbb{R},\ j \in \mathbb{N}.$$

We can now write the kernel q_λ of the operator $\pi_\lambda(q) = \exp(d\pi(L))$ where $q = q_1$. From 2.6.1 it follows that

(2.6.1.3) $$\pi_\lambda(q_t)h_{j,\lambda} = e^{-2\pi|\lambda|(2j+1)t}h_{j,\lambda}, \qquad \forall \lambda, j, t > 0.$$

2.6.2. Going back to H_n, it suffices to consider the Schwartz function $\underline{q} = q_1^1 * \ldots * q_1^n$ on H_n, where q_1^i, $i = 1, \ldots, n$, is the smooth measure defined on H_n by

$$\langle q_1^i, f\rangle = \int_{\mathbb{R}^3} f(\exp(s_iX_i + t_iY_i + u_iZ))q_1(s_i, t_i, u_i)\,ds_i\,dt_i\,du_i, \qquad f \in \mathcal{S}(H_n).$$

For $\underline{j} = (j_1, \ldots, j_n) \in \mathbb{N}^n$, let

$$h_{\underline{j},\lambda}(v) = |\lambda|^{n/4}h_{j_1}(v_1\sqrt{|\lambda|})\ldots h_{j_n}(v_n\sqrt{|\lambda|}), \qquad v \in \mathbb{R}^n,$$

and

$$|\underline{j}| = j_1 + \ldots + j_n.$$

It follows from (2.6.1.3) that the kernel $\underline{q}_\lambda$ of $\pi_\lambda(\underline{q})$ can be written as

$$\underline{q}_\lambda(v,p) = \sum_{\underline{j}\in\mathbb{N}^n} e^{-2\pi|\lambda|(2|\underline{j}|+n)}h_{\underline{j},\lambda}(v)h_{\underline{j},\lambda}(p), \qquad v, p \in \mathbb{R}^n,$$

and

$$\pi_\lambda(\underline{q}) = \sum_{\underline{j}\in\mathbb{N}^n} e^{-2\pi|\lambda|(2|\underline{j}|+n)}q_{\underline{j},\lambda}$$

is the sum of $e^{-2\pi|\lambda|(2|\underline{j}|+n)}$-times the one-dimensional orthogonal projections $q_{\underline{j},\lambda}$ onto $\mathbb{C}h_{\underline{j},\lambda}$. Since $\underline{q}$ is selfadjoint, we may apply the functional calculus of C^∞ functions to $\underline{q}$. Therefore if $\varphi \in C^\infty(\mathbb{R})$ and $\varphi(0) = 0$, then

$$\pi_\lambda(\varphi(\underline{q})) = \varphi(\pi_\lambda(\underline{q})) = \sum_{\underline{j}\in\mathbb{N}^n}\varphi(e^{-2\pi|\lambda|(2|\underline{j}|+n)})q_{\underline{j},\lambda}$$

and so

(2.6.2.1) $$\varphi(\underline{q})_\lambda(v,p) = \sum_{\underline{j}\in\mathbb{N}^n}\varphi(e^{-2\pi|\lambda|(2|\underline{j}|+n)})h_{\underline{j},\lambda}(v)h_{\underline{j},\lambda}(p), \qquad v, p \in \mathbb{R}^n.$$

Let now $\chi = \chi_{a,b}$ be a unitary character of H_n. Multiplication with χ defines an automorphism of $\mathcal{S}(H_n)$ and we have

$$\varphi(\chi f) = \chi(\varphi(f))$$

for every selfadjoint $f \in \mathcal{S}(H_n)$ and $\varphi \in C^\infty(\mathbb{R})$ with $\varphi(0) = 0$. Furthermore, it follows from (2.2.2) that

(2.6.2.2) $$\begin{aligned}(\chi f)_\lambda(v,p) &= e^{-2i\pi a(v-p)}f^{2,3}(v-p, (p+v)\lambda/2 - b, \lambda)\\ &= e^{-2i\pi a(v-p)}f_\lambda(v - b/\lambda, p - b/\lambda), \qquad v, p \in \mathbb{R}^n,\end{aligned}$$

for every $f = f^* \in \mathcal{S}(H_1)$. In particular,

$$(\chi\underline{q})_\lambda(p,v) = \sum_{\underline{j}\in\mathbb{N}^n} e^{-2\pi|\lambda|(2|\underline{j}|+n)}h^\chi_{\underline{j},\lambda}(v)\overline{h^\chi_{\underline{j},\lambda}(p)}, \qquad v, p \in \mathbb{R}^n,$$

where

(2.6.2.3) $$h^\chi_{\underline{j},\lambda}(v) = e^{-2i\pi av}h_{\underline{j},\lambda}(v - b/\lambda), \qquad v \in \mathbb{R}^n.$$

2.7.1. Definition. Let C be a closed subset of $\widehat{H}_n$ not containing $\mathcal{CHA}$. For every $\chi = (a,b) \in \mathbb{R}^{2n} \setminus C$, let

$$d(\chi, C) \doteq \min(\text{distance of } \chi \text{ to } C_0, \delta),$$

where δ is as in (2.6.1.1).

2.7.2. Definition. We say that a function $\varphi \in C_c^\infty(\mathbb{R})$ is *adapted to* $\chi \in \mathcal{CHA}$ if $\varphi(1) = 1$ and $\varphi(t) = 0$ whenever $|t| < e^{-(2\pi d(\chi,C))^2}$.

2.7.3. Definition. We say that a function $f \in \mathcal{S}(H_n)$ is *elementary for* χ if for any $\lambda \in \mathbb{R}^*$,

$$f_\lambda = \sum_{\underline{j}\in\mathbb{N}^n}\varphi(e^{-2\pi|\lambda|(2|\underline{j}|+n)})\xi_{\underline{j},\lambda}\otimes\overline{\eta}_{\underline{j},\lambda},$$

where φ is adapted to χ and $\xi_{\underline{j},\lambda}, \overline{\eta}_{\underline{j},\lambda}$ are Schwartz functions such that the functions F, G defined on $\mathbb{R}^{2n} \times \mathbb{R}^*$ by

$$F(\lambda) = \sum_{\underline{j}\in\mathbb{N}^n}\xi_{\lambda,\underline{j}}\otimes\overline{h}^{\overline{\chi}}_{\underline{j},\lambda}, \qquad G(\lambda) = \sum_{\underline{j}\in\mathbb{N}^n}\overline{h}^{\overline{\chi}}_{\underline{j},\lambda}\otimes\eta_{\underline{j},\lambda}, \qquad \lambda \in \mathbb{R}^*,$$

are in $\widehat{\mathcal{S}}(H_n)$.

2.8. Proposition. *Let C be a closed subset of $\widehat{H}_n$ not containing all characters of H_n. The minimal ideal $j(C)$ is the span of all the f's in $\mathcal{S}(H_n)$*

which are elementary for some $\chi \in \mathcal{CHA} \setminus C$ or which are elementary in $j(C \cup \mathcal{CHA})$.

Proof. Let $\chi = \chi_{a,b} \in \mathcal{CHA} \setminus C$ and let φ be adapted to χ. It follows from the definition of $d(\chi, C)$ that for the Schwartz function $\underline{q}$ of 2.6,

$$(\varphi(\overline{\chi}\underline{q}))^\wedge(u,v) = (\varphi(\underline{q}))^\wedge(u-a, v-b, 0) = \varphi(e^{-4\pi^2\|(u-a,v-b)\|^2}) = 0$$

whenever $(u,v) \in C \cap \mathcal{CHA}$, since $\|(u-a, v-b)\|^2 > d(\chi, C)^2$. Furthermore, $\varphi(e^{-2\pi|\lambda|(2|\underline{j}|+n)}) = 0$ for every $\underline{j} \in \mathbb{N}^n$ and $|\lambda| > d(\chi, C)^2$. Hence if we take ψ adapted to χ such that $\psi \cdot \varphi = \varphi$, then we see that $\psi(\overline{\chi}\underline{q}) \in \ker(C \cup \mathbb{R}_\delta)$ where $\mathbb{R}_\delta = \{t \in \mathbb{R} : |t| \geq \delta\}$, $\varphi(\overline{\chi}\underline{q})^\wedge(\chi) = \varphi(\widehat{\underline{q}}(0)) = 1$ and $\psi(\overline{\chi}\underline{q}) * \varphi(\overline{\chi}\underline{q}) = \varphi(\overline{\chi}\underline{q})$. Therefore $\varphi(\overline{\chi}\underline{q}) \in j(C \cup \mathbb{R}_\delta)$ and so $\varphi(\overline{\chi}\underline{q}) \in j(C)$ and the elements $\varphi(\overline{\chi}\underline{q})$ generate $j(C \cup \mathbb{R}_\delta)$. The function $\varphi'(t) = \varphi(t)/t$, $t \in \mathbb{R}^*$, extends to an element of $C^\infty_{c,0}(\mathbb{R})$, since φ vanishes in a neighbourhood of 0. Hence we can write

$$\varphi(\overline{\chi}\underline{q})_\lambda(p,v) = \sum_{\underline{j} \in \mathbb{N}^n} \varphi'(e^{-2\pi|\lambda|(2|\underline{j}|+n)}) e^{-2\pi|\lambda|(2|\underline{j}|+n)} h^{\overline{\chi}}_{\underline{j},\lambda}(v) \overline{h^{\overline{\chi}}_{\underline{j},\lambda}(p)}, \qquad v, p \in \mathbb{R}^n,$$

and so

$$\xi_{\underline{j},\lambda} = e^{-2\pi|\lambda|(2|\underline{j}|+n)} h^{\overline{\chi}}_{\underline{j},\lambda}, \; \eta_{\underline{j},\lambda} = e^{-2\pi|\lambda|(2|\underline{j}|+n)} h^{\overline{\chi}}_{\underline{j},\lambda}$$

i.e.

$$F = \overline{G} = \widehat{q}.$$

If f, g are in $\mathcal{S}(H_n)$ then

$$\begin{aligned}(f * \varphi(\overline{\chi}\underline{q}) * g)^\wedge(\lambda) &= f_\lambda \circ \varphi(\overline{\chi}\underline{q})_\lambda \circ g_\lambda \\ &= \sum_{\underline{j} \in \mathbb{N}^n} \varphi(e^{-2\pi(2|\underline{j}|+n)}) \xi_{\underline{j},\lambda} \otimes \overline{\eta}_{\underline{j},\lambda}\end{aligned}$$

where

$$\eta_{\underline{j},\lambda} = \int_{\mathbb{R}^n} \overline{h}^{\overline{\chi}}_{\underline{j},\lambda}(u) g_\lambda(u, \cdot)\, du, \qquad \xi_{\underline{j},\lambda} = \int_{\mathbb{R}^n} \overline{h}^{\overline{\chi}}_{\underline{j},\lambda}(u) f_\lambda(\cdot, u)\, du.$$

On the other hand,

$$f_\lambda(v,p) = \sum_{\underline{j} \in \mathbb{N}^n} f_{\underline{j},\lambda}(v) \overline{h}^{\overline{\chi}}_{\underline{j},\lambda}(p), \qquad g_\lambda(v,p) = \sum_{\underline{j} \in \mathbb{N}^n} g_{\underline{j},\lambda}(p) \overline{h}^{\overline{\chi}}_{\underline{j},\lambda}(v)$$

where

$$f_{\underline{j},\lambda}(v) = \int_{\mathbb{R}^n} f_\lambda(v,p) h^{\overline{\chi}}_{\underline{j},\lambda}(p)\, dp = \xi_{\underline{j},\lambda}(v),$$

$$g_{\underline{j},\lambda}(p) = \int_{\mathbb{R}} g_\lambda(v,p) h^{\overline{\chi}}_{\underline{j},\lambda}(v)\, dv = \eta_{\underline{j},\lambda}(p),$$

since $(h^{\overline{\chi}}_{\underline{j},\lambda})_{\underline{j} \in \mathbb{N}^n}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ and a basis of $\mathcal{S}(H_n)$ and so $F, G \in \mathcal{S}(H_n)^\wedge$.

Conversely, if F, G are as above, then there exist $f, g \in \mathcal{S}(H_n)$ such that $\widehat{f} = F$, $\widehat{g} = G$ and so

$$\sum_{\underline{j} \in \mathbb{N}^n} \varphi(e^{-2\pi|\lambda|(2|\underline{j}|+n)}) \xi_{\underline{j},\lambda} \otimes \overline{\eta}_{\underline{j},\lambda} = (f * \varphi(\overline{\chi}\underline{q}) * g)^\wedge(\lambda) \in j(C)^\wedge. \;\blacksquare$$

2.9. Corollary. *Let C, C' be two closed subsets of $\mathcal{CHA}$ such that $C \cup C' = \mathcal{CHA}$, $C \not\subset C'$ and $C' \not\subset C$. Then $j(C) * j(C') \neq j(C \cup C') = j(\mathcal{CHA})$.*

Proof. Choose χ in $C' \setminus C$ and χ' in $C \setminus C'$. Choose also $\varphi \in C^\infty_{c,0}(\mathbb{R})$ adapted to χ and χ'. We see that $\varphi(\overline{\chi'}\underline{q}) \in j(C')$, $\varphi(\overline{\chi}\underline{q}) \in j(C)$ and

$$\begin{aligned}&\varphi(\overline{\chi}\underline{q})_\lambda \circ \varphi(\overline{\chi'}\underline{q})_\lambda \\ &= \sum_{\underline{j},\underline{k} \in \mathbb{N}^n} \varphi(e^{-2\pi|\lambda|(2|\underline{j}|+n)}) \varphi(e^{-2\pi|\lambda|(2|\underline{k}|+1)}) \langle h^{\overline{\chi}}_{\underline{j},\lambda}, h^{\overline{\chi'}}_{\underline{k},\lambda}\rangle_{L^2} h^{\overline{\chi}}_{\underline{j},\lambda} \otimes h^{\overline{\chi'}}_{\underline{k},\lambda}.\end{aligned}$$

Since $h^{\overline{\chi}}_{\underline{j},\lambda}$ is not 0, we see that $\langle h^{\overline{\chi}}_{\underline{j},\lambda}, h^{\overline{\chi'}}_{\underline{k},\lambda}\rangle_{L^2}$ is not 0 for one $\underline{k}$ and so the analytic function $\lambda \mapsto \langle h^{\overline{\chi}}_{\underline{j},\lambda}, h^{\overline{\chi'}}_{\underline{k},\lambda}\rangle_{L^2}$, $\lambda \in \mathbb{R}^*$, is not 0. Hence the mapping

$$\lambda \mapsto \varphi(e^{-2\pi|\lambda|(2|\underline{j}|+n)}) \varphi(e^{-2\pi|\lambda|(2|\underline{k}|+1)}) \langle h^{\overline{\chi}}_{\underline{j},\lambda}, h^{\overline{\chi'}}_{\underline{k},\lambda}\rangle_{L^2} h^{\overline{\chi}}_{\underline{j},\lambda} \otimes h^{\overline{\chi'}}_{\underline{k},\lambda}$$

is not 0 in any neighbourhood of 0. Furthermore, the functions $h^{\overline{\chi}}_{\underline{j},\lambda}, h^{\overline{\chi'}}_{\underline{k},\lambda}$ being linearly independent, we see that also the mapping $\varphi(\overline{\chi}\underline{q})_\lambda \circ \varphi(\overline{\chi'}\underline{q})_\lambda$ is not identically 0 in any neighbourhood of 0. Hence $\varphi(\chi\underline{q}) * \varphi(\chi'\underline{q})$ is in $j(C) * j(C')$ but not in $j(C \cup C') = j(\mathcal{CHA})$, since for $g \in j(\mathcal{CHA})$, g_λ vanishes in a neighbourhood of 0 by 2.5. $\blacksquare$

2.10. *Questions and remarks*

2.10.1. According to Mehler's formula the function q_λ on H_1 is equal to

$$\begin{aligned}q_\lambda(v,p) &= \left(\frac{2|\lambda|}{e^{2|\lambda|} - e^{-2|\lambda|}}\right)^{1/2} \\ &\quad \times \exp\left(\frac{-\pi(e^{2|\lambda|} + e^{-2|\lambda|})(|\lambda|(v^2+p^2) + 4\pi|\lambda|vp)}{e^{2|\lambda|} - e^{-2|\lambda|}}\right) \\ &\to \sqrt{1/2}\, e^{-\pi(v-p)^2/2} \quad \text{as } \lambda \to 0,\end{aligned}$$

and so

$$q_\lambda\left(\frac{x}{2}+\frac{y}{\lambda},-\frac{x}{2}+\frac{y}{\lambda}\right)$$
$$=\left(\frac{2|\lambda|}{e^{2|\lambda|}-e^{-2|\lambda|}}\right)^{1/2}$$
$$\times\exp\left(\frac{-2\pi(e^{2|\lambda|}+e^{-2|\lambda|})((x^2|\lambda|/4+y^2/|\lambda|)+2(-x^2|\lambda|/4+y^2/|\lambda|))}{e^{2|\lambda|}-e^{-2|\lambda|}}\right)$$
$$=\left(\frac{2|\lambda|}{e^{2|\lambda|}-e^{-2|\lambda|}}\right)^{1/2}$$
$$\times\exp\left(-\frac{\pi}{2}\left(\frac{(e^{|\lambda|}+e^{-|\lambda|})^2}{e^{2|\lambda|}-e^{-2|\lambda|}}|\lambda|x^2+4\frac{(e^{|\lambda|}-e^{-|\lambda|})^2}{(e^{2|\lambda|}-e^{-2|\lambda|})|\lambda|}y^2\right)\right)$$
$$\to\sqrt{1/2}\,e^{-\pi x^2/2-4y^2}\quad\text{as }\lambda\to 0.$$

2.10.2. By the Riemann–Lebesgue lemma for a nilpotent Lie group G, for any $f\in L^1(G)$ we have $\lim_{\pi\to\infty}\|\pi(f)\|_{\mathrm{op}}=0$ on $\widehat{G}$ and so
$$\pi(\varphi(f))=0$$
for any $f=f^*\in\mathcal{S}(G)$, $\varphi\in C^\infty(\mathbb{R})$ vanishing in a neighbourhood of 0, and π which is far enough from the origin. Hence for every closed subset $C\subset\widehat{G}$, $j(C)$ is contained in the ideal
$$I_{\text{bounded supp.}}=\{f\in\mathcal{S}(G):\operatorname{supp}(\widehat{f})\text{ is compact in }\widehat{G}\}.$$
Since for every $\pi\in\widehat{H}_n$, the operators $\pi(f)$, $f\in L^1(G)$, are compact, we see that $j(C)$ is always contained in the ideal
$$I_{\text{finite rank}}=\{f\in\mathcal{S}(G):\operatorname{rank}(\pi(f))\text{ is finite for any }\pi\in\widehat{G}\}.$$

2.10.3. Let C be a closed subset in the dual of a nilpotent Lie group containing $\widehat{G}_{\text{sing}}$. The description of $j(C)$ which we have given in 2.5 is not yet intrinsic enough. It would be nicer if $j(C)$ was just the ideal
$$I_{\text{finite}}(C)=\{f\in\mathcal{S}(G):\operatorname{supp}(\widehat{f})\text{ is compact and disjoint from }C,$$
$$\text{there exists }M>0\text{ such that }\operatorname{rank}(\pi(f))\le M,\ \forall\pi\}.$$
Do we have $I_{\text{finite}}(C)=j(C)$?

2.10.4. Let C be a closed subset of $\widehat{H}_n$ which is contained in $\mathcal{CHA}$. Let $I_{1/\lambda}(C)$ be the ideal
$$I_{1/\lambda}(C)=\{f\in\mathcal{S}(H_n):\operatorname{rank}(\pi_\lambda(f))\le\text{const}/|\lambda|^n,\ \lambda\in\mathbb{R}^*,$$
$$\operatorname{supp}(\widehat{f})\text{ compact and disjoint from }C\}.$$
It follows from the description of $j(C)$ that $\overline{\chi}q$ is contained in $I_{1/\lambda}$, hence also $j(C)\subset I_{1/\lambda}(C)$ and so $h(I_{1/\lambda}(C))=C$. Question: Is $j(C)$ equal to $I_{1/\lambda}(C)$?

2.10.5. Let $C\subset\mathcal{CHA}$ be a closed subset not containing 0. The description of $j(C)$ given in 2.8 is not very precise. Choose a real function $\xi\in\mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}(\xi)\subset[-1/4,1/4]$. Choose $\varphi\in\mathcal{S}(\mathbb{R})$ such that $\widehat{\varphi}$ is compactly supported and vanishes on C and $\varphi(0)=1$. For any $\lambda\neq 0$ let
$$g_\lambda(v,p)=\sum_{j\in\mathbb{Z}}\varphi(\lambda j)e^{-i2\pi\lambda j(v-p)}|\lambda|\xi(\lambda v)\xi(\lambda p)$$
$$=\sum_{j\in\mathbb{Z}}(|\lambda|\varphi(\lambda j)e^{-i2\pi\lambda j(v-p)})\xi(\lambda v)\xi(\lambda p)$$
$$=\widehat{\varphi}(v-p,\lambda)\xi(\lambda v)\xi(\lambda p),$$
where
$$\widehat{\varphi}(x,\lambda)=\sum_{j\in\mathbb{Z}}\widehat{\varphi}(x+j/\lambda).$$
Hence
$$g_\lambda(u/\lambda+r/2,u/\lambda-r/2)=\widehat{\varphi}(r,\lambda)\xi(u+\lambda r/2)\xi(u-\lambda r/2).$$
Therefore
$$\lim_{\lambda\to 0}g_\lambda(u/\lambda+r/2,u/\lambda-r/2)=\widehat{\varphi}(r)\xi^2(u)$$
and since $\xi(u+\lambda r/2)\xi(u-\lambda r/2)=0$ for any $u\in\mathbb{R}$ whenever $|\lambda r|>1/2$ we see that the function
$$\widetilde{f}(r,u,\lambda)=\widehat{\varphi}(r,\lambda)\xi(u+\lambda r/2)\xi(u-\lambda r/2)\varphi(\lambda),\qquad r,v,\lambda\in\mathbb{R},$$
is a Schwartz function. Hence there exists $f\in\mathcal{S}(H_1)$ such that $f_\lambda=g_\lambda$ for any $\lambda\in\mathbb{R}^*$. We see that for any $\lambda\neq 0$, $\pi_\lambda(f)$ has finite rank and that
$$\operatorname{rank}(\pi_\lambda(f))\le C/|\lambda|$$
for any $\lambda\neq 0$ and some constant independent of λ.

Is f contained in $j(C)$?

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Corrigenda to: "Generalizations of theorems of Fejér and Zygmund on convergence and boundedness of conjugate series"

(Studia Math. 57 (1976), 241–249)

by

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Abstract. Proposition 4.1(i) of [1] is incorrect, i.e. the sequence of Cesàro-sections $\{\sigma_n x\}$ of a sequence x in a translation invariant BK-space is not necessarily bounded. Theorem 4.2(ii) of [1] and the proof of Proposition 4.3 of [1] are corrected. All other statements of [1], including Proposition 4.3 itself, are correct.

1. Notations and definitions. We use the notations and definitions of [1]. Two more definitions:

$$\widehat{L}^2 := \Big\{x \in \Omega : \|x\|_2 := \Big(\sum_{k=-\infty}^{\infty} |x_k|^2\Big)^{1/2} < \infty\Big\},$$

$$\widehat{M}^{\mathrm{d}} := \{x \in \widehat{M} : x = \widehat{\mu},\ \mu \in M_{2\pi} \text{ is discrete}\}.$$

2. The error in Proposition 4.1(i) of [1]. The error in the proof of this proposition consists in the assumption of the existence of the E-valued Riemann integral $\int_0^{2\pi} K_n(t)x \cdot e(-t)\,dt$, where $x \in E$ and K_n is the nth Fejér-kernel. This is pointed out in detail in [2].

A counterexample to 4.1(i) of [1] is $E = \widehat{M}^{\mathrm{d}}$. In fact, since $\Phi \cap \widehat{M}^{\mathrm{d}} = \{0\}$, evidently $\widehat{M}^{\mathrm{d}} \not\subset (\widehat{M}^{\mathrm{d}})_{\sigma\mathrm{B}} = \{0\}$.

A less trivial counterexample is $E = \widehat{M}^{\mathrm{d}} + \widehat{L}^2 = \{x \in \Omega : x = a + b,\ a \in \widehat{M}^{\mathrm{d}},\ b \in \widehat{L}^2\}$ with $\|x\|_E := \|a\|_{\widehat{M}} + \|b\|_{\widehat{L}^2}$. Through this example Ulf Boettcher brought the incorrectness of 4.1(i) in [1] to our attention. Evidently E is a translation invariant BK-space. If $x \in \widehat{M}^{\mathrm{d}} + \widehat{L}^2$ then $\sigma_n x \in \widehat{L}^2$ for all $n = 0, 1, 2, \ldots$ Hence $E \not\subset E_{\sigma\mathrm{B}} = \widehat{L}^2$, since $\widehat{M}^{\mathrm{d}} \not\subset \widehat{L}^2$.

3. Corrected version of Theorem 4.2(ii) of [1]. *If E is a translation invariant* BK*-space with $E \subset E_{\sigma\mathrm{B}}$, then $E_{\mathrm{AB}} \cap \widetilde{E} \subset \widetilde{E}_{\mathrm{AB}}$.*

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