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Two-parameter maximal functions associated with homogeneous surfaces in $\mathbb{R}^n$

by

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Abstract. Given a hypersurface $x_n = \Gamma(x_1, \ldots, x_{n-1})$ in $\mathbb{R}^n$, where Γ is homogeneous of degree $d > 0$, we define the two-parameter maximal operator

$$Mf(x) = \sup_{a,b>0} \int_{s\in\mathbb{R}^{n-1},\, |s|<1} |f(x - (as, b\Gamma(s)))|\, ds.$$

We prove that if $d \neq 1$ and the hypersurface has non-vanishing Gaussian curvature away from the origin, then M is bounded on L^p if and only if $p > n/(n-1)$. If $d = 1$, i.e. if the surface is a cone, the same conclusion holds in dimension $n \geq 3$ if the surface has $n-1$ non-vanishing principal curvatures away from the origin and it intersects the hyperplane $x_n = 0$ only at the origin.

Maximal operators defined by averages on curves or surfaces have been extensively considered. Restricting our attention to translation invariant operators in $\mathbb{R}^n$, the usual way to construct such operators is to take the surface measure on some bounded part of the manifold and then act on it by a one-parameter family of dilations. If μ is the basic measure and μ_δ is the same measure dilated by $\delta > 0$ and appropriately normalised, the operator is

$$Mf(x) = \sup_{\delta>0} |f| * \mu_\delta(x).$$

Two different situations can arise. If the manifold is homogeneous under the given dilations, one obtains basically the same operator by restricting the supremum to $\delta = 2^j$. Under appropriate assumptions on the manifold, one then proves that M is bounded on L^p for $p > 1$ (see [SW]).

If the manifold is not homogeneous under the given dilations, then the various μ_δ are supported on different manifolds and the problem becomes much more subtle. Most of the results available concern the case where

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the dilations are isotropic and the manifold has dimension $n-1$. If the Gaussian curvature does not vanish, then M turns out to be bounded on L^p for $p > n/(n-1)$ [S, B, MSS]. The range of p may be further restricted if the curvature vanishes at some point [I] unless one introduces a damping factor [CDMM, M].

The maximal operators we study involve two-parameter dilations. Operators of this kind have been considered in [C]. Our operators combine both features of the one-parameter operators described above. The simplest example is the following: given $d > 0$, $d \neq 1$, define the maximal operator on $\mathbb{R}^2$

$$Mf(x,y) = \sup_{a,b>0} \int_1^2 f(x-as, y-bs^d)\,ds.$$

If we modify the definition of M by imposing the relation $b = a^d$ on the two parameters, we obtain a one-parameter maximal operator M_1 of the first type described above and which is bounded on L^p for $p > 1$. If we impose instead the relation $b = a$, we obtain an operator M_2 of the second type, which is bounded only for $p > 2$.

Since M dominates both M_1 and M_2, it can only be bounded for $p > 2$. In Section 1 we prove that this is in fact the case by combining Bourgain's theorem with a Littlewood–Paley decomposition.

The rôle of curvature is rather clear. If we take $d = 1$, the operator M becomes basically the Kakeya maximal operator, which is bounded only for $p = \infty$.

The homogeneity of the curve allows one of the two parameters a, b to be taken dyadic, i.e. equal to 2^j, but not both. If we restrict the supremum to dyadic values of both parameters, then the corresponding maximal operator is bounded for all $p > 1$ (see [RS]).

In Section 2 we consider variants of these operators in higher dimensions. We take a hypersurface $x_n = \Gamma(x_1, \ldots, x_{n-1})$ where Γ is homogeneous of degree $d > 0$ and define the two-parameter maximal operator

$$Mf(x) = \sup_{a,b>0} \int_{|s|<1} |f(x-(as, b\Gamma(s)))|\,ds.$$

We prove that if the hypersurface has non-vanishing Gaussian curvature away from the origin (which forces the condition $d \neq 1$), then M is bounded on L^p if and only if $p > n/(n-1)$. It is interesting to observe that this is also true if $d = 1$, i.e. when the surface is a cone, if one assumes that it intersects the hyperplane $x_n = 0$ only at the origin and that it has $n-1$ non-vanishing principal curvatures away from the origin. In other words, within these assumptions, the lack of curvature in the radial directions does not affect the range of L^p-boundedness. This phenomenon depends on the fact that the crucial estimate on the Fourier transform of the measure does not involve the pointwise decay, but the decay of certain quadratic averages. The same condition appears in recent work by Iosevich and Sawyer [IS] on a slightly different problem.

In a separate paper [MRZ] we investigate the case of surfaces in higher dimensions with other kinds of degeneracy in the curvature.

We finally comment on some consequences of our results. By applying transference to the maximal operator related to the parabola $y = x^2$ in the plane, one concludes that the maximal operator on the real line

$$\mathcal{M}f(x) = \sup_{a,b} \int_0^1 f(x-as-bs^2)\,ds$$

is bounded on L^p for $p > 2$. This has been proved directly by A. Carbery, J. Wright and the second author [CRW]. They also prove that the restriction $p > 2$ is sharp and that, more generally, the operator

$$\mathcal{M}_n f(x) = \sup_{a_1,\ldots,a_n} \int_0^1 f(x-a_1s-\ldots-a_ns^n)\,ds$$

is bounded on L^p if and only if $p > n$. It is then natural to ask for which values of p the maximal operator on $\mathbb{R}^n$

$$Mf(x) = \sup_{a_1,\ldots,a_n} \int_0^1 f(x-(a_1s,\ldots,a_ns^n))\,ds.$$

is bounded on L^p. A necessary condition is obviously $p > n$, but attacking this problem would require completely new ideas.

1. Homogeneous curves in $\mathbb{R}^2$.

In $\mathbb{R}^2$ consider the family of curves $\gamma_b(s) = (s, bs^d)$ depending on the parameter $b > 0$. We assume $d \in \mathbb{R}$ and $d \neq 0, 1$. If on each curve we consider the arcs corresponding to $s \in [0, a]$, $a > 0$, we can construct the maximal operator

$$Mf(x) = \sup_{a,b>0} \frac{1}{a} \int_0^a |f(x-(s, bs^d))|\,ds, \tag{1.1}$$

defined initially for f continuous with compact support. Clearly

$$Mf(x) = \sup_{a,b>0} \int_0^1 |f(x-(as, bs^d))|\,ds = \sup_{a,b>0} |f| * \mu_{ab}(x),$$

where μ is a fixed positive measure supported on the arc (s, s^d), $s \in [0,1]$. Therefore we can regard M as naturally associated with the two-parameter dilation structure on $\mathbb{R}^2$.

The homogeneity of the curve allows us to restrict one of the parameters for the dilations to dyadic values. In fact,

$$\begin{aligned}\frac{1}{a}\int_0^a |f(x-(s,bs^d))|\,ds &\le \frac{1}{a}\sum_{2^i<a}\int_{2^i}^{2^{i+1}} |f(x-(s,bs^d))|\,ds\\ &\le 2\sup_{i\in\mathbb{Z}} 2^{-i}\int_{2^i}^{2^{i+1}} |f(x-(s,bs^d))|\,ds\\ &= 2\sup_{i\in\mathbb{Z}}\int_1^2 |f(x-(2^is,b2^{id}s^d))|\,ds.\end{aligned}$$

We can now take the supremum over a and b to see that it suffices to control

$$\sup_{i\in\mathbb{Z},\,b>0}\int_1^2 |f(x-(2^is,bs^d))|\,ds.$$

We can now insert a smooth cut-off $\varphi(s)\ge 0$ into our integrals, with $\varphi(s)=1$ if $s\in[1,2]$ and $\operatorname{supp}\varphi\subset[1/2,3]$, and define

$$M'f(x)=\sup_{i\in\mathbb{Z},\,b>0}\Big|\int f(x-(2^is,bs^d))\varphi(s)\,ds\Big|.$$

Clearly the ratio $Mf(x)/M'f(x)$ is bounded pointwise from above and from below by absolute positive constants.

THEOREM 1.1. *The operator M is bounded on L^p if and only if $p>2$.*

Proof. It is clear that M dominates the maximal operator

$$f\mapsto \sup_{a>0}\int_1^2 |f(x-a(s,s^d))|\,ds,$$

which is bounded on L^p only if $p>2$. Therefore M is not bounded for $p\le 2$.

To prove the converse, we consider M' instead of M. If we denote by m the multiplier

$$m(\xi)=\int e^{-2\pi i(s,s^d)\cdot\xi}\varphi(s)\,ds,$$

we can write our maximal function as

$$M'f(x)=\sup_{i\in\mathbb{Z},\,b>0}|T_{i,b}f(x)|,$$

where

$$\mathcal{F}(T_{i,b}f)(\xi)=m(2^i\xi_1,b\xi_2)\widehat{f}(\xi),$$

and where $\mathcal{F}$ denotes the Fourier transform in x, as usual. A standard integration by parts argument shows that outside the cone

$$\bigcup_{s\in[1/2,3]}\{\xi:|\xi\cdot(1,ds^{d-1})|\le c|\xi|\}$$

$m(\xi)$ decays faster than any power of $|\xi|$, where c is some small constant depending only on d. It is easy to see, however, that the above cone may be "widened" slightly and still be contained in the conical region

$$C=\{\xi:c_d\le|\xi_2/\xi_1|\le C_d\},$$

for certain non-zero constants c_d and C_d dependent only on d. Hence, for $\xi\in\mathbb{R}^2\setminus C$, $m(\xi)$ has rapid decay. In order that we may take advantage of the good decay of $m(\xi)$ outside the cone, we decompose it smoothly into two parts:

$$m(\xi)=m_0(\xi)+m_1(\xi),$$

where

$$m_0(\xi)=m(\xi)\varrho(|\xi_2/\xi_1|)\sigma(|\xi|);$$

here ϱ is a C^∞ function such that $\varrho(\lambda)=0$ if $\lambda\ge 2C_d$ or $\lambda\le c_d/2$ and $\varrho(\lambda)=1$ if $\lambda\in[c_d,C_d]$ and σ is a C^∞ function equal to 0 in a neighborhood of 0 and equal to 1 in a neighborhhod of infinity. Because of the smoothness of m and of its rapid decay outside the cone C, we can dominate the Fourier transform of m_1 by $\eta(x)=C(1+|x|)^{-3}$, so that the corresponding maximal function

$$\sup_{i\in\mathbb{Z},\,b>0}|\mathcal{F}^{-1}(m_1(2^i\xi_1,b\xi_2)\widehat{f}(\xi))(x)|$$

is controlled by

$$\sup_{i\in\mathbb{Z},\,b>0} C2^{-i}b^{-1}\int|f(x-y)|\eta(2^{-i}y_1,b^{-1}y_2)\,dy,$$

which, in turn, is controlled by the strong maximal function in $\mathbb{R}^2$, and is hence a bounded operator on L^p whenever $p>1$.

We now treat m_0 using a Littlewood–Paley type argument. We have to control

$$M_0f(x)=\sup_{i\in\mathbb{Z},\,b>0}|A_{i,b}f(x)|,$$

where $A_{i,b}f$ is defined by

$$\mathcal{F}(A_{i,b}f)(\xi)=m_0(2^i\xi_1,b\xi_2)\widehat{f}(\xi).$$

We first observe that if $b\in I_j=[2^j,2^{1+j}]$, the definition of $A_{i,b}f$ only involves the values of $\widehat{f}$ on the cone where

$$c_d'2^{i-j}\le|\xi_2/\xi_1|\le C_d'2^{i-j},$$

for appropriate values of the positive constants c_d',C_d'.

Thus, we define the usual Littlewood–Paley operators S_k corresponding to these cones, i.e.

$$\mathcal{F}(S_k f)(\xi) = \widetilde{\varrho}(2^{-k}|\xi_2/\xi_1|)\widehat{f}(\xi),$$

where $\widetilde{\varrho}$ is a smooth cut-off function, like ϱ above, though perhaps with a slightly wider support. Clearly we may write

$$\begin{aligned} M_0 f(x) &= \sup_{j\in\mathbb{Z}} \sup_{i\in\mathbb{Z},\, b\in I_j} |A_{i,b} S_{i-j} f(x)| = \sup_{k\in\mathbb{Z}} \sup_{i\in\mathbb{Z},\, b\in I_{i-k}} |A_{i,b} S_k f(x)| \\ &= \sup_{k\in\mathbb{Z}} N_k S_k f(x), \end{aligned}$$

where

$$N_k g(x) = \sup_{i\in\mathbb{Z},\, b\in I_{i-k}} |A_{i,b} g(x)|.$$

Suppose we know that

$$\|N_k g\|_p \le C\|g\|_p \tag{1.2}$$

for $p > 2$, with C independent of k. We could then conclude that, for $p > 2$,

$$\begin{aligned} \|M_0 f\|_p^p &\le \sum_k \int |N_k S_k f(x)|^p\,dx \le C^p \int \sum_{k\in\mathbb{Z}} |S_k f(x)|^p\,dx \\ &\le C^p \int \Big(\sum_{k\in\mathbb{Z}} |S_k f(x)|^2\Big)^{p/2} dx \le C^p \|f\|_p^p. \end{aligned}$$

Therefore it remains to prove (1.2). Take first $k = 0$. Writing $m_0(\xi) = m(\xi) - m_1(\xi)$, we can control $N_0 g$ with the sum of two terms, one of them being

$$\widetilde{N}_0 g(x) = \sup_{i\in\mathbb{Z},\, b\in I_i} \int_{1/2}^{3} |f(x - (2^i s, b s^d))|\,ds,$$

and the other controlled by the strong maximal function. It is therefore sufficient to show that $\widetilde{N}_0$ is bounded on L^p for $p > 2$. Changing variable $s = \alpha t$, we have

$$\int_{1/2}^{3} |f(x - (2^i s, b s^d))|\,ds = \alpha \int_{1/(2\alpha)}^{3/\alpha} |f(x - (2^i \alpha t, b\alpha^d t^d))|\,dt.$$

We take $\alpha = (2^i/b)^{1/(d-1)}$ so that $2^i\alpha = b\alpha^d = \beta$. Since $b \in I_i$, we obtain values of α ranging between two positive absolute constants $c_1 < c_2$. Therefore

$$\int_{1/2}^{3} |f(x - (2^i s, b s^d))|\,ds \le c_2 \int_{1/(2c_2)}^{3/c_1} |f(x - \beta(t, t^d))|\,dt,$$

so that

$$\widetilde{N}_0 g(x) \le c_2 \sup_{\beta>0} \int_{1/(2c_2)}^{3/c_1} |f(x - \beta(t, t^d))|\,dt.$$

Since the arc (t, t^d) with $t \in [1/(2c_2), 3/c_1]$ has non-zero curvature, the boundedness of $\widetilde{N}_0$, and hence of N_0 for $p > 2$ follows by Bourgain's theorem.

Finally, we must prove (1.2) for generic k. But N_k can be obtained directly from N_0 by conjugating with dilation by 2^k in the first variable. Therefore $\|N_k\|_{p,p} = \|N_0\|_{p,p}$, and this completes the proof. ∎

We finally remark that if we take $d = 1$ in (1.1), the corresponding operator M is essentially the Kakeya maximal operator [dG], and therefore it is bounded only for $p = \infty$. The same is true, but for a simpler reason, if $d = 0$.

2. Homogeneous hypersurfaces in higher dimensions. In higher dimensions the critical exponent will be smaller than 2. We shall then need an efficient way of controlling our maximal functions on L^2. All our results will be obtained by interpolating an L^2 result with a "trivial estimate" for p near 1.

We denote $x \in \mathbb{R}^n$ by $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$, and consider the hypersurface $x_n = \Gamma(x')$, where Γ is smooth away from the origin and homogeneous of degree d, i.e. $\Gamma(tx') = t^d\Gamma(x')$. Here we do not exclude the case $d = 1$, i.e. that the surface is a cone.

In what follows we assume for simplicity that Γ is defined on all of $\mathbb{R}^{n-1}$, but the arguments extend easily to homogeneous functions defined on proper cones.

The maximal operator we want to discuss is

$$Mf(x) = \sup_{a,b>0} \frac{1}{a^{n-1}} \int_{|s'|<a} |f(x - (s', b\Gamma(s')))|\,ds'. \tag{2.1}$$

As in Section 1, we can replace M in (2.1) with

$$M'f(x) = \sup_{i\in\mathbb{Z},\, b>0} \Big| \int f(x - (2^i s', b\Gamma(s')))\varphi(s')\,ds' \Big|,$$

where φ is a smooth cut-off function supported where $1/2 \le |s'| \le 3$. If

$$m(\xi', \xi_n) = \int e^{-2\pi i(\xi'\cdot s' + \xi_n\Gamma(s'))}\varphi(s')\,ds', \tag{2.2}$$

and $T_{i,b}$ is defined by

$$\mathcal{F}(T_{i,b} f)(\xi) = m(2^i\xi', b\xi_n)\widehat{f}(\xi), \tag{2.3}$$

then

$$M'f(x)=\sup_{i\in\mathbb{Z},\,b>0}|T_{i,b}f(x)|. \tag{2.4}$$

We assume that $\nabla\Gamma$ does not vanish away from the origin. This implies that $m(\xi)$ coincides with a Schwartz function away from a conical region

$$C=\{(\xi',\xi_n):c<|\xi_n|/|\xi'|<c'\},$$

where $c,c'>0$ depend on Γ.

Let $\psi(\xi)$ be a smooth function supported on the set where $1/2\le|\xi'|\le 3$ and $c/2\le|\xi_n|/|\xi'|\le 2c'$ and such that $\sum_{j\in\mathbb{Z}}\psi(2^{-j}\xi)=1$ for $\xi\in C$. Then

$$m(\xi)-\sum_{j\ge0}m(\xi)\psi(2^{-j}\xi)=\widetilde{m}(\xi)\in\mathcal{S}.$$

Let $m_j(\xi)=m(\xi)\psi(2^{-j}\xi)$. We define $\widetilde{T}_{i,b}$ and $T^j_{i,b}$ by (2.3), replacing m with $\widetilde{m}$ and m_j respectively. Accordingly, we define $\widetilde{M}f$ and M_jf by (2.4) replacing $T_{i,b}$ with $\widetilde{T}_{i,b}$ and $T^j_{i,b}$ respectively. Obviously,

$$M'f(x)\le\widetilde{M}f(x)+\sum_{j\ge0}M_jf(x). \tag{2.5}$$

Since $\widetilde{m}\in\mathcal{S}$, $\widetilde{M}$ is controlled by the strong maximal operator and therefore it is bounded for all $p>1$.

LEMMA 2.1. *Assume that Γ, m and M_j are as above and that*

$$\int\limits_{c|\xi'|/2<|\xi_n|<2c'|\xi'|}\left(|m(\xi',\xi_n)|^2+\left|\frac{\partial m}{\partial\xi_n}(\xi',\xi_n)\right|^2\right)\frac{d\xi_n}{|\xi_n|}\le A^2|\xi'|^{-(n-1)}.$$

Then $\|M_jf\|_{2,2}\le CA2^{-j(n-2)/2}$.

The proof of this lemma is very similar to that in [RdF].

Proof. It suffices to consider $f\in\mathcal{S}$. We use the following inequality: if $g\in C^1(\mathbb{R}^+)$ is such that $g(0)=0$, then

$$\sup_{b>0}|g(b)|\le C\Big(\int_0^\infty|g(b)|^2\,\frac{db}{b}\Big)^{1/4}\Big(\int_0^\infty|bg'(b)|^2\,\frac{db}{b}\Big)^{1/4},$$

which follows by writing $|g(b)|^2$ as the integral of its derivative, and applying Hölder's inequality. We want to apply this to the function $g(b)=T^j_{i,b}f(x)$. Notice that since f is Schwartz, and since $m_j(2^{-i}\xi',b\xi_n)$ has support where $|2^{-i}\xi'|\sim|b\xi_n|\sim 2^j$, this multiplier is identically zero when $b=0$, and so

$T^j_{i,0}f(x)=0$. Hence

$$M_jf(x)\le C\sup_{i\in\mathbb{Z}}\Big(\int_0^\infty|T^j_{i,b}f(x)|^2\,\frac{db}{b}\Big)^{1/4}\Big(\int_0^\infty\Big|b\frac{d}{db}T^j_{i,b}f(x)\Big|^2\,\frac{db}{b}\Big)^{1/4}.$$

We now control the supremum in i by a sum over i and then apply Cauchy–Schwarz to get

$$\begin{aligned}M_jf(x)^2&\le C\sum_{i\in\mathbb{Z}}\Big(\int_0^\infty|T^j_{i,b}f(x)|^2\,\frac{db}{b}\Big)^{1/2}\Big(\int_0^\infty\Big|b\frac{d}{db}T^j_{i,b}f(x)\Big|^2\,\frac{db}{b}\Big)^{1/2}\\&\le\Big(\sum_{i\in\mathbb{Z}}\int_0^\infty|T^j_{i,b}f(x)|^2\,\frac{db}{b}\Big)^{1/2}\Big(\sum_{i\in\mathbb{Z}}\int_0^\infty\Big|b\frac{d}{db}T^j_{i,b}f(x)\Big|^2\,\frac{db}{b}\Big)^{1/2}.\end{aligned}$$

Now we integrate in dx and apply Hölder's inequality to the right hand side to obtain

$$\begin{aligned}\|M_jf\|_2\le C&\Big\|\Big(\sum_{i\in\mathbb{Z}}\int_0^\infty|T^j_{i,b}f(x)|^2\,\frac{db}{b}\Big)^{1/2}\Big\|_2^{1/2}\\&\times\Big\|\Big(\sum_{i\in\mathbb{Z}}\int_0^\infty\Big|b\frac{d}{db}T^j_{i,b}f(x)\Big|^2\,\frac{db}{b}\Big)^{1/2}\Big\|_2^{1/2}.\end{aligned}$$

We can now apply Plancherel's theorem:

$$\begin{aligned}\Big\|\Big(\int_0^\infty\sum_{i\in\mathbb{Z}}|T^j_{i,b}f(x)|^2\,\frac{db}{b}\Big)^{1/2}\Big\|_2&=\Big(\int\int_0^\infty\sum_{i\in\mathbb{Z}}|T^j_{i,b}f(x)|^2\,\frac{db}{b}\,dx\Big)^{1/2}\\&=\Big(\int_0^\infty\sum_{i\in\mathbb{Z}}\int|T^j_{i,b}f(x)|^2\,dx\,\frac{db}{b}\Big)^{1/2}\\&=\Big(\int_0^\infty\sum_{i}\int|m_j(2^{-i}\xi',b\xi_n)|^2|\widehat{f}(\xi)|^2\,d\xi\,\frac{db}{b}\Big)^{1/2}\\&=\Big(\int|\widehat{f}(\xi)|^2\sum_{i\in\mathbb{Z}}\int_0^\infty|m_j(2^{-i}\xi',b\xi_n)|^2\,\frac{db}{b}\,d\xi\Big)^{1/2}\\&\le\Big(\int|\widehat{f}(\xi)|^2\sum_{i\in\mathbb{Z}}\int_{-\infty}^\infty|m_j(2^{-i}\xi',b)|^2\,\frac{db}{|b|}\,d\xi\Big)^{1/2}.\end{aligned}$$

The main point to notice is that because $m_j(2^{-i}\xi',b)$ has support where $|2^{-i}\xi'|\sim|b|\sim 2^j$, for fixed ξ there can be at most a finite number of non-zero terms in the sum over i, this finite number depending only on the constants defining the cone C and not on ξ. Explicitly, we have

$$\Big(\int |\widehat{f}(\xi)|^2 \sum_{i\in\mathbb{Z}} \int_{-\infty}^{\infty} |m_j(2^{-i}\xi',b)|^2 \frac{db}{|b|}\,d\xi\Big)^{1/2}$$
$$\le C\|f\|_2 \sup_{|\xi'|\sim 2^j} \Big(\int |m_j(\xi',b)|^2 \frac{db}{|b|}\Big)^{1/2}$$
$$\le C\|f\|_2 \sup_{|\xi'|\sim 2^j} \Big(\int_{c|\xi'|/2<|b|<2c'|\xi'|} |m(\xi',b)|^2 \frac{db}{|b|}\Big)^{1/2}$$
$$\le C\|f\|_2 A 2^{-j(n-1)/2}.$$

A similar computation works for the other term, the only difference being the presence of an extra factor b^2 in the integral. This gives

$$\Big\|\Big(\sum_{i\in\mathbb{Z}} \int_0^{\infty} \Big|b\frac{d}{db}T^j_{i,b}f(x)\Big|^2 \frac{db}{b}\Big)^{1/2}\Big\|_2 \le C\|f\|_2 A 2^{-j(n-3)/2}.$$

Combining the two estimates together, we obtain the conclusion. ∎

We now make the following curvature assumptions:

(1) if $d\ne 1$, the Gaussian curvature of the surface $x_n=\Gamma(x')$ is non-zero away from the origin;

(2) if $d=1$, away from the origin the surface (a cone in this case) does not intersect the hyperplane $x_n=0$ and has $n-1$ non-vanishing principal curvatures.

Observe that both conditions (1) and (2) imply that $\nabla\Gamma(x')\ne 0$ for $x'\ne 0$. This is not hard to verify using Euler's equation. Condition (2) is equivalent to saying that the level set $\Gamma(x')=\pm 1$ (according to the signum of Γ) is compact and has non-zero Gaussian curvature as a hypersurface in $\mathbb{R}^{n-1}$.

THEOREM 2.2. *Assume that $\Gamma(x')$ is smooth away from the origin, homogeneous of degree $d>0$, and satisfies the appropriate condition* (1) *or* (2), *according to the value of d. Then the maximal operator M defined by* (2.1) *is bounded on L^p if and only if $p>n/(n-1)$.*

Proof. If $f(x)=1/(|x|^{n-1}\log|x|)$ in a neighborhood of the origin, then $Mf(x)=\infty$ on a set of positive measure. This shows the necessity of the condition $p>n/(n-1)$. We now prove that it is also sufficient.

Consider the operators M_j that appear in (2.5) together with the corresponding multipliers $m_j=m\psi_j$, where we have set $\psi_j(\xi)=\psi(2^{-j}\xi)$. If μ is the positive measure supported on the hypersurface whose Fourier transform is the multiplier m in (2.2), then m_j is the Fourier transform of $\mu*(\mathcal{F}^{-1}\psi_j)$.

We can think of $\mathcal{F}^{-1}\psi_j$ as essentially being a smooth bump function of height 2^{nj}, supported in a 2^{-j} neighbourhood of the origin; hence $\mu*(\mathcal{F}^{-1}\psi_j)$ is approximately equal to $C2^{nj}$ times the measure of a ball of radius 2^{-j} on the hypersurface, whenever x is within a 2^{-j} neighbourhood of the surface, and zero when x is more distant than 2.2^{-j} from the surface. Clearly we can dominate such a function by the characteristic function of a ball of radius equal to the maximum distance of the surface from the origin, times $C2^j$. This means to dominate M_j by the strong maximal function times the constant $C2^j$.

Hence, for every $p>1$ we have

$$\|M_j\|_{p,p}\le C_p 2^j.$$

It is then sufficient to prove that m satisfies the assumptions of Lemma 2.1. The conclusion will then follow by applying the Marcinkiewicz interpolation theorem to the M_j. We distinguish the case $d\ne 1$ from the case $d=1$.

If $d\ne 1$, the curvature assumption (1) implies, by stationary phase, the pointwise estimates

$$|m(\xi)|\le C|\xi|^{-(n-1)/2},\qquad \Big|\frac{\partial m}{\partial\xi_n}(\xi)\Big|\le C|\xi|^{-(n-1)/2},$$

which are stronger than the quadratic estimate required.

Assume now that $d=1$ and that condition (2) holds. We can assume with loss of generality that $\Gamma(x')>0$ away from the origin.

Let E be the level set where $\Gamma=1$. Then each point $x'\ne 0$ can be written in a unique way as $x'=rs'$ with $s'\in E$ and $r=\Gamma(x')>0$. Also, by the implicit function theorem, there is a measure ν supported on E with a smooth density such that the Lebesgue measure dx' decomposes as $dx'=r^{n-2}\,dr\,d\nu(s')$ if $x'=rs'$.

We choose the function φ in (2.2) as a function of r only. Then the multiplier m takes the form

$$m(\xi',\xi_n)=\iint e^{-2\pi i(rs'\cdot\xi'+r\xi_n)}\varphi(r)\,dr\,d\nu(s')$$
$$=\int\widehat{\varphi}(s'\cdot\xi'+\xi_n)\,d\nu(s'),$$

where we have incorporated the factor r^{n-2} in the function φ. Similarly,

$$\frac{\partial m}{\partial\xi_n}(\xi',\xi_n)=-2\pi i\int\widehat{r\varphi}(s'\cdot\xi'+\xi_n)\,d\nu(s').$$

In order to verify the hypotheses of Lemma 2.1 we can then apply Plancherel's formula. We discuss only the part concerning m, the part concerning $\partial m/\partial\xi_n$ being completely analogous. We have

$$(2.6)\quad \int\limits_{c|\xi'|/2<|\xi_n|<2c'|\xi'|} |m(\xi',\xi_n)|^2 \frac{d\xi_n}{|\xi_n|} \le \frac{C}{|\xi'|} \int\limits_{-\infty}^{\infty} |m(\xi',\xi_n)|^2\,d\xi_n = \frac{C}{|\xi'|} \int\limits_{-\infty}^{\infty} |(\mathcal{F}_{\xi_n}^{-1} m)(\xi',t)|^2\,dt.$$

Now,

$$(\mathcal{F}_{\xi_n}^{-1} m)(\xi',t) = \varphi(t) \int e^{-2\pi i t s'\cdot\xi'}\,d\nu(s').$$

Since E has non-vanishing Gaussian curvature, we call upon the standard stationary phase estimate to get

$$|(\mathcal{F}_{\xi_n}^{-1} m)(\xi',t)| \le \frac{C\varphi(t)}{|t\xi'|^{(n-2)/2}}.$$

Inserting this estimate into (2.6) and observing that the support of φ is compact and does not contain 0, we obtain the conclusion. ∎

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