STUDIA MATHEMATICA 130 (1) (1998)

α-Equivalence

by

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Abstract. We define the α-relation between discrete systems and between continuous systems. We show that it is an equivalence relation. α-Equivalence vs. even α-equivalence is analogous to Kakutani equivalence vs. even Kakutani equivalence.

1. Introduction. Classification of ergodic dynamical systems has been one of the central research areas in Ergodic Theory. Kakutani equivalence which is stronger than Dye's orbit equivalence and weaker than isomorphism has been studied by many people [Ka], [ORW], [dJR]. The equivalences of $\mathbb{R}$-actions and $\mathbb{Z}$-actions are defined using Ambrose's representation of an ergodic flow [Am].

We say that two $\mathbb{R}$-actions (flows), $(\Omega_1, \mathcal{L}_1, \lambda_1, S_1^t)$ and $(\Omega_2, \mathcal{L}_2, \lambda_2, S_2^t)$, are *Kakutani equivalent* if there exists a $\mathbb{Z}$-action which is isomorphic to both cross sections in Ambrose's representations of the given two flows [Am]. Two $\mathbb{Z}$-actions, $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$, are *Kakutani equivalent* if there exists an $\mathbb{R}$-action which can be built under functions with $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$ as cross sections.

This equivalence relation can be described via inducing and exducing. Two actions $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$ are Kakutani equivalent if and only if there exist subsets $A_1 \subset X_1$ and $A_2 \subset X_2$ such that T_1 induced on A_1 is isomorphic to T_2 induced on A_2. If A_1 and A_2 can be chosen to have the same measure, we say that T_1 and T_2 are *evenly Kakutani equivalent*. It is clear that if T_1 and T_2 are evenly Kakutani equivalent, then they belong to the same entropy class.

If $(X, \mathcal{F}, \mu, T)$ is a cross section in a representation of a flow $(\Omega, \mathcal{L}, \lambda, S^t)$, we sometimes denote the flow by $[X, \mathcal{F}, \mu, T, f]$ where f is a ceiling function in the representation. Given a partition $P = \{P_0, P_1, \ldots, P_{k-1}\}$, we say that the *$n$-long P-name of a point x satisfies the ergodic theorem within* ε if $|n^{-1}\sum_{i=0}^{n-1} \chi_{P_j}(T^i x) - \nu P_j| < \varepsilon$ for each $P_j \in P$.

1991 *Mathematics Subject Classification*: Primary 28D15.
This research is supported in part by BSRI 96-1441 and KOSEF 95-0701-03-3.

We define two flows $(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$ and $(\Omega_2,\mathcal{L}_2,\lambda_2,S_2^t)$ to be *evenly α-equivalent* if there exists $(X,\mathcal{F},\mu,T)$ such that both flows can be represented under functions taking values 1 and $1+\alpha$ only with $(X,\mathcal{F},\mu,T)$ as a cross section. That is, $(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$ is isomorphic to $[X,\mathcal{F},\mu,T,f_1]$ and $(\Omega_2,\mathcal{L}_2,\lambda_2,S_2^t)$ is isomorphic to $[X,\mathcal{F},\mu,T,f_2]$ where each of f_1 and f_2 takes values 1 and $1+\alpha$ only. We require $\int f_1\,d\mu=\int f_2\,d\mu$ for "even" equivalence. Even α-equivalences between two transformations $(X_1,\mathcal{F}_1,\mu_1,T_1)$ and $(X_2,\mathcal{F}_2,\mu_2,T_2)$ are likewise defined. We say two $\mathbb{Z}$-actions $(X_1,\mathcal{F}_1,\mu_1,T_1)$ and $(X_2,\mathcal{F}_2,\mu_2,T_2)$ are *evenly α-equivalent* if there exists a flow $(\Omega,\mathcal{L},\lambda,S^t)$ such that $(\Omega,\mathcal{L},\lambda,S^t)$ is isomorphic to $[X_1,\mathcal{F}_1,\mu_1,T_1,f_1]$ and $[X_2,\mathcal{F}_2,\mu_2,T_2,f_2]$ where f_1 and f_2 are functions taking values 1 and $1+\alpha$ only. This is proven to be an equivalence relation, which is stronger than even Kakutani equivalence [Pa3], [dJFR].

We will define α-equivalence of $\mathbb{R}$-actions and $\mathbb{Z}$-actions and prove that it is in fact an equivalence relation. It will be clear that α-equivalence vs. even α-equivalence is analogous to Kakutani equivalence vs. even Kakutani equivalence. As in the case of even α-equivalence, it is not difficult to show that Kakutani equivalence can be described via α- and β-equivalences. That is, if $(X_1,\mathcal{F}_1,\mu_1,T_1)$ and $(X_2,\mathcal{F}_2,\mu_2,T_2)$ are Kakutani equivalent, then there exists $(X_3,\mathcal{F}_3,\mu_3,T_3)$ such that $(X_3,\mathcal{F}_3,\mu_3,T_3)$ is α-equivalent to $(X_1,\mathcal{F}_1,\mu_1,T_1)$ and β-equivalent to $(X_2,\mathcal{F}_2,\mu_2,T_2)$ [Pa2].

2. α-equivalence

(I) *Flow case.* Even α-equivalence, in the case of an $\mathbb{R}$-action, can be described via the change of orbits in a measurable way through inserting and/or cutting out intervals whose lengths are multiples of α. Two flows having the same measure (the same integral of the ceiling functions) means that the measure of the inserted set is the same as the measure of the set removed.

By relaxing the condition that the measures of the sets being added and being removed are the same, we get a different equivalence relation. When we say a flow is a *$\{1,1+\alpha\}$-flow*, we indicate it is represented as a flow under a function taking values 1 and $1+\alpha$ only. We say that a flow $(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$ is an *α-change* of another flow S_2^t if S_1^t is isomorphic to a flow whose orbits are obtained from orbits of S_2^t by inserting or taking out intervals whose lengths are multiples of α. We note that S_1^t and S_2^t may not be represented as $\{1,1+\alpha\}$-flows over the same base: Suppose S_2^t is an $\{1,1+\alpha\}$-flow over $(X_2,\mathcal{F}_2,\mu_2,T_2)$. If S_1^t is obtained from S_2^t by removing a set of measure greater than αu where $u=\mu(\{x: f(x)=1+\alpha\})$, then it is clear that the induced flow S_1^t cannot be represented as a $\{1,1+\alpha\}$-flow over $(X_2,\mathcal{F}_2,\mu_2,T_2)$ because any $\{1,1+\alpha\}$-flow over $(X_2,\mathcal{F}_2,\mu_2,T_2)$ has to have integral greater than $\mu_2(X_2)$.

For the flow equivalence of this section, we assume that a flow acts on a probability space. We need to fix the speed of the flow to discuss the equivalence. We say that a measure $\widehat{\mu}$ on X is a *rescale* of μ on X if $\widehat{\mu}E=r\mu E$ for a fixed real number r and for all $E\in\mathcal{F}$.

We say that a flow $(\Omega,\mathcal{L},\lambda,S^t)$ has a *generalized α-flow representation* over $(X,\mathcal{F},\mu,T)$ if $(\Omega,\mathcal{L},\lambda,S^t)$ is isomorphic to $[X,\mathcal{F},\mu,T,f]$ where $f(x)=1+k\alpha$ for $k=0,1,2,\ldots$ In particular, if $(\Omega,\mathcal{L},\lambda,S^t)$ is a $\{1,1+\alpha\}$-flow over $(X,\mathcal{F},\mu,T)$, then it can be considered as a generalized α-flow over $(X,\mathcal{F},\mu,T)$. Henceforth we assume that all ceiling functions only take values of the form $1+k\alpha$ for $k=0,1,2,\ldots$

We recall that an ergodic flow has a representation under a function with values $1+k\alpha$, $k=0,1,2,\ldots$ (see [Ru1]). Let $(X_1,\mathcal{F}_1,\mu_1,T_1)$ and $(X_2,\mathcal{F}_2,\mu_2,T_2)$ denote the cross sections in generalized α-flow representations of $(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$ and $(\Omega_2,\mathcal{L}_2,\lambda_2,S_2^t)$, respectively. Motivated by the observations above, we make the following

DEFINITION 2.1. We say that two flows $(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$ and $(\Omega_2,\mathcal{L}_2,\lambda_2,S_2^t)$ are *α-related* if there exists a flow $(\widehat{\Omega},\widehat{\mathcal{L}},\widehat{\lambda},\widehat{S})$ which can be represented as a generalized α-flow over $(X_1,\mathcal{F}_1,\widehat{\mu}_1,T_1)$ and $(X_2,\mathcal{F}_2,\widehat{\mu}_2,T_2)$ where $\widehat{\mu}_1$ and $\widehat{\mu}_2$ are rescales of μ_1 and μ_2 respectively.

REMARK 2.1. Note that $\mu_1(X_1)$ and $\mu_2(X_2)$ are less than 1, because we assume $\lambda_1(\Omega_1)=\lambda_2(\Omega_2)=1$.

REMARK 2.2. Note that $[X_1,\mathcal{F}_1,\widehat{\mu}_1,T_1,f_1]$ is an α-change of $[X_1,\mathcal{F}_1,\widehat{\mu}_1,T_1,\widehat{f}_1]$ and $[X_2,\mathcal{F}_2,\widehat{\mu}_2,T_2,f_2]$ is an α-change of $[X_2,\mathcal{F}_2,\widehat{\mu}_2,T_2,\widehat{f}_2]$ where $\widehat{f}_i$'s denote the ceiling functions in the representations of $(\widehat{\Omega},\widehat{\mathcal{L}},\widehat{\lambda},\widehat{S}^t)$.

We have the following theorem.

THEOREM 2.1. *The α-relation is an equivalence relation.*

Proof. Assume that $(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$ is α-related to $(\Omega_2,\mathcal{L}_2,\lambda_2,S_2^t)$ and $(\Omega_2,\mathcal{L}_2,\lambda_2,S_2^t)$ is α-related to $(\Omega_3,\mathcal{L}_3,\lambda_3,S_3^t)$. Let $(X_1,\mathcal{F}_1,\mu_1,T_1)$, $(X_2,\mathcal{F}_2,\mu_2,T_2)$ and $(X_3,\mathcal{F}_3,\mu_3,T_3)$ be the bases in the representations of $(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$, $(\Omega_2,\mathcal{L}_2,\lambda_2,S_2^t)$ and $(\Omega_3,\mathcal{L}_3,\lambda_3,S_3^t)$ as generalized α-flows respectively. Let $(\widehat{\Omega},\widehat{\mathcal{L}},\widehat{\lambda},\widehat{S}^t)$ denote a flow which can be represented as a generalized α-flow over $(X_1,\mathcal{F}_1,\widehat{\mu}_1,T_1)$, $(X_2,\mathcal{F}_2,\widehat{\mu}_2,T_2)$. We denote by $(\widetilde{\Omega},\widetilde{\mathcal{L}},\widetilde{\lambda},\widetilde{S}^t)$ the flow which can be represented as a generalized α-flow over $(X_2,\mathcal{F}_2,\widetilde{\mu}_2,T_2)$ and $(X_3,\mathcal{F}_3,\widetilde{\mu}_3,T_3)$ where $\widehat{\mu}_i$ and $\widetilde{\mu}_i$ are suitable rescales of the respective measures.

Let $f(x)=\max\{\widehat{f}_2(x),\widetilde{f}_2(x)\}$ where $\widehat{f}_2$ and $\widetilde{f}_2$ denote the ceiling functions of $\widehat{S}^t$ and $\widetilde{S}^t$ over $(X_2,\mathcal{F}_2,\widehat{\mu}_2,T_2)$ and $(X_2,\mathcal{F}_2,\widetilde{\mu}_2,T_2)$, respectively. We may assume $\widehat{f}_2$ and $\widetilde{f}_2$ are different. It is enough to show that the flow on $\Omega=\{(x,t): x\in X_2,\ 0\le t<f(x)\}$ with a suitably rescaled measure

$\overline{\mu}_2$ on X_2 satisfying $\int f\,d\overline{\mu}_2 = 1$ has a representation as a generalized α-flow over X_1 and X_3. We note that the values of $f(x)$ are also of the form $1+k\alpha$, $k = 0,1,2,\ldots$ We will also find the respective measures on X_1 and X_3.

We denote by Q the partition of X_2 according to the heights of $\widehat{f}_2$, $\widetilde{f}_2$ and f. That is, x and y are in the same atom of the partition Q if and only if $\widehat{f}(x) = \widehat{f}(y)$, $\widetilde{f}(x) = \widetilde{f}(y)$ and $f(x) = f(y)$. We now build a skyscraper of X_2. We assume that the skyscraper is long enough so that the names of the base of the skyscraper satisfy the ergodic theorem within small ε with respect to the partition Q. We partition the base of the skyscraper according to the Q-names along the tower. We call a tower over each atom of the base a *column*. We sometimes refer to an atom of the base by the name of the column. We also build a flow skyscraper of $\widehat{\Omega}$ using the base skyscraper of X_2. A *flow column* is naturally defined. Each level set of X_2, which we call an X_2-*cut*, has height $1+k\alpha$ for some $k = 0,1,\ldots$ By the *height* of a level set, we mean the distance from the level set to the level set directly above it in the flow column.

Let φ be an isomorphism between $[X_1,\mathcal{F}_1,\widehat{\mu}_1,T_1,\widehat{f}_1]$ and $[X_2,\mathcal{F}_2,\widehat{\mu}_2, T_2,\widehat{f}_2]$. Without confusion we denote the image $\varphi(X_1)$ of X_1 in $[X_2,\mathcal{F}_2,\widehat{\mu}_2, T_2,\widehat{f}_2]$ by X_1. We also refer to the level sets of X_1 as X_1-*cuts*. By subdividing the columns if necessary, we may assume that the 0th X_1-cut in a column has a unique P-name where P is the partition of X_1 according to the values of $\widehat{f}_1$. Hence each X_1-cut in a column has constant height $1+k\alpha$ for some k. We also assume that the skyscraper is long enough so that the P-name of the 0th X_1-cut in each column satisfies the ergodic theorem within ε.

We also build the flow skyscraper of $[X_2,\mathcal{F}_2,\overline{\mu}_2,T_2,\widehat{f}_2]$ using the skyscraper of X_2. The difference between the tower of $[X_2,\mathcal{F}_2,\widehat{\mu}_2,T_2,\widehat{f}_2]$ and the tower of $[X_2,\mathcal{F}_2,\overline{\mu}_2,T_2,\widehat{f}_2]$ is the measure of the bottom level set of each column of X_2. That is, for each bottom level set B of the skyscraper of X_2, $\overline{\mu}_2(B) = r\widehat{\mu}_2(B)$ if $\overline{\mu}_2(X_2) = r\widehat{\mu}_2(X_2)$. Since $[X_1,\mathcal{F}_1,\widehat{\mu}_1,T_1,\widehat{f}_1]$ and $[X_2,\ \mathcal{F}_2,\ \widehat{\mu}_2,\ T_2,\ \widehat{f}_2]$ are isomorphic, there exists an isomorphism between $[X_1,\mathcal{F}_1,r\widehat{\mu}_1,T_1,\widehat{f}_1]$ and $[X_2,\mathcal{F}_2,\overline{\mu}_2,T_2,\widehat{f}_2]$.

Let $l_1\alpha$ denote the difference between the heights of the first flow columns of $[X_2,\mathcal{F}_2,\overline{\mu}_2,T_2,f]$ and $[X_2,\mathcal{F}_2,\overline{\mu}_2,T_2,\widehat{f}_2]$. Since the skyscraper satisfies the ergodic theorem with respect to the partition Q within ε for sufficiently small ε, we may assume that the height of each flow column of $[X_2,\mathcal{F}_2,\mu_2,T_2,f]$ is greater than the height of each flow column of $[X_2,\mathcal{F}_2,\overline{\mu}_2,T_2,\widehat{f}_2]$ and their differences are multiples of α. We raise the flow height by $l_1\alpha$ to the first flow column of $[X_2,\mathcal{F}_2,\overline{\mu}_2,T_2,\widehat{f}_2]$ and push up the top X_1-cut by $l_1\alpha$. The height of the X_1-cut located right below the top X_1-cut is raised by $l_1\alpha$. We repeat this for each column and the new flow is isomorphic to $[X_2,\mathcal{F}_2,\overline{\mu}_2,T_2,f] = (\Omega,\mathcal{L},\lambda,S^t)$. Now it is clear that the flow $(\Omega,\mathcal{L},\lambda,S^t)$ has a generalized α-flow representation over $(X_1,\mathcal{F}_1,r\widehat{\mu}_1,T_1)$.

Likewise the flow S^t can have a representation as a generalized α-flow over the base $(X_3,\mathcal{F}_3,\widetilde{r}\,\widetilde{\mu}_3,T_3)$ for some $\widetilde{r}$. Hence S_1^t and S_3^t are α-related.

REMARK 2.3. Although l_i depends on the columns and the number of columns may be infinite, we may choose a representation of S^t over $(X_1,\mathcal{F}_1,r\widehat{\mu}_1,T_1)$ under a bounded function whose values are of the form $1+k\alpha$, $k = 0,1,\ldots,K$, for some K. We can do this by spreading out the added extra height $l_i\alpha$ to many different X_1-cuts in the column instead of putting the whole extra height on the X_1-cut just below the top X_1-cut. We accomplish this by successively pushing up the necessary X_1-cuts by multiples of α. Since we assume that each column satisfies the ergodic theorem within ε, $\{l_i/q_i\}$ is bounded where q_i denotes the height of the ith column of the skyscraper of X_2.

COROLLARY 2.2. *$(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$ and $(\Omega_2,\mathcal{L}_2,\lambda_2,S_2^t)$ are α-equivalent if and only if there exists a discrete action $(X,\mathcal{F},\mu,T)$ such that S_1^t is isomorphic to a generalized α-flow $[X,\mathcal{F},\widehat{\mu},T,f_1]$ and S_2^t is isomorphic to a generalized α-flow $[X,\mathcal{F},\widehat{\widehat{\mu}},T,f_2]$ where $\widehat{\mu}$ and $\widehat{\widehat{\mu}}$ are rescales of μ.*

Proof. We let $(X_1,\mathcal{F}_1,\mu_1,T_1)$ and $(X_2,\mathcal{F}_2,\mu_2,T_2)$ be bases in the representation of $(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$ and $(\Omega_2,\mathcal{L}_2,\lambda_2,S_2^t)$ as generalized α-flows, respectively. We denote the ceiling functions by g_1 and g_2 respectively. Let $(\widehat{\Omega},\widehat{\mathcal{L}},\widehat{\lambda},\widehat{S}^t)$ be a flow which can be represented as a generalized α-flow as $[X_1,\mathcal{F}_1,\widehat{\mu}_1,T_1,\widehat{f}_1]$ and $[X_2,\mathcal{F}_2,\widehat{\mu}_2,T_2,\widehat{f}_2]$.

Let $[X,\mathcal{F},\mu,T,f]$ denote a representation of $(\widehat{\Omega},\widehat{\mathcal{L}},\widehat{\lambda},\widehat{S}^t)$ as a generalized α-flow. We may choose the base $(X,\mathcal{F},\mu,T)$ so that

$$\mu(X) < \min\{\widehat{\mu}_1(X_1),\widehat{\mu}_2(X_2)\}$$

(see [Ru1]). We claim that $(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$ and $(\Omega_2,\mathcal{L}_2,\lambda_2,S_2^t)$ can be represented as generalized α-flows over $(X,\mathcal{F},T)$ with suitably rescaled measures.

We denote by P the partition of X_1 according to the heights of the ceiling function g_1. We build a long enough skyscraper of $(X_1,\mathcal{F}_1,\mu_1,T_1)$ and divide it into columns so that each column has a unique P-name. We build a flow skyscraper of $(\Omega_1,\mathcal{L}_1,\lambda_1,S_1^t)$ using the skyscraper of $(X_1,\mathcal{F}_1,\mu_1,T_1)$, and also the flow skyscraper of $(\widehat{\Omega},\widehat{\mathcal{L}},\widehat{\lambda},\widehat{S}^t)$ using the skyscraper of $(X_1,\mathcal{F}_1, \widehat{\mu}_1,T_1)$.

We use an isomorphism $\varphi : [X,\mathcal{F},\mu,T,f] \to [X_1,\mathcal{F}_1,\widehat{\mu}_1,T_1,\widehat{f}_1]$ so that $[X,\mathcal{F},\mu,T,f]$ has an isomorphic image in the flow skyscraper of $[X_1,\mathcal{F}_1, \widehat{\mu}_1,T_1,\widehat{f}_1]$. We subdivide each column, if necessary, so that the 0th X-cut contained in each column has a unique Q-name where Q is the partition of X according to the heights of f. Let h and $\widehat{h}$ denote the heights of the first flow columns of $[X_1, \mathcal{F}_1, \mu_1, T_1, g_1]$ and $[X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1, \widehat{f}_1]$, and let $l_1\alpha$ denote the difference $\widehat{h} - h$. We may assume $l_1 > 0$. We denote the height of the ith X-cut by $1 + \tau_i\alpha$. If $0 < \tau_0 < l_1$, then we push down all the X-cuts above the 0th X-cut by $\tau_0\alpha$. We keep pushing down the X-cuts successively above the ith X-cut by $\tau_i\alpha$ for $i = 0, 1, \ldots, i_1$, where i_1 is the largest integer satisfying

$$\sum_{i=0}^{i_1} \tau_i < l_1.$$

Let j_1 be the first X-cut above the i_1th X-cut with $\sum_{i=0}^{j_1} \tau_i \geq l_1$.

We push down all the X-cuts above the j_1th X-cut by $(l_1 - \sum_{i=0}^{i_1} \tau_i)\alpha$ so that the top X-cut now has height $l_1\alpha$ greater than before. We remove the top section of length $l_1\alpha$ from the column. The height of the new flow column is the same as the height of the first flow column of $[X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1, g_1]$. If $l_1 < 0$, then we add the extra height $l_1\alpha$ to the first column of $[X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1, \widehat{f}_1]$ so that the new flow column is the same as the corresponding flow column of $[X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1, g_1]$. The top X-cut in the column has height $l_1\alpha$ greater than before.

We repeat this for each column so that the flow $[X_1, \mathcal{F}_1, \mu_1, T_1, g_1]$ is represented as a generalized α-flow over $(X, \mathcal{F}, \widehat{\mu}, T)$, where $\widehat{\mu}E = r\mu E$ for all $E \in \mathcal{F}$ if $\widehat{\mu}_1 F = (1/r)\mu_1 F$ for all $F \in \mathcal{F}_1$.

In the case $l_1 > 0$, we need to check that we have enough X-cuts of height greater than 1 so that we can reduce the height of the column by $l_1\alpha$. Recall that each cut has height $1 + k\alpha$ for some $k = 0, 1, \ldots$ Let $m + n\alpha$ denote the height between the 0th X-cut and the top X-cut in the column. Let $m_1 + n_1\alpha$ denote the height of the first column of $[X_1, \mathcal{F}_1, T_1, \widehat{\mu}_1, \widehat{f}_1]$. By our choice of $\mu(X)$ which is smaller than $\widehat{\mu}_1(X_1)$, we may assume that $n > n_1$. Since we reduce the height by at most $n_1\alpha$, we have enough X-cuts whose heights can be reduced by multiples of α.

For the other direction, we let the height be $f(x) = \max\{f_1(x), f_2(x)\}$. Since $(\Omega_1, \mathcal{F}_1, \lambda_1, S_1^t)$ is isomorphic to $[X_1, \mathcal{F}_1, \mu_1, T_1, g_1]$ and $[X, \mathcal{F}, \widehat{\mu}, T, f_1]$, it is not hard to show that $[X, \mathcal{F}, \widehat{\mu}, T, f]$ has a representation $[X_1, \mathcal{F}_1, \mu_1, T_1, h_1]$ where h_1 and g_1 differ by multiples of α. Likewise $[X, \mathcal{F}, \widehat{\widehat{\mu}}, T, f]$ has a representation $[X_2, \mathcal{F}_2, \mu_2, T_2, h_2]$ where h_2 and g_2 differ by multiples of α. If we choose a rescale $\widetilde{\mu}$ of μ so that $\int f\,d\widetilde{\mu} = 1$, then the flow $[X, \mathcal{F}, \widetilde{\mu}, T, f]$ has representations $[X_1, \mathcal{F}_1, \widetilde{\mu}_1, T_1, h_1]$ and $[X_2, \mathcal{F}_2, \widetilde{\mu}_2, T_2, h_2]$ where $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are rescales of μ_1 and μ_2 respectively satisfying $\int h_1 d\widetilde{\mu}_1 = 1 = \int h_2\,d\widetilde{\mu}_2$.

Before proceeding to the discrete case, we would like to compare α-equivalence and 2α-equivalence. S_1^t and S_2^t are 2α-equivalent if and only if S_1^t and S_2^t are orbit changes of each other where insertions in and removals from an orbit are done by means of intervals whose lengths are multiples

of 2α. Clearly if S_1^t and S_2^t are 2α-equivalent, then they are α-equivalent. We see that 2α-equivalence is finer than α-equivalence as follows: If S_1^t has a periodic function with period 2α, that is, if there exists a function $p(x)$ such that $S_1^{2\alpha}p(x) = p(S^{2\alpha}x) = p(x)$ for all x, and S_2^t is 2α-equivalent to S_1^t, then there exists a periodic function $q(x)$ under S_2^t with period 2α. However, the existence of a periodic function of period 2α is not an α-equivalence invariant.

REMARK 2.4. If S_1^t and S_2^t are α-equivalent and there exists $f_1(\omega_1)$ such that $S_1^t f_1(\omega_1) = e^{(2\pi ik/\alpha)t} f_1(\omega_1)$ for all $\omega_1 \in \Omega_1$, then it is clear from our Corollary 2.2 that there exists f_2 such that $S_2^t f_2(\omega_2) = e^{(2\pi ik/\alpha)t} f_2(\omega_2)$ for all $\omega_2 \in \Omega_2$.

(II) *Transformation case*. In this section we define α-equivalence of discrete actions using the α-equivalence of continuous actions, and we characterize these properties. A *generalized α-cross section* means a cross section in a representation of a flow as a generalized α-flow. For this part (II), we assume that $(X_i, \mathcal{F}_i, \mu_i)$ is a probability space for $i = 1, 2$, or 3.

DEFINITION 2.2. We say $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$ are *α-related* if $(X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1)$ is isomorphic to a generalized α-cross section of $(\Omega_1, \mathcal{L}_1, \lambda_1, S_1^t)$ and $(X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2)$ is isomorphic to a generalized α-cross section of $(\Omega_2, \mathcal{L}_2, \lambda_2, S_2^t)$ where $(\Omega_1, \mathcal{L}_1, \lambda_1, S_1^t)$ and $(\Omega_2, \mathcal{L}_2, \lambda_2, S_2^t)$ are α-equivalent flows.

REMARK 2.5. If $(X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1)$ and $(X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2)$ are isomorphic to generalized α-cross sections of a flow $(\Omega, \mathcal{L}, \lambda, S^t)$, then clearly they are α-related.

The following theorem is more or less obvious from the definition.

THEOREM 2.3. *The α-relation is an equivalence relation.*

Proof. Let $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$ be isomorphic to generalized α-cross sections of $(\Omega_1, \mathcal{L}_1, \lambda_1, S_1^t)$ and $(\Omega_2, \mathcal{L}_2, \lambda_2, S_2^t)$ respectively. We also let $(X_2, \mathcal{F}_2, \mu_2, T_2)$ and $(X_3, \mathcal{F}_3, \mu_3, T_3)$ be isomorphic to generalized α-cross sections of $(\Omega_2', \mathcal{L}_2', \lambda_2', S_2^{t\prime})$ and $(\Omega_3, \mathcal{L}_3, \lambda_3, S_3^t)$. Note that $(\Omega_2, \mathcal{L}_2, \lambda_2, S_2^t)$ and $(\Omega_2', \mathcal{L}_2', \lambda_2', S_2^{t\prime})$ are α-equivalent. Since $(\Omega_1, \mathcal{L}_1, \lambda_1, S_1^t)$ and $(\Omega_2, \mathcal{L}_2, \lambda_2, S_2^t)$ are α-equivalent and $(\Omega_2', \mathcal{L}_2', \lambda_2', S_2^{t\prime})$ and $(\Omega_3, \mathcal{L}_3, \lambda_3, S_3^t)$ are α-equivalent, $(\Omega_1, \mathcal{L}_1, \lambda_1, S_1^t)$ and $(\Omega_3, \mathcal{L}_3, \lambda_3, S_3^t)$ are α-equivalent. Hence $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_3, \mathcal{F}_3, \mu_3, T_3)$ are α-related.

COROLLARY 2.4. *If $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$ are α-equivalent, then there exists a flow $(\Omega, \mathcal{L}, \lambda, S^t)$ which can be represented as a generalized α-flow over both $(X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1)$ and $(X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2)$ where $\widehat{\mu}_1$ and $\widehat{\mu}_2$ are rescales of μ_1 and μ_2 respectively.*

Proof. This is a consequence of the definition of α-equivalence of flows. Let $(\Omega_1, \mathcal{L}_1, \lambda_1, S_1^t)$ and $(\Omega_2, \mathcal{L}_2, \lambda_2, S_2^t)$ be α-equivalent flows whose generalized α-cross sections are $(X_1, \mathcal{F}_1, \widetilde{\mu}_1, T_1)$ and $(X_2, \mathcal{F}_2, \widetilde{\mu}_2, T_2)$ respectively. By the definition of α-equivalence of flows, there exists a flow $(\Omega, \mathcal{L}, \lambda, S^t)$ which can be represented as a generalized α-flow over each of $(X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1)$ and $(X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2)$.

REMARK 2.6. If $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$ are α-equivalent, then $\min\{k \in \mathbb{Z} : e^{2\pi i k/\alpha} = e^{2\pi i/(\alpha/k)}$ is in the point spectrum of $T_1\} = \min\{k \in \mathbb{Z} : e^{2\pi i k/\alpha}$ is in the point spectrum of $T_2\}$.

THEOREM 2.5. *If $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$ are α-equivalent and $h(T_1) > h(T_2)$, then any generalized α-flow over T_2 can be built as a generalized α-flow over T_1.*

Proof. First we prove that the flow on $\Omega_2 = X_2 \times [0, 1)$ can be built as a generalized α-flow over T_1. Let $(\widehat{\Omega}, \widehat{\mathcal{L}}, \widehat{\lambda}, \widehat{S}^t)$ denote a flow which has a generalized α-flow representation over each of $(X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1)$ and $(X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2)$. We assume that the ceiling functions $\widehat{f}_1$ and $\widehat{f}_2$ are bounded. Let $\widehat{P} = \{\widehat{P}_0, \widehat{P}_1, \ldots, \widehat{P}_l\}$ denote the partition of X_1 according to the heights of $\widehat{f}_1$, and $\widehat{Q} = \{\widehat{Q}_0, \widehat{Q}_1, \ldots, \widehat{Q}_{l'}\}$ denote the partition of X_2 according to the heights of $\widehat{f}_2$.

Build a skyscraper of X_2 and divide the skyscraper so that each column has a unique $\widehat{Q}$-name. We build the flow skyscraper of $\widehat{\Omega}$ using the skyscraper of X_2. Let φ be an isomorphism from $[X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1, \widehat{f}_1]$ to $[X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2, \widehat{f}_2]$. We subdivide the columns, if necessary, so that the bottom X_1-cut in each column has a unique $\widehat{P}$-name.

Let $m_2 + n_2\alpha$ denote the height of the first flow column and $m_1 + n_1\alpha$ denote the height of the section of the first column from the 0th X_1-cut till the top X_1-cut. Since $h(T_2) < h(T_1)$, by Abramov's formula we have $\widehat{\mu}_1(X_1) < \widehat{\mu}_2(X_2)$. If we assume that the skyscraper is sufficiently long, since m_2 and m_1 are the numbers of the respective cuts in the column, we have $m_2 > m_1$. Hence we may assume $n_1 > n_2$.

We move down the X_2-cuts one by one reducing the heights of X_2-cuts to 1 until we have the top X_2-cut of height $1 + n_2\alpha$. We also push down X_1-cuts by multiples of α successively so that we have an at least $1 + n_2\alpha$ long flow section at the top without X_1-cuts. We remove the top flow section of height $n_2\alpha$. Note that when we move down a cut by $\tau\alpha$, the height of the shifted cut becomes $\tau\alpha$ greater.

If we repeat this for each column, then the new flow columns form a new flow skyscraper which is isomorphic to $[X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2, \widehat{f}]$ where $\widehat{f} \equiv 1$. Since $n_1 > n_2$, we have enough X_1-cuts to shorten the heights and the flow can be built as a generalized α-flow over $(X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1)$. Hence $[X_2, \mathcal{F}_2, \mu_2, T_2, \widehat{f}]$

can be built as a generalized α-flow over $(X_1, \mathcal{F}_1, \widetilde{\mu}_1, T_1)$, where $\widetilde{\mu}_1$ is an appropriate rescale of $\widehat{\mu}_1$.

Let $(\Omega, \mathcal{L}, \lambda, S^t)$ denote a generalized α-flow over T_2 and Q denote the partition of X_2 according to the heights of the ceiling function f. Using the skyscraper of $(X_2, \mathcal{F}_2, \mu_2, T_2)$, we build the flow skyscraper of $[X_2, \mathcal{F}_2, \mu_2, T_2, f]$. We subdivide each column so that the bottom X_2-cut in each column has a unique Q-name. We consider the isomorphic image of $[X_1, \mathcal{F}_1, \widetilde{\mu}_1, T_1, \widetilde{f}]$ in the flow skyscraper of $[X_2, \mathcal{F}_2, \mu_2, T_2, \widehat{f}]$. Let $m_2' + n_2'\alpha = \sum_{i=0}^{m_2'-1} f(T_2^i x)$ where x is a point in the 0th X_2-cut in a column and m_2' denotes the number of X_2-cuts in the column. The difference in the heights of the first flow columns of $[X_2, \mathcal{F}_2, \mu_2, T_2, \widehat{f}]$ and $[X_2, \mathcal{F}_2, \mu_2, T_2, f]$ is $n_2'\alpha$. By adding an extra flow section of length $n_2'\alpha$ to the first column of $[X_2, \mathcal{F}_2, \mu_2, T_2, \widehat{f}]$, we get the first flow column of $[X_2, \mathcal{F}_2, \mu_2, T_2, f]$. We lengthen the column by $n_2'\alpha$ by adding a flow section of length $n_2'\alpha$ at the top.

If we repeat this for each column, then it is clear that the new flow skyscraper is isomorphic to $[X_2, \mathcal{F}_2, \mu_2, T_2, f]$. By putting the added length to the height of the top X_1-cut, the flow $[X_2, \mathcal{F}_2, \mu_2, T_2, f]$ can be built as a generalized α-flow over $(X_1, \mathcal{F}_1, \widetilde{\mu}_1, T_1)$. Hence the flow $[X_2, \mathcal{F}_2, \widetilde{\mu}_2, T_2, f]$ where $\widetilde{\mu}_2$ is a rescale of μ_2 satisfying $\int f\, d\widetilde{\mu}_2 = 1$ can be built as a generalized α-flow over $(X_1, \mathcal{F}_1, \widetilde{\widetilde{\mu}}_1, T_1)$ for some rescale $\widetilde{\widetilde{\mu}}_1$ of $\widetilde{\mu}_1$.

REMARK 2.8. If we want to represent $(\Omega, \mathcal{L}, \lambda, S^t)$ under a bounded function, then we need to spread out the added length of each column to more X_1-cuts instead of putting the whole added length to the top X_1-cut.

REMARK 2.9. Given a transformation T, it is not hard to see that each entropy class contains a transformation which is α-equivalent to T.

REMARK 2.10. Since there are uncountably many evenly Kakutani equivalent maps no two of which are α-equivalent, there are uncountably many Kakutani equivalent maps no two of which are α-equivalent [dJFR].

Given two points x and y which are in an orbit equivalence class of T, we let $T(x, y) = n$ if $y = T^n x$. We let $[\beta]$ denote the nearest integer to β, and $\|\beta\|$ denote the difference between β and its nearest integer. We reformulate the α-equivalence in terms of the orbit equivalence map.

THEOREM 2.6. *The following are equivalent:*

(1) *T_1 and T_2 are α-equivalent.*

(2) *For any $\varepsilon > 0$, there exist subsets $A \subset X_1$ and $B \subset X_2$ and a map $U : A \to B$ such that*

$$(*) \qquad \left\| \frac{1}{\alpha}(T_1(x, y) - T_2 \circ U(x, y)) \right\| < \varepsilon \quad \textit{for all } x, y \in A$$

where U is an isomorphism with suitably rescaled measures on A and B.

Proof. (1)⇒(2). Let $(\Omega, \mathcal{L}, \lambda, S^t)$ denote a flow which can be built as a generalized α-flow over $(X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1)$ and $(X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2)$. Without loss of generality we may assume $\widehat{\mu}_1(X_1) < \widehat{\mu}_2(X_2)$. Let φ be an isomorphism from $[X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2, f_2]$ to $[X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1, f_1]$. We denote the φ-image of X_2 in $[X_1, \mathcal{F}_1, \widehat{\mu}_1, T_1, f_1]$ again by X_2. Given $\delta < \alpha\varepsilon/2$, find $t_0 \in \mathbb{R}$ and a subset $A \subset X_1$ of positive measure such that

$$r_{X_2}(x) \le r_{A(x)} \quad \text{and} \quad |r_{X_2}(x) - t_0| < \delta$$

where $r_B(y) = \min\{t \ge 0 : S^t(y) \in B\}$ for any B and y. Then $U(x) = S^{r_{X_2}(x)}(x)$ is an isomorphism between A and $U(A) = B$ with the measure $\widehat{\mu}_2$ on B and $\widehat{\mu}_1$ on A, which makes the map U measure preserving. By the choice of A and the definition of U, for all $x, y \in A$, if $y = S^l(x)$ and $U(y) = S^{l'}(Ux)$, then $|l - l'| < 2\delta$. Since both l and l' are linear sums of 1's and α's,

$$\begin{aligned}|l - l'| &= |(T_1(x,y) + n\alpha) - (T_2(Ux, Uy) + m\alpha)| \\ &= |T_1(x,y) - T_2 \circ U(x,y) + (n-m)\alpha|.\end{aligned}$$

Hence

$$\left\|\frac{1}{\alpha}(T_1(x,y) - T_2 \circ U(x,y))\right\| = \frac{1}{\alpha}|l - l'| < \frac{2\delta}{\alpha} < \varepsilon.$$

(2)⇒(1). Although the proof is analogous to that of Proposition 2.2 of [dJFR], we reproduce it here for completeness. We may assume that $\mu_1(A) = r\mu_2(B)$, where $r < 1$. We extend the definition of $\widehat{\mu}_2$ to all elements of $\mathcal{F}_2$ by $\widehat{\mu}_2(E) = r\mu_2(E)$. Let $[X_1, \mathcal{F}_1, \mu_1, T_1, f_1]$ be a generalized α-flow where $\int f_1\, d\mu_1 = 1 + \gamma$ for some number γ. We show that $[X_1, \mathcal{F}_1, \mu_1, T_1, f_1]$ can be constructed as a generalized α-flow over $(X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2)$.

Let $y = T_1^{n(y)}x$ be the point of first return to A. We may assume that

(i) $$g_1(x) = \sum_{i=0}^{n(y)-1} f_1(T_1^i x) = T_1(x,y) + l\alpha = n(y) + l\alpha,$$

where $(\gamma/\alpha - \varepsilon)n(y) < l < (\gamma/\alpha + \varepsilon)n(y)$.

Since we assume that $\mu_2(B) > \mu_1(A)$, we take a subset of A if necessary and the corresponding subset of B so that

(ii) $$T_1(x,y) > T_2 \circ U(x,y) = T_2(Ux, Uy).$$

Moreover, we may assume that

$$\begin{aligned}\left(\frac{1}{r} - 1\right)(1-\varepsilon)T_2 \circ U(x,y) &< T_1(x,y) - T_2 \circ U(x,y) \\ &< \left(\frac{1}{r} - 1\right)(1+\varepsilon)T_2 \circ U(x,y).\end{aligned}$$

By condition $(*)$, it is easy to see that

$$c(x,y) = \left[\frac{1}{\alpha}\Big(\sum_{i=0}^{n(y)-1} f_1(T_1^i x) - T_2 \circ U(x,y)\Big)\right] = \left[\frac{1}{\alpha}(T_1(x,y) - T_2 \circ U(x,y))\right] + l$$

is a cocycle.

Let τ be the integer satisfying

$$\tau = \left[\frac{2}{\alpha}\left(\frac{1}{r} - 1\right) + \left(\frac{\gamma}{\alpha} + 1\right)\left(\frac{1}{r} + 1\right)\right].$$

We build a flow over $(X_2, \mathcal{F}_2, \mu_2, T_2)$ by defining a ceiling function f_2 on X_2 as follows:

$$f_2(T_2^i(Ux)) = \begin{cases} 1 + 2\tau\alpha, & 0 \le i \le \left[\frac{c(x,y)}{2\tau}\right], \\ 1 + \left(c(x,y) - \left[\frac{c(x,y)}{2\tau}\right]2\tau\right)\alpha, & i = \left[\frac{c(x,y)}{2\tau}\right], \\ 1, & \left[\frac{c(x,y)}{2\tau}\right] + 1 \le i < T_2 \circ U(x,y). \end{cases}$$

To show that the flow is well defined, we need to check that

$$\left[\frac{c(x,y)}{2\tau}\right] + 1 < T_2 \circ U(x,y).$$

It is easy to check that

$$\begin{aligned}\frac{c(x,y)}{2\tau} &\le \frac{1}{2\tau}\left[\frac{1}{\alpha}\Big(\sum_{i=0}^{n(y)-1} f_1(T_1^i x) - T_2 \circ U(x,y)\Big)\right] \\ &\le \frac{1}{2\tau}\left[\frac{1}{\alpha}(T_1(x,y) + l\alpha - T_2 \circ U(x,y))\right] \\ &< \frac{1}{2\tau}\left(\frac{1}{\alpha}(1+\varepsilon)\left(\frac{1}{r} - 1\right)(T_2 \circ U(x,y)) + \left(\frac{\gamma}{\alpha} + \varepsilon\right)n(y)\right) \\ &< \frac{1}{2\tau}\left(\frac{1}{\alpha}(1+\varepsilon)\left(\frac{1}{r} - 1\right)(T_2 \circ U(x,y)) \right. \\ &\qquad \left. + \left(\frac{\gamma}{\alpha} + \varepsilon\right)\left(\frac{1}{r} + \varepsilon\right)(T_2 \circ U(x,y))\right) \\ &= \frac{1}{2\tau}\left(\frac{1}{\alpha}(1+\varepsilon)\left(\frac{1}{r} - 1\right) + \left(\frac{\gamma}{\alpha} + \varepsilon\right)\left(\frac{1}{r} + \varepsilon\right)\right)(T_2 \circ U(x,y)) \\ &< \frac{1}{2}T_2 \circ U(x,y).\end{aligned}$$

What is now left is to prove that $[X_1, \mathcal{F}_1, \mu_1, T_1, f_1]$ and $[X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2, f_2]$ are isomorphic where $\widehat{\mu}_2$ is the given rescale of μ_2. Note that $[X_1, \mathcal{F}_1, \mu_1, T_1, f_1]$ is isomorphic to the flow $[A, \mathcal{F}_1|_A, \mu_1|_A, g_1]$, and $[X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2, f_2]$ is isomorphic to $[B, \mathcal{F}_2|_B, \widehat{\mu}_2|_B, T_2|_B, g_2]$ where

$$g_2(Ux) = c(x,y)\alpha + T_2 \circ U(x,y).$$

By our construction of g_2, we have

$$\begin{aligned} |g_1(x) - g_2(Ux)| &= \Big| \sum_{i=0}^{n(y)-1} f_1(T_1^i x) - c(x,y)\cdot\alpha - T_2 \circ U(x,y) \Big| \\ &= \Big| \sum_{i=0}^{n(y)-1} f_1(T_1^i x) - T_2 \circ U(x,y) - c(x,y)\cdot\alpha \Big| \\ &= \Big| \frac{1}{\alpha}\Big(\sum_{i=0}^{n(y)-1} f_1(T_1^i x) - T_2 \circ U(x,y)\Big) - c(x,y) \Big| \cdot \alpha \\ &< \varepsilon\alpha. \end{aligned}$$

That is, $g_1(x) - g_2(Ux)$ is a bounded cocycle, hence g_1 and g_2 are cohomologous. Therefore $[A, \mathcal{F}_1|_A, \mu_1|_A, T_1|_A, g_1]$ is isomorphic to $[B, \mathcal{F}_2|_B, \widehat{\mu}_2|_B, T_2|_B, g_2]$. We know that $[X_1, \mathcal{F}_1, \mu_1, T_1, f_1]$ is isomorphic to $[X_2, \mathcal{F}_2, \widehat{\mu}_2, T_2, f_2]$. If we take a rescale $\widetilde{\mu}_1$ of μ_1 so that $\int f_1\, d\widetilde{\mu}_1 = 1$, then $[X_1, \mathcal{F}_1, \widetilde{\mu}_1, T_1, f_1]$ is isomorphic to $[X_2, \mathcal{F}_2, \widetilde{\mu}_2, T_2, f_2]$ where $\widetilde{\mu}_2$ is a rescale of $\widehat{\mu}_2$ so that $\int f_2\, d\widetilde{\mu}_2 = 1$.

Hence $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$ are α-equivalent.

We denote the unit circle by π and the rotation by β on the circle by R_β. We have the following corollary whose proof we omit (see [dJFR]).

COROLLARY 2.8. *$(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$ are α-equivalent if and only if there exist subsets $A \subset X_1$ and $B \subset X_2$, not necessarily of the same measure, such that there exists an isomorphism φ between $T_1 \times R_{\alpha^{-1}}|_{A\times\pi}$ and $T_2 \times R_{\alpha^{-1}}|_{B\times\pi}$ of the form*

$$\varphi(x,t) = (Ux, a(x) + t)$$

where U is an isomorphism between A and B with rescaled measures and a is a map from A to π.

REMARK 2.11. Let $(X, \mathcal{F}, \mu, T)$ be a $\mathbb{Z}$-action and let S_{p} denote the point spectrum. It is easy to see that the following are equivalent.

(i) $k\alpha^{-1} \in S_{\mathrm{p}} S^t$ where S^t is a generalized α-flow over $(X, \mathcal{F}, \mu, T)$.
(ii) $k\alpha^{-1} \in S_{\mathrm{p}} T^t$ where T^t is the flow on $X \times [0,1)$.
(iii) $e^{2\pi k/\alpha} \in S_{\mathrm{p}} T$.

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Received July 3, 1996 (3704)
Revised version October 21, 1997