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On a generalization of Lumer–Phillips' theorem for dissipative operators in a Banach space

by

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Abstract. Using [1], which is a local generalization of Gelfand's result for power-bounded operators, we first give a quantitative local extension of Lumer–Phillips' result that states conditions under which a quasi-nilpotent dissipative operator vanishes. Secondly, we also improve Lumer–Phillips' theorem on strongly continuous semigroups of contraction operators.

1. Introduction. Formally, $u(\cdot) = T(\cdot)f$ solves the initial value problem

$$u'(t) = Au(t), \quad u(0) = f, \quad t \geq 0,$$

where A is the generator of the semigroup $\{T_t, t \geq 0\}$. Thus, from the point of view of solving initial value problems (or abstract Cauchy problems), it is natural to ask: which operators A generate $(\mathcal{C}_0)$ semigroups? Conditions on the behaviour of the resolvent of an operator A which are necessary and sufficient for A to be the infinitesimal generator of a $(\mathcal{C}_0)$ semigroup of contractions were given by E. Hille and K. Yosida. A different characterization of the generator of a $(\mathcal{C}_0)$ semigroup was also given by G. Lumer and R. S. Phillips in [7]. In this paper, we give another characterization which improves the latter result. In Section 2, the main result provides certain growth conditions on the iterates of T, which give a local extension of Lumer and Phillips' result on quasi-nilpotent dissipative operators defined in a Banach space X.

2. Results. Let A be a linear operator with domain $D(A)$ in a Banach space X, and let $D(x,1) = \{f \in X^* : f(x) = \|f\| = 1\}$. If for any $x \in D(A)$, there exists an $f \in D(x,1)$ satisfying $\mathrm{Re}(f(Ax)) \leq 0$, then A is called a

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dissipative operator. It is easy to see that A is dissipative if

$$\|tx - Ax\| \geq t\|x\| \quad (t \in \mathbb{R}_+,\ x \in D(A)).$$

Let us denote by $\mathcal{C}_{n!}(A)$ the set of all vectors $x \in D(A^\infty) = \bigcap_{n=1}^\infty D(A^n)$ such that for some $\lambda_x > 0$, $\|A^n x\| \leq \lambda_x^{n+1} n!$ $(n \geq 0)$. We have the following result.

THEOREM 1. *Let A be a closed dissipative operator on a Banach space X. Suppose that $\mathcal{C}_{n!}(A)$ is dense in $D(A)$. Then A generates a strongly continuous semigroup of contraction operators.*

Proof. For any $x \in \mathcal{C}_{n!}(A)$, let

$$S(t)x = \sum_{n=0}^\infty \frac{t^n}{n!} A^n x. \tag{1}$$

The series in (1) converges for all t such that $|t| < \lambda_x^{-1}$. Thus $S(t)x$ is analytic as a function of t in the interval $(-\lambda_x^{-1}, \lambda_x^{-1})$. If $t_0 \in (-\lambda_x^{-1}, \lambda_x^{-1})$, we define

$$S_{t_0}(t)x = \sum_{n=0}^\infty \frac{(t-t_0)^n}{n!} A^n x. \tag{2}$$

The function $t \to S_{t_0}(t)x$ is analytic in $(t_0 - \lambda_x^{-1}, t_0 + \lambda_x^{-1})$. Clearly

$$(S_{t_0}(t)x)^{(n)}_{t=t_0} = A^n x = (S(t)x)^{(n)}_{t=t_0}$$

for $n \geq 0$. Hence for $t = t_0$,

$$S_{t_0}(t)x = S(t)x. \tag{3}$$

By the principle of analytic continuation, $S(t)x$ is defined for all $t \in \mathbb{R}_+$ as a real-analytic function having a local representation (2). Clearly $S(t)$ is a linear operator defined on $\mathcal{C}_{n!}(A)$.

If $x \in \mathcal{C}_{n!}(A)$, then $S(t)x \in \mathcal{C}_{n!}(A)$. In fact, if we denote by $x_N(t)$ the N-th partial sum of the series in (2), then $x_N(t) \to S(t)x$. Also

$$Ax_N(t) \to \sum_{n=0}^\infty \frac{(t-t_0)^n}{n!} A^{n+1} x$$

if $|t - t_0| < 1/(2\lambda_x)$. But A is closed. Hence

$$AS(t)x = \sum_{n=0}^\infty \frac{(t-t_0)^n}{n!} A^{n+1} x \tag{4}$$

and, by induction,

$$A^j S(t)x = \sum_{n=0}^\infty \frac{(t-t_0)^n}{n!} A^{n+j} x \tag{5}$$

for $|t - t_0| < 1/(2\lambda_x)$ and $j = 1, 2, \ldots$ Moreover,

$$\|A^j S(t)x\| \leq 2^j \lambda_x^{j+1} j!$$

for $|t - t_0| < 1/(4\lambda_x)$ and $j = 1, 2, \ldots$ It follows that if $|t - t_0| < 1/(4\lambda_x)$ and $|u - t_0| < 1/(4\lambda_x)$, then

$$\begin{aligned} S(t)S(u)x &= \sum_{j=0}^\infty \frac{(t-t_0)^j}{j!} S(u)x = \sum_{j=0}^\infty \frac{(t-t_0)^j}{j!} \sum_{k=0}^\infty \frac{(u-t_0)^k}{k!} A^{j+k} x \\ &= \sum_{j=0}^\infty \frac{(t-t_0)^j}{j!} \sum_{n=j}^\infty \frac{(u-t_0)^{n-j}}{(n-j)!} A^n x \\ &= \sum_{n=0}^\infty \left(\sum_{j=0}^n \frac{(t-t_0)^j (u-t_0)^{n-j}}{j!(n-j)!} \right) A^n x \\ &= \sum_{n=0}^\infty \frac{(u+t-t_0)^n}{n!} A^n x = S(u+t)x. \end{aligned}$$

We have thus shown that $S(u)S(t) = S(u+t)$ locally on $\mathbb{R}_+$.

We now proceed to show that this relation holds globally on $\mathbb{R}_+$. For fixed t such that $|t - t_0| < 1/(4\lambda_x)$ and for any $x \in \mathcal{C}_{n!}(A)$, set

$$\begin{aligned} \phi_{x,t}(u) &= S(u+t)x - S(u)S(t)x \\ &= \sum_{n=0}^\infty \frac{(u+t-t_0)^n}{n!} A^n x - \sum_{n=0}^\infty \frac{(u-t_0)^n}{n!} A^n x. \end{aligned}$$

The first series converges for $|u - t_0| < 1/(4\lambda_x)$, whereas the second for $|u - t_0| < 1/(4\lambda_{S(t)x})$. Thus $\phi_{x,t}(u)$ is locally analytic for all $u \in \mathbb{R}_+$. But $\phi_{x,t}(u) = 0$ for all $|u| < 1/(2\lambda_x)$. Hence $\phi_{x,t} \equiv 0$, i.e. for a fixed t such that $|t| < 1/(4\lambda_x)$ and for all $u \in \mathbb{R}_+$, $S(u+t)x = S(u)S(t)x$. Similarly, if we put

$$\psi_{x,u}(t) = S(u+t)x - S(u)S(t)x$$

and reason as before, we conclude that $\psi_{x,u}(t)$ is analytic for all $t \in \mathbb{R}_+$. But $\psi_{x,u} = 0$ for all t such that $|t| \leq 1/(4\lambda_x)$. Thus $S(u+t)x = S(u)S(t)x$ for all $u, t \in \mathbb{R}_+$.

Next, we show that $S(t)$ is a contraction operator on $\mathcal{C}_{n!}(A)$. We apply the reasoning of [7, Theorem 3.2] to our situation. Since

$$\begin{aligned} \|S(u)S(t)x\| &= \left\| \sum_{j=0}^\infty \frac{u^j}{j!} A^j S(t)x \right\| \\ &= \left\| S(t)x + \sum_{j=1}^\infty \frac{u^j}{j!} A^j S(t)x \right\| \\ &= \|S(t)x + uAS(t)x\| + O(u^2) \end{aligned}$$

we have

$$\limsup_{u\to 0^+} \frac{\|S(u+t)x\| - \|S(t)x\|}{u} = \limsup_{u\to 0^+} \frac{\|S(t)x + uAS(t)x\| - \|S(t)x\|}{u} \le \theta(S(t)) < 0,$$

since A is dissipative, where

$$\theta(S(t)) = \sup\{\mathrm{Re}[Ay,y] : \|y\| = \|S(t)x\|,\ y \in D(A)\}.$$

(For the notation $[Ay, y]$, see [7].) Since the first derived number in the sense of Denjoy is negative for all $t \in \mathbb{R}_+$, it follows that $\|S(t)x\|$ is a decreasing function of t. Hence

$$\|S(t)x\| \le \|S(0)x\| = \|x\| \tag{6}$$

for all $t \in \mathbb{R}_+$.

We now show that the generator B of this semigroup coincides with A. If $x \in D(A)$, then

$$\lim_{t\to 0^+} \frac{S(t)x - x}{t} = Ax.$$

This shows that the restriction of B to $D(A)$ is A. If $x \in D(B)$, it is known that

$$S(t)x - x = \int_0^t S(u)Bx\,du.$$

For $x \in D(A)$, $S(u)Ax = AS(u)x$ and since A is closed,

$$\int_0^t S(u)Ax\,du = A\int_0^t S(u)x\,du.$$

For each $x \in D(B)$, there exists a sequence $\{x_n\} \subset D(A)$ such that $x_n \to x$. Hence

$$S(t)x - x = \int_0^t S(u)Bx\,du = \lim_{n\to\infty} \int_0^t S(u)Ax_n\,du = \lim_{n\to\infty} A\int_0^t S(u)x_n\,du = A\int_0^t S(u)x\,du, \qquad x \in D(B).$$

Thus

$$Bx = \lim_{t\to 0^+} t^{-1}[S(t)x - x] = \lim_{t\to 0^+} A\Big[t^{-1}\int_0^t S(t)x\,du\Big]$$

and, since $t^{-1}\int_0^t S(t)x\,du \to x$, we conclude that $x \in D(A)$ and that $Ax = Bx$; in other words $A \supset B$. However, A is dissipative and B is maximal dissipative and therefore $A = B$.

As a corollary, we get the following theorem of Lumer and Phillips:

THEOREM 2. *Let A be a closed dissipative operator on a Banach space. Suppose that $\|A^n x\|^{1/n} = o(n)$ on a dense subset of $D(A^\infty)$. Then A generates a strongly continuous semigroup of contraction operators.*

A closed linear operator A on a Banach space X is called *skew-Hermitian* if A and $-A$ are both dissipative (when $A \in \mathcal{B}(X)$, A is skew-Hermitian if and only if iA is Hermitian, see [3]). An obvious modification of the proof of Theorem 1 readily gives the following theorem:

THEOREM 3. *Let A be a closed skew-Hermitian operator on a Banach space X. For any $x \in \mathcal{C}_{n!}(A)$, $S(t)x$ is an analytic function of t defined on $\mathbb{R}$, $\|S(t)x\| = \|x\|$ and*

$$\frac{d^n}{dt^n} S(t)x = S(t)A^n x, \quad n = 1, 2, \ldots$$

Moreover, for all $t \in (-\lambda_x^{-1}, \lambda_x^{-1})$,

$$\left\|\frac{d^n}{dt^n} S(t)x\right\| = \|S(t)A^n x\| = \|A^n x\| \le \lambda_x^{n+1} n!.$$

Let $T \in \mathcal{B}(X)$ and $x \in X$. We define Ω_x to be the set of $\alpha \in \mathbb{C}$ for which there exists a neighbourhood V_α of α and a function u analytic on V_α having values in X such that $(\lambda - T)u(\lambda) = x$ on V_α. This set is open and contains the complement of the spectrum of T. The function u is called a *local resolvent* of T on V_α. By definition the *local spectrum* of T at x, denoted by $\mathrm{Sp}_x(T)$, is the complement of Ω_x, so it is a compact subset of $\mathrm{Sp}(T)$.

In general, this set may be empty even for $x \ne 0$ (take the left shift operator on l^2 with $x = e_1 = (1, 0, \ldots)$). But for $x \ne 0$, the local spectrum of T at x is non-empty if T has the uniqueness property for the local resolvent. That is, $(\lambda - T)v(\lambda) = 0$ implies $v = 0$ for any analytic function v defined on any domain D of $\mathbb{C}$ with values in a Banach space X. It is easy to see that an operator T having spectrum without interior points has this property (for more details see [2]). For operators with this property there is a unique local resolvent which is the analytic extension of $(\lambda - T)^{-1}x$ to Ω_x. Also in this case the local spectral radius $r_x(T) = \max\{|z| : z \in \mathrm{Sp}_x(T)\}$ is equal to $\limsup_{k\to\infty} \|T^k x\|^{1/k}$. In general, we only have $r_x(T) \le \limsup_{k\to\infty} \|T^k x\|^{1/k}$.

In 1941, I. Gelfand [4] proved that if T is a linear bounded operator on a complex Banach space X which satisfies $\mathrm{Sp}(T) = \{1\}$ and $\sup_{k\in\mathbb{Z}} \|T^k\| < \infty$, then $T = I$. This result was generalized by E. Hille in 1944 (see [5] or [6], Theorem 4.10.1), who proved that if $\mathrm{Sp}(T) = \{1\}$ and $\|T^k\| = o(|k|)$ for $k \in \mathbb{Z}$, then $T = I$. In [1] and in [8], the following generalization of Gelfand–Hille's result was proved.

THEOREM 4. *Let $T \in \mathcal{B}(X)$ and $x \in X$. Suppose that*

(i) $\mathrm{Sp}_x(T) = \{1\}$,
(ii) $\|T^n x\| = o(n^p)$ *as* $n \to -\infty$, *and*
(iii) $\|T^n x\| = o(n^q)$ *as* $n \to \infty$.

Then $(T - I)^{\max(p,q)}x = 0$. *However, if* $\min(p, q) = 1$, *then we obtain* $Tx = x$.

As a corollary, we get the following local version of G. Lumer and R. S. Phillips' theorem.

THEOREM 5. *Let $S \in \mathcal{B}(X)$ be a locally dissipative operator (i.e. $\|e^{tS}x\| \le 1$ for all $t \ge 0$) and let $x \in X$ be such that*

(i) *S is locally quasi-nilpotent, and*
(ii) $\|e^{tS}x\| = O(t^k)$ *as* $t \to -\infty$, *for some* $k \ge 0$.

Then $Sx = 0$.

Proof. Apply Theorem 4 with $T = e^S$. In fact, the condition (i) implies, using the Riesz–Dunford functional calculus, that $\mathrm{Sp}_x(T) = \{1\}$. Since S is locally dissipative, we have

$$\|T^n x\| = \|e^{nS}x\| = o(n) \quad \text{as } n \to \infty.$$

So the conditions of Theorem 4 are satisfied with $\min(p, q) = 1$.

Here we give another local extension of Gelfand's theorem which improves Theorem 4 as well as Lumer–Phillips' theorem.

THEOREM 6. *Let $T \in \mathcal{B}(X)$ and $x \in X$. Suppose that*

(i) $\mathrm{Sp}_x(T) = \{1\}$,
(ii) $\|T^n x\| = o(n^p)$ *as* $n \to -\infty$, *for some integer* $p \ge 3$, *and*
(iii) $\|T^n x\| = o(n^2)$ *as* $n \to \infty$.

Then $(T - I)^2 x = 0$.

Proof. By Theorem 4, we have $(T-I)^p x = 0$. Suppose that $(T - I)^r x = 0$ for some $r \ge 3$. Let $y = (T - I)^{r-2}x$. Then $(T - I)^2 y = 0$. So

$$\frac{M_n(T)y}{n} = \left(\frac{n-1}{2n}\right)(Ty - y) + \frac{y}{n} \to \frac{1}{2}(Ty - y) \quad (\text{as } n \to \infty)$$

where $M_n(T) = (I + T + \ldots + T^{n-1})/n$. On the other hand, from (iii), we have

$$\frac{M_n(T)y}{n} = (T - I)^{r-3}\left(\frac{T^n - I}{n^2}\right)(T - I)x \to 0 \quad (\text{as } n \to \infty).$$

Hence $(T - I)y = 0$, which implies $(T - I)^{r-1}x = 0$. By induction, we obtain $(T - I)^2 x = 0$.

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