On a generalization of Lumer–Phillips’ theorem for dissipative operators in a Banach space

by

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Abstract. Using [1], which is a local generalization of Gelfand’s result for power-bounded operators, we first give a quantitative local extension of Lumer–Phillips’ result that states conditions under which a quasi-nilpotent dissipative operator vanishes. Secondly, we also improve Lumer–Phillips’ theorem on strongly continuous semigroups of contraction operators.

1. Introduction. Formally, \( u(\cdot) = T(\cdot)f \) solves the initial value problem

\[
\begin{align*}
    u'(t) &= Au(t), \\
    u(0) &= f, \quad t \geq 0,
\end{align*}
\]

where \( A \) is the generator of the semigroup \( \{T_t, t \geq 0\} \). Thus, from the point of view of solving initial value problems (or abstract Cauchy problems), it is natural to ask: which operators \( A \) generate \( (C_0) \) semigroups? Conditions on the behaviour of the resolvent of an operator \( A \) which are necessary and sufficient for \( A \) to be the infinitesimal generator of a \( (C_0) \) semigroup of contractions were given by E. Hille and K. Yosida. A different characterization of the generator of a \( (C_0) \) semigroup was also given by G. Lumer and R. S. Phillips in [7]. In this paper, we give another characterization which improves the latter result. In Section 2, the main result provides certain growth conditions on the iterates of \( T \), which give a local extension of Lumer and Phillips’ result on quasi-nilpotent dissipative operators defined in a Banach space \( X \).

2. Results. Let \( A \) be a linear operator with domain \( D(A) \) in a Banach space \( X \), and let \( D(x,1) = \{ f \in X^*: f(x) = \| f \| = 1 \} \). If for any \( x \in D(A) \), there exists an \( f \in D(x,1) \) satisfying \( \text{Re}(f(Ax)) \leq 0 \), then \( A \) is called a...
dissipative operator. It is easy to see that $A$ is dissipative if

$$\|tx - Ax\| \geq t\|x\| \quad (t \in \mathbb{R}_+, \ x \in D(A)).$$

Let us denote by $C_n(A)$ the set of all vectors $x \in D(A^\infty) = \bigcap_{n=1}^{\infty} D(A^n)$ such that for some $\lambda_x > 0$, $\|A^n x\| \leq \lambda_x^{n+1} n! \ (n \geq 0)$. We have the following result.

**Theorem 1.** Let $A$ be a closed dissipative operator on a Banach space $X$. Suppose that $C_n(A)$ is dense in $D(A)$. Then $A$ generates a strongly continuous semigroup of contraction operators.

**Proof.** For any $x \in C_n(A)$, let

$$S(t)x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x.$$  \hfill (1)

The series in (1) converges for all $t$ such that $|t| < \lambda_x^{-1}$. Thus $S(t)x$ is analytic as a function of $t$ in the interval $(-\lambda_x^{-1}, \lambda_x^{-1})$. If $t_0 \in (-\lambda_x^{-1}, \lambda_x^{-1})$, we define

$$S_{t_0}(t)x = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} A^n x.$$  \hfill (2)

The function $t \to S_{t_0}(t)x$ is analytic in $(t_0 - \lambda_x^{-1}, t_0 + \lambda_x^{-1})$. Clearly

$$(S_{t_0}(t)x)_{t=t_0} = A^n x = (S(t)x)_{t=t_0}$$

for $n \geq 0$. Hence for $t = t_0$,

$$(3) \quad S_{t_0}(t_0)x = S(t_0)x.$$

By the principle of analytic continuation, $S(t)x$ is defined for all $t \in \mathbb{R}_+$ as a real-analytic function having a local representation (2). Clearly $S(t)$ is a linear operator defined on $C_n(A)$.

If $x \in C_n(A)$, then $S(t)x \in C_n(A)$. In fact, if we denote by $x_N(t)$ the $N$-th partial sum of the series in (2), then $x_N(t) \to S(t)x$. Also

$$ Ax_N(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} A^{n+1} x $$

if $|t - t_0| < 1/(2\lambda_x)$. But $A$ is closed. Hence

$$(4) \quad AS(t)x = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} A^{n+1} x $$

and, by induction,

$$(5) \quad A^j S(t)x = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} A^{n+j} x$$

for $|t - t_0| < 1/(2\lambda_x)$ and $j = 1, 2, \ldots$. Moreover,

$$\|A^j S(t)x\| \leq 2^j \lambda_x^{j+1} t^j$$

for $|t - t_0| < 1/(4\lambda_x)$ and $j = 1, 2, \ldots$. It follows that if $|t - t_0| < 1/(4\lambda_x)$ and $|u - t_0| < 1/(4\lambda_x)$, then

$$S(t)S(u)x = \sum_{j=0}^{\infty} \frac{(t-t_0)^j}{j!} \sum_{k=0}^{\infty} \frac{(u-t_0)^k}{k!} A^{j+k} x$$

$$= \sum_{j=0}^{\infty} \frac{(t-t_0)^j}{j!} \sum_{n=0}^{\infty} \frac{(u-t_0)^j}{n!} A^n x$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(t-t_0)^j}{j!} (u-t_0)^{n-j} A^n x$$

$$= \sum_{n=0}^{\infty} \frac{(u + t - t_0)^n}{n!} A^n x = S(u + t)x.$$

We have thus shown that $S(u)S(t) = S(u + t)$ locally on $\mathbb{R}_+$. We now proceed to show that this relation holds globally on $\mathbb{R}_+$. For fixed $t$ such that $|t - t_0| < 1/(4\lambda_x)$ and for any $x \in C_n(A)$, set

$$\phi_{x,t}(u) = S(u + t)x - S(u)S(t)x$$

$$= \sum_{n=0}^{\infty} \frac{(u + t - t_0)^n}{n!} A^n x - \sum_{n=0}^{\infty} \frac{(u-t_0)^n}{n!} A^n x.$$

The first series converges for $|u - t_0| < 1/(4\lambda_x)$, whereas the second for $|u - t_0| < 1/(4\lambda_x S(t))$. Thus $\phi_{x,t}(u)$ is locally analytic for all $u \in \mathbb{R}_+$. But $\phi_{x,t}(u) = 0$ for all $|u| < 1/(2\lambda_x)$. Hence $\phi_{x,t} \equiv 0$, i.e., for a fixed $t$ such that $|t| < 1/(4\lambda_x)$ and for all $u \in \mathbb{R}_+$, $S(u + t)x = S(u)S(t)x$. Similarly, if we put

$$\psi_{x,u}(t) = S(u + t)x - S(u)S(t)x$$

and reason as before, we conclude that $\psi_{x,u}(t)$ is analytic for all $t \in \mathbb{R}_+$. But $\psi_{x,u}(u) = 0$ for all $t$ such that $|t| \leq 1/(4\lambda_x)$. Thus $S(u + t)x = S(u)S(t)x$ for all $u, t \in \mathbb{R}_+$.

Next, we show that $S(t)$ is a contraction operator on $C_n(A)$. We apply the reasoning of [7, Theorem 3.2] to our situation. Since

$$\|S(u)S(t)x\| = \left\| \sum_{j=0}^{\infty} \frac{u^j}{j!} A^j S(t)x \right\|$$

$$= \left\| S(t)x + \sum_{j=1}^{\infty} \frac{u^j}{j!} A^j S(t)x \right\|$$

$$= \|S(t)x + u A S(t)x\| + O(u^2)$$
we have
\[
\limsup_{u \to 0^+} \frac{\|S(u + t)x\| - \|S(t)x\|}{u} = \limsup_{u \to 0^+} \frac{\|S(t)x + uAS(t)x\| - \|S(t)x\|}{u} \\
\leq \theta(S(t)) < 0,
\]

since \(A\) is dissipative, where
\[
\theta(S(t)) = \sup \{\text{Re}(Ay, y) : \|y\| = \|S(t)x\|, \ y \in D(A)\}.
\]

(For the notation \([Ay, y]\), see [7].) Since the first derived number in the sense of Denjoy is negative for all \(t \in \mathbb{R}_+\), it follows that \(\|S(t)x\|\) is a decreasing function of \(t\). Hence
\[
(6)\quad \|S(t)x\| \leq \|S(0)x\| = \|x\|
\]
for all \(t \in \mathbb{R}_+\).

We now show that the generator \(B\) of this semigroup coincides with \(A\). If \(x \in D(A)\), then
\[
\lim_{t \to 0^+} \frac{S(t)x - x}{t} = Ax.
\]
This shows that the restriction of \(B\) to \(D(A)\) is \(A\). If \(x \in D(B)\), it is known that
\[
S(t)x - x = \int_0^t S(u)Bx \, du.
\]
For \(x \in D(A)\), \(S(u)Ax = AS(u)x\) and since \(A\) is closed,
\[
\int_0^t S(u)Ax \, du = A \int_0^t S(u)x \, du.
\]
For each \(x \in D(B)\), there exists a sequence \(\{x_n\} \subset D(A)\) such that \(x_n \to x\).

Hence
\[
S(t)x - x = \int_0^t S(u)Bx \, du = \lim_{n \to \infty} \int_0^t S(u)Ax_n \, du \\
= \lim_{n \to \infty} \left[ \int_0^t S(u)x_n \, du \right] = A \left[ \int_0^t S(u)x \, du \right], \quad x \in D(B).
\]

Thus
\[
Bx = \lim_{t \to 0^+} t^{-1}[S(t)x - x] = \lim_{t \to 0^+} \left[ t^{-1} \int_0^t S(t)x \, du \right]
\]
and, since \(t^{-1} \int_0^t S(t)x \, du \to x\) \(t \to 0^+\), we conclude that \(x \in D(A)\) and that \(Ax = Bx\); in other words \(A \supseteq B\). However, \(A\) is dissipative and \(B\) is maximal dissipative and therefore \(A = B\).

As a corollary, we get the following theorem of Lumer and Phillips:

**Theorem 2.** Let \(A\) be a closed dissipative operator on a Banach space. Suppose that \(\|A^n x\|^{1/n} \to 0\) on a dense subset of \(D(A^{\infty})\). Then \(A\) generates a strongly continuous group of contraction operators.

A closed linear operator \(A\) on a Banach space \(X\) is called skew-Hermitian if \(A\) and \(-A\) are both dissipative (when \(A \in B(X)\), \(A\) is skew-Hermitian if and only if \(i A\) is Hermitian, see [3]). An obvious modification of the proof of Theorem 1 readily gives the following theorem:

**Theorem 3.** Let \(A\) be a closed skew-Hermitian operator on a Banach space \(X\). For any \(x \in C_0(A), S(t)x\) is an analytic function of \(t\) defined on \(\mathbb{R}\), \(\|S(t)x\| = \|x\|\), and
\[
\frac{d^n}{dt^n} S(t)x = S(t)A^n x, \quad n = 1, 2, \ldots
\]
Moreover, for all \(t \in (-\lambda_{n-1}^{-1}, \lambda_n^{-1})\)
\[
\left\| \frac{d^n}{dt^n} S(t)x \right\| = \|S(t)A^n x\| = \|A^n x\| \leq \lambda_n^{n+1} \|x\|.
\]

Let \(T \in B(X)\) and \(x \in X\). We define \(\Omega_x\) to be the set of \(\alpha \in \mathbb{C}\) for which there exists a neighbourhood \(V_\alpha\) of \(0\) and a function \(u\) analytic on \(V_\alpha\) having values in \(X\) such that \((\lambda - T)u(\lambda) = x\) on \(V_\alpha\). This set is open and contains the complement of the spectrum of \(T\). The function \(u\) is called a local resolvent of \(T\) on \(V_\alpha\). By definition the local spectrum of \(T\) at \(x\), denoted by \(S_{\alpha}(T)\), is the complement of \(\Omega_x\), so it is a compact subset of \(S(T)\).

In general, this set may be empty even for \(x \neq 0\) (take the left shift operator on \(l^2\) with \(x = e_1 = (1, 0, \ldots)\)). But for \(x \neq 0\), the local spectrum of \(T\) at \(x\) is non-empty if \(T\) has the uniqueness property for the local resolvent. That is, \((\lambda - T)u(\lambda) = 0\) implies \(u = 0\) for any analytic function \(u\) defined on any domain \(D\) of \(X\) with values in a Banach space \(X\). It is easy to see that an operator \(T\) having spectrum without interior points has this property (for more details see [2]). For operators with this property there is a unique local resolvent which is the analytic extension of \((\lambda - T)^{-1}x\) to \(\Omega_x\). Also in this case the local spectral radius \(r_\alpha(T) = \max \{|x| : x \in S_{\alpha}(T)\}\) is equal to \(\limsup_{k \to \infty} \|T^k x\|^{1/k}\). In general, we only have \(r_\alpha(T) \leq \limsup_{k \to \infty} \|T^k x\|^{1/k}\).

In 1941, I. Gelfand [4] proved that if \(T\) is a linear bounded operator on a complex Banach space \(X\) which satisfies \(S(T) = \{1\}\) and \(\sup_{k \in \mathbb{N}} \|T^k\| < \infty\), then \(T = I\). This result was generalized by E. Hille in 1944 (see [5] or [6],...
Theorem 4.10.1), who proved that if $\text{Sp}(T) = \{1\}$ and $\|T^k\| = o(|k|)$ for $k \in \mathbb{Z}$, then $T = I$. In [1] and in [8], the following generalization of Gelfand–Hille's result was proved.

**Theorem 4.** Let $T \in B(X)$ and $x \in X$. Suppose that

(i) $\text{Sp}_a(T) = \{1\}$,
(ii) $\|T^nx\| = o(n^p)$ as $n \to -\infty$, and
(iii) $\|T^nx\| = o(n^2)$ as $n \to \infty$.

Then $(T - I)^{\max(p, 2)}x = 0$. However, if $\min(p, q) = 1$, then we obtain $T^nx = x$.

As a corollary, we get the following local version of G. Lumer and R. S. Phillips’ theorem.

**Theorem 5.** Let $S \in B(X)$ be a locally dissipative operator (i.e. $\|e^{itS}x\| \leq 1$ for all $t \geq 0$) and let $x \in X$ be such that

(i) $S$ is locally quasi-nilpotent, and
(ii) $\|e^{is}x\| = O(s^k)$ as $s \to -\infty$, for some $k \geq 0$.

Then $Sx = 0$.

**Proof.** Apply Theorem 4 with $T = e^{iS}$. In fact, the condition (i) implies, using the Riesz–Dunford functional calculus, that $\text{Sp}_a(T) = \{1\}$. Since $S$ is locally dissipative, we have

$$\|T^nx\| = \|e^{is}x\| = o(n) \quad \text{as } n \to \infty.$$ 

So the conditions of Theorem 4 are satisfied with $\min(p, q) = 1$.

Here we give another local extension of Gelfand’s theorem which improves Theorem 4 as well as Lumer–Phillips’ theorem.

**Theorem 6.** Let $T \in B(X)$ and $x \in X$. Suppose that

(i) $\text{Sp}_a(T) = \{1\}$,
(ii) $\|T^nx\| = o(n^p)$ as $n \to -\infty$, for some integer $p \geq 3$, and
(iii) $\|T^nx\| = o(n^2)$ as $n \to \infty$.

Then $(T - I)^2x = 0$.

**Proof.** By Theorem 4, we have $(T - I)^px = 0$. Suppose that $(T - I)^r x = 0$ for some $r \geq 3$. Let $y = (T - I)^{r-2} x$. Then $(T - I)^2 y = 0$. So

$$\frac{M_n(T)y}{n} = \frac{n-1}{2n} (Ty - y) + \frac{y}{n} \to \frac{1}{2} (Ty - y) \quad (\text{as } n \to \infty)$$

where $M_n(T) = (I + T + \ldots + T^{n-1})/n$. On the other hand, from (iii), we have

$$\frac{M_n(T)y}{n} = (T - I)^{r-2} \left( \frac{T^n - I}{n^2} \right) (T - I)x \to 0 \quad (\text{as } n \to \infty).$$

Hence $(T - I)y = 0$, which implies $(T - I)^{r-1} x = 0$. By induction, we obtain $(T - I)^2 x = 0$.

**References**


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