Denote by $\{\alpha_n\}$ the sequence of all positive integers such that

$$\left|\int_{-T/2}^{T/2} e^{\alpha_n u} f(\tfrac{1}{2}T-u)\,du\right| \leqslant M \qquad (n=1,2,\ldots)$$

and by $\{\beta_n\}$ the sequence of all positive integers such that

$$\left|\int_{-T/2}^{T/2} e^{\beta_n v} g(\tfrac{1}{2}T-v)\,dv\right| \leqslant M \qquad (n=1,2,\ldots).$$

By (1), one at least of the relations

$$\sum_{n=1}^{\infty}\frac{1}{\alpha_n}=\infty \qquad \text{or} \qquad \sum_{n=1}^{\infty}\frac{1}{\beta_n}=\infty$$

must hold. Suppose the first does so.

Since

$$\left|\int_{0}^{T/2} e^{\alpha_n u} f(\tfrac{1}{2}T-u)\,du\right| \leqslant M+\left|\int_{-T/2}^{0} f(\tfrac{1}{2}T-u)\,du\right| = N \qquad (n=1,2,\ldots),$$

we have, by the Theorem on bounded moments, $f(\frac{1}{2}T-t)=0$ a. e. in $[0,\frac{1}{2}T]$, that is $f(t)=0$ a. e. in $[0,\frac{1}{2}T]$. Thus, the theorem (II) and, consequently, the theorem (I) are proved.

References.

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[3] — *Autour du théorème de Phragmén-Lindelöf*, Bulletin des Sciences Mathématiques 72 (1948), p. 17-22.

[4] J. G.-Mikusiński and C. Ryll-Nardzewski, *A theorem on bounded moments*, this volume.

[5] E. C. Titchmarsh, *The zeros of certain integral functions*, Proceedings of the London Mathematical Society 25 (1926), p. 283-302.

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Remarks on a moment problem

by

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J. G.-Mikusiński[2]) recently gave an elementary proof of the following generalization of Lerch's theorem:

If $f(t)$ is integrable over the finite interval $0\leqslant a<b$ and if for some $\delta>0$ and every $\varepsilon>0$

$$\text{(1)} \qquad \int_a^b t^{n\delta} f(t)\,dt = O[(a+\varepsilon)^{n\delta}],$$

then $f(t)=0$ almost everywhere in (a,b).

He raised the question of whether the theorem can be extended by replacing the arithmetic progression $\{n\delta\}$ by a more general sequence $\{\lambda_n\}$. I shall show that the theorem can be proved by less elementary methods, one of which leads to a generalization of the desired kind.

By a change of variable we can make $\delta=1$ in (1), and we suppose this done. We remark first that if $f(t)$ is non-negative the conclusion is immediate, since if $f(t)$ does not vanish almost everywhere in a neighbourhood of b, we have[3])

$$\text{(2)} \qquad \overline{\lim_{n\to\infty}}\left|\int_a^b t^n f(t)\,dt\right|^{1/n} = b.$$

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[2]) J. G.-Mikusiński, *Remarks on the moment problem and a theorem of Picone*, Colloquium Mathematicum 2 (1951), p. 138-141.

[3]) G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge 1934, p. 143.

It should be possible to establish (2) by set-theoretic methods in the general case, but I cannot do this. However, it can be done by complex variable methods. Let

$$F(z)=\int_a^b e^{izt}f(t)\,dt; \tag{3}$$

then

$$|F^{(n)}(0)|=\left|\int_a^b t^n f(t)\,dt\right|,$$

so that (1) implies that $F(z)$ is an entire function of exponential type not exceeding a. On the other hand it is well known that if $f(t)$ does not vanish almost everywhere in a neighbourhood of b, then $f(z)$ is of exponential type precisely b, so that (2) holds and we have a contradiction.

To prove that (3) is of type b is not trivial; the following argument is perhaps as short as any. Put

$$G(t)=\int_a^t f(u)\,du.$$

Then by integration by parts,

$$F(z)=F(0)e^{izb}-iz\int_a^b e^{izt}G(t)\,dt. \tag{4}$$

On the other hand, if $F(z)$ were of type at most c, $0<c<b$, the function $z^{-1}\{F(z)-F(0)e^{iza}\}$ would be of exponential type at most c, and would belong to L^2 on the real axis. By a theorem of Paley and Wiener[4]), then, we should have

$$F(z)=F(0)e^{izc}-iz\int_{-c}^{c} e^{izt}H(t)\,dt, \qquad H(t)\in L^2. \tag{5}$$

Comparing (4) and (5), we see that

$$F(0)(e^{izb}-e^{izc})=iF(0)z\int_c^b e^{izt}\,dt=iz\int_{-c}^{c} e^{izt}H(t)\,dt-iz\int_a^b e^{izt}G(t)\,dt.$$

[4]) R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, New York 1934, p. 13. Many proofs have been given.

By the uniqueness theorem for Fourier transforms, we therefore should have $G(t)=-F(0)$ almost everywhere on (c,b), hence $f(t)=0$ almost everywhere on (c,b).

We now give a more sophisticated proof which establishes more, namely that we can replace $n\delta$ in (1) by λ_n, where λ_n are complex, $R\lambda_n\to\infty$, $\arg\lambda_n\to 0$, $\Sigma 1/|\lambda_n|=\infty$, and $|\lambda_n-\lambda_m|\geqslant|n-m|h$, $h>0$.

Put

$$G(z)=\int_a^b t^z f(t)\,dt;$$

then $G(z)$ is a regular function for $x>0$, satisfying

$$|G(x)|\leqslant b^x\int_a^b |f(t)|\,dt,$$

$$|G(iy)|=O(1),$$

$$\overline{\lim_{n\to\infty}}\frac{\log|G(\lambda_n)|}{\lambda_n}\leqslant\log a.$$

Then by a theorem of Levinson[5])

$$\overline{\lim_{x\to\infty}}\frac{\log|G(x)|}{x}\leqslant\log a. \tag{6}$$

By putting in particular $x=n$ in (6) we are back in the original case where $\lambda_n=n$, and the proof is completed by establishing the special case by any method.

[5]) N. Levinson, *On the growth of analytic functions*, Transactions of the American Mathematical Society 43 (1938), p. 240-257.

Levinson requires $n/\lambda_n\to D\geqslant 0$, but we can always thin out $\{\lambda_n\}$ to make this hold with $D=0$ if $\{\lambda_n\}$ satisfies the other condition imposed on it. The condition $\arg\lambda_n\to 0$ is omitted by an oversight in Levinson's book, *Gap and density theorems*, New York 1940, p. 107. I am indebted to Prof. Mikusiński for the substance of these remarks and for an example showing the necessity of the original condition.

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