$$S_3 < \frac{1}{\beta_n} \sum_{\nu=n+1}^{\infty} \left(\log \frac{\beta_n + \varepsilon(\nu - n)}{\varepsilon(\nu - n)} - \frac{\beta_n}{\beta_n + \varepsilon(\nu - n)} \right)$$

$$= \frac{1}{\beta_n} \sum_{\nu=1}^{\infty} \left(\log \frac{\beta_n + \varepsilon\nu}{\varepsilon\nu} - \frac{\beta_n}{\beta_n + \varepsilon\nu} \right)$$

(5)

$$< \frac{1}{\beta_n} \int_0^{\infty} \left(\log \frac{\beta_n + \varepsilon x}{\varepsilon x} - \frac{\beta_n}{\beta_n + \varepsilon x} \right) dx$$

$$= \frac{1}{\varepsilon} \int_0^{\infty} \left(\log \frac{1+t}{t} - \frac{1}{1+t} \right) dt = \frac{1}{\varepsilon}.$$

From (3), (4) and (5) follows (2). From (2) and (i) follows (1) which completes the proof.

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A theorem on bounded moments

by

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1. The well known theorem of MÜNTZ [6] can be formulated as follows:

(I) *If* $\beta_1, \beta_2, \ldots$ *is an increasing sequence such that* $\sum_{n=1}^{\infty} 1/\beta_n = \infty$ *and* $f(x)$ *a function integrable in* $[a,b]$ *(where* $a \geqslant 0$*) such that*

$$\int_a^b x^{\beta_n} f(x)\,dx = 0 \qquad (n=1,2,\ldots),$$

then $f(x)=0$ *almost everywhere in* $[a,b]$.

If particularly $\beta_n = n$, this theorem reduces itself to the well known theorem of LERCH [1]. On the other hand, the following theorem holds [2]:

(II) *If* $f(x)$ *is integrable in* $[1,b]$ *and there exists a number* M *such that*

(1)
$$\left| \int_1^b x^n f(x)\,dx \right| < M \qquad (n=1,2,\ldots).$$

then $f(x)=0$ *almost everywhere in* $[1,b]$.

It is easy to see that the lower bound of the integral cannot be diminued. Indeed, all moments of any function which vanishes for $x>1$ are always commonly bounded.

The theorem (II) can be generalized by replacing the natural sequence of exponents n by any sequence $\{n^a\}$ where $0<a\leqslant 1$ [4]. The question arises if the sequence of exponents may be replaced

by more general sequences. It is easy to show that the condition $\sum_{n=1}^{\infty} 1/\beta_n = \infty$ alone does not suffice. In fact, there exists a function $f(x)$, continuous and non-vanishing identically in $[1,2]$ such that $\int_1^2 x^{n^2} f(x)\,dx = 0$. Its transform

$$F(\beta) = \int_1^2 x^\beta f(x)\,dx$$

is a continuous function which vanishes for $\beta = n^2$ $(n=1,2,\ldots)$. Thus, if β is near to n^2, we have $|F(\beta)|<1$. Consequently we can complete the sequence $\{1/n^2\}$ by so much terms, that the new sequence should have the property $\sum_{n=1}^{\infty} 1/\beta_n = \infty$ and that the inequalities $|F(\beta_n)|<1$ hold.

On the other hand, the theorem on bounded moments will be still true if we add, to the condition $\sum_{n=1}^{\infty} 1/\beta_n = \infty$, a supplementary condition $\beta_{n+1} - \beta_n > \varepsilon > 0$ $(n=1,2,\ldots)$. This is the chief result of our paper. It can be explicitly written as follows:

Theorem. *If $\beta_1, \beta_2, \ldots$ is a sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty \quad \text{and} \quad \beta_{n+1} - \beta_n > \varepsilon > 0 \quad \text{for} \quad n=1,2,\ldots$$

and $f(x)$ is a function, integrable in $[1,b]$, such that

$$\left| \int_1^b x^{\beta_n} f(x)\,dx \right| < M \quad \text{for} \quad n=1,2,\ldots,$$

then $f(x)=0$ almost everywhere in $[1,b]$.

The proof will be based on a *discontinuity factor*, analogous to that used by Phragmén [7]

$$\varphi(x) = \lim_{n\to\infty} \exp(-e^{nx}).$$

This method was extended by Picone [8] and Mikusiński [4]. In the sequel, we shall use a very general form of discontinuity factor, by replacing the function $\exp(-e^{nx})$ by a suitable sequence of generalized exponential functions.

2. Before the proof we shall give some corollaries of the Theorem.

Corollary 1. *If the sequence $\beta_1, \beta_2, \ldots$ satisfies the conditions of the Theorem and $f(x)$ is a function, integrable in $[0,b]$, such that*

$$\left| \int_0^b x^{\beta_n} f(x)\,dx \right| < M q^{\beta_n} \quad \text{for} \quad n=1,2,\ldots,$$

then $f(x)=0$ a. e.[1] *in $q<x<b$.*

Indeed, we have

$$q^{\beta_n} \int_0^{b/q} x^{\beta_n} q f(qx)\,dx = \int_0^b x^{\beta_n} f(x)\,dx.$$

Thus, if $0<q<b$,

$$\left| \int_1^{b/q} x^{\beta_n} q f(qx)\,dx \right| \leqslant M + \int_0^1 q\,|f(qx)|\,dx$$

and, by the Theorem, $qf(qx)=0$ a. e. in $[1,b/q]$, that is $f(x)=0$ a. e. in $[q,b]$.

Corollary 2. *If the sequence $\beta_1, \beta_2, \ldots$ satisfies the conditions of the Theorem and $g(t)$ is a function, integrable in $[0,T]$, such that*

$$\left| \int_0^T e^{\beta_n t} g(t)\,dt \right| < M$$

then $g(t)=0$ a. e. in $[0,T]$.

This Corollary follows from the Theorem by the substitution

$$x=e^t, \qquad b=e^T, \qquad f(x)=g(t).$$

3. Now, we approach the proof. Write

$$\varphi_m(x) = 1 - a_1^{(m)} x^{\beta_m} + a_2^{(m)} x^{\beta_{2m}} - \ldots,$$

where

$$a_n^{(m)} = \frac{1}{e} \prod_{\nu=1}^{\infty}{}' \frac{\beta_{m\nu}}{|\beta_{m\nu} - \beta_{mn}|} \exp\left(-\frac{\beta_{mn}}{\beta_{m\nu}}\right).$$

By the preceeding paper [5], the functions $\varphi_m(x)$ have the following properties:

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[1]) a. e. = almost everywhere.

1^0 $\varphi_m(x)$ decreases in the interval $0<x<\infty$ from 1 to 0;

2^0 $\int_0^\infty \varphi_m(x)\,dx = \prod_{\nu=1}^\infty \frac{\beta_{m\nu}}{\beta_{m\nu}+1}\exp\left(\frac{1}{\beta_{m\nu}}\right)$;

3^0 $\frac{\log \alpha_n^{(m)}}{\beta_{m\nu}} < \frac{2}{m\varepsilon}$ $(m,n=1,2,\ldots)$.

We are going to show that

(2) $$\lim_{m\to\infty}\varphi_m(x) = \begin{cases} 1 & \text{for } 0\leqslant x<1, \\ 0 & \text{for } 1<x<\infty. \end{cases}$$

Let $0\leqslant\theta<\lambda<1$. Since $\alpha_n^{(m)} < \exp\frac{2\beta_{mn}}{m\varepsilon}$, we have

(3) $$\begin{aligned} |1-\varphi_m(\theta)| &\leqslant \sum_{n=1}^\infty \alpha_n^{(m)}\theta^{\beta_{mn}} \\ &\leqslant \sum_{n=1}^\infty \left(\theta\exp\frac{2}{m\varepsilon}\right)^{\beta_{mn}} < \sum_{n=1}^\infty \lambda^{\varepsilon(mn-1)} = \frac{\lambda^{\varepsilon(m-1)}}{1-\lambda^{m\varepsilon}} \end{aligned}$$

for sufficiently large m.

This proves that

$$\lim_{m\to\infty}\varphi_m(x)=1 \quad \text{for} \quad 0\leqslant x<1.$$

To prove

$$\lim_{m\to\infty}\varphi_m(x)=0 \quad \text{for} \quad 1<x<\infty,$$

it suffices to show that

$$\lim_{m\to\infty}\int_0^\infty \varphi_m(x)\,dx = 1,$$

for $\varphi_m(x)$ are positive and decreasing. But this follows from the convergence of the infinite product

$$\prod_{\nu=1}^\infty \frac{\beta_\nu}{\beta_\nu+1}\exp\left(\frac{1}{\beta_\nu}\right).$$

Thus, the formula (2) is proved.

The formula (2) enables us to use the sequence $\varphi_m(x)$ as a discontinuity factor; to this purpose, we write

$$\int_1^b [1-\varphi_m(\theta x)]f(x)\,dx = \sum_{n=1}^\infty (-1)^{n+1}\alpha_n^{(m)}\theta^{\beta_{mn}}\int_1^b x^{\beta_{mn}}f(x)\,dx.$$

If $1<\theta^{-1}<b$, the left member approaches the limit $\int_{\theta^{-1}}^b f(x)\,dx$.

On the other hand, the right member tends to 0, for its absolute value is, by (1), less than

$$M\sum_{n=1}^\infty \alpha_n^{(m)}\theta^{\beta_{mn}}$$

and the last expression tends to 0, by (3). In this way we have

$$\int_{\theta^{-1}}^b f(x)\,dx = 0$$

and, as θ^{-1} can be fixed arbitrarily in $(1,b)$, $f(x)=0$ a. e. in $[1,b]$.

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