On generalized exponential functions

by

J. G.-MIKUSIŃSKI (Wrocław).

The purpose of this paper is to prove the following

Theorem. *If $\beta_1, \beta_2, \ldots$ is any sequence of positive numbers such that*

(i) $$\sum_{\nu=1}^{\infty} \frac{1}{\beta_\nu} = \infty,$$

(ii) $$\beta_{\nu+1} - \beta_\nu > \varepsilon > 0 \quad (\nu = 1, 2, \ldots),$$

then the series

$$f(x) = 1 - a_1 x^{\beta_1} + a_2 x^{\beta_2} - a_3 x^{\beta_3} + \ldots,$$

where

$$a_n = \frac{1}{e} \prod_{\nu=1}^{\infty}{}' \frac{\beta_\nu}{|\beta_\nu - \beta_n|} \exp\left(-\frac{\beta_n}{\beta_\nu}\right) \qquad (n = 1, 2, \ldots),$$

is convergent for every non-negative x and its sum decreases in the interval $0 \leqslant x < \infty$ monotonically from 1 to 0. Moreover we have

$$\int_0^\infty x^{p-1} f(x)\, dx = \frac{1}{p} \prod_{\nu=1}^{\infty} \frac{\beta_\nu}{\beta_\nu + p} \exp\left(\frac{p}{\beta_\nu}\right) \quad \textit{for} \quad p > 0.$$

This theorem is stricly related with the results of our earlier paper[1]) and extends the theorem 3 given there. In view of the theorems 1 and 2 of that paper, it suffices here to prove that

(1) $$\lim_{n \to \infty} \sqrt[\beta_n]{a_n} = 0.$$

[1]) J. G.-Mikusiński, *On generalized power series*, Studia Mathematica 12 (1951), p. 181-190.

We are going to show that

(2) $$\frac{\log a_n}{\beta_n} < \frac{1 + \log 2}{\varepsilon} - \sum_{\nu=1}^{n} \frac{1}{\beta_\nu} \qquad (n = 1, 2, \ldots).$$

We have evidently

(3) $$\frac{\log a_n}{\beta_n} = -\sum_{\nu=1}^{n} \frac{1}{\beta_\nu} + \frac{1}{\beta_n} \sum_{\nu=1}^{n-1} \log \frac{\beta_\nu}{\beta_n - \beta_\nu} + \frac{1}{\beta_n} \sum_{\nu=n+1}^{\infty} \left(\log \frac{\beta_\nu}{\beta_\nu - \beta_n} - \frac{\beta_n}{\beta_\nu} \right) = S_1 + S_2 + S_3$$

(if $n = 1$, one admits that $S_2 = 0$). Since the function $x/(\beta_n - x)$ is increasing for $x < \beta_n$, we can write, in view of (ii),

(4) $$S_2 < \frac{1}{\beta_n} \sum_{\nu=1}^{n-1} \log \frac{\beta_n - \varepsilon(n - \nu)}{\varepsilon(n - \nu)} = \frac{1}{\beta_n} \sum_{\nu=1}^{n-1} \log \frac{\beta_n - \varepsilon\nu}{\varepsilon\nu} < \frac{1}{\beta_n} \int_0^{n-1} \log \frac{\beta_n - \varepsilon x}{\varepsilon x}\, dx = \frac{1}{\varepsilon} \int_0^{\varepsilon(n-1)/\beta_n} \log\left(\frac{1}{t} - 1\right) dt;$$

the last integral may be obtained by substituting $t = \varepsilon x/\beta_n$. The function $\log\left(\frac{1}{t} - 1\right)$ being positive for $0 < t < 1/2$ and negative for $t > 1/2$ we have *a fortiori*

$$S_2 < \frac{1}{\varepsilon} \int_0^{1/2} \log\left(\frac{1}{t} - 1\right) dt = \frac{\log 2}{\varepsilon}.$$

Now, the function

$$\log \frac{x}{x - \beta_n} - \frac{\beta_n}{x}$$

is decreasing for $x > \beta_n$, because its derivative

$$\frac{\beta_n}{x^2} - \frac{\beta_n}{x(x - \beta_n)}$$

is negative. Hence

$$S_3 < \frac{1}{\beta_n}\sum_{\nu=n+1}^{\infty}\left(\log\frac{\beta_n+\varepsilon(\nu-n)}{\varepsilon(\nu-n)} - \frac{\beta_n}{\beta_n+\varepsilon(\nu-n)}\right)$$

(5)
$$= \frac{1}{\beta_n}\sum_{\nu=1}^{\infty}\left(\log\frac{\beta_n+\varepsilon\nu}{\varepsilon\nu} - \frac{\beta_n}{\beta_n+\varepsilon\nu}\right)$$

$$< \frac{1}{\beta_n}\int_0^{\infty}\left(\log\frac{\beta_n+\varepsilon x}{\varepsilon x} - \frac{\beta_n}{\beta_n+\varepsilon x}\right)dx$$

$$= \frac{1}{\varepsilon}\int_0^{\infty}\left(\log\frac{1+t}{t} - \frac{1}{1+t}\right)dt = \frac{1}{\varepsilon}.$$

From (3), (4) and (5) follows (2). From (2) and (i) follows (1) which completes the proof.

PAŃSTWOWY INSTYTUT MATEMATYCZNY
STATE INSTITUTE OF MATHEMATICS

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A theorem on bounded moments

by

J. G.-MIKUSIŃSKI (Wrocław) and C. RYLL-NARDZEWSKI (Warszawa).

1. The well known theorem of MÜNTZ [6] can be formulated as follows:

(I) *If $\beta_1, \beta_2, \ldots$ is an increasing sequence such that $\sum_{n=1}^{\infty} 1/\beta_n = \infty$ and $f(x)$ a function integrable in $[a,b]$ (where $a \geqslant 0$) such that*

$$\int_a^b x^{\beta_n} f(x)\,dx = 0 \qquad (n=1,2,\ldots),$$

then $f(x)=0$ almost everywhere in $[a,b]$.

If particularly $\beta_n = n$, this theorem reduces itself to the well known theorem of LERCH [1]. On the other hand, the following theorem holds [2]:

(II) *If $f(x)$ is integrable in $[1,b]$ and there exists a number M such that*

(1)
$$\left|\int_1^b x^n f(x)\,dx\right| < M \qquad (n=1,2,\ldots).$$

then $f(x)=0$ almost everywhere in $[1,b]$.

It is easy to see that the lower bound of the integral cannot be diminued. Indeed, all moments of any function which vanishes for $x>1$ are always commonly bounded.

The theorem (II) can be generalized by replacing the natural sequence of exponents n by any sequence $\{n^a\}$ where $0<a\leqslant 1$ [4]. The question arises if the sequence of exponents may be replaced