D. Blackwell's conjecture on power series with random coefficients

by

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E. BOREL[1]) was the first to assert that Taylor's series with mutually independent random coefficients have, with probability 1, a *coupure* on their circle of convergence, which is to say that almost certainly the circle is a singular line for the function defined by such a series. This assertion has been proved since for certain classes of random power series[2]), nevertheless its general validity is easily refuted by the following example:

Let $\{\vartheta_n\}$ be a sequence of random variables, each ϑ_n being chosen independently from the interval $(0,2\pi)$ with uniform probability. The power series

$$\sum_{n=1}^{\infty} a_n z^n \quad \text{with} \quad a_n = e^{i\vartheta_n}$$

has 1 as its radius of convergence; it has independent random coefficients. The series

$$\sum_{n=1}^{\infty} \beta_n z^n \quad \text{with} \quad \beta_n = a_n + 2^n$$

has also independent random coefficients. Its radius of convergence is 1/2 and as the first series is regular for $|z|=1/2$ the only singular point of the second series on the circle $|z|=1/2$ is the pole $z=1/2$ of the series $\sum 2^n z^n$, against Borel's assertion.

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[1]) *Sur les séries de Taylor,* C. R. 123 (1896), p. 1051-1052.

[2]) Cf. H. Steinhaus, *Über die Wahrscheinlichkeit dafür, dass der Konvergentkreis einer Potenzreihe ihre natürliche Grenze ist,* Math. Zeitschrift 31 (1930), p. 408-416.

It is obvious that it is sufficient to add to the second series the series $\sum(-2^n)z^n$ to get the first one; now, it is known that the first series has, with probability 1, the circle $|z|=1$ as its singular line.

The conjecture we have to prove in this Note is, roughly speaking, that the example given above exhibits already the features of the general case: given any power series with random coefficients we have only to add to it an appropriate series with fix coefficients (in our example the series $\sum(-2^n)z^n$) to get a series obeying Borel's assertion[3]).

Preliminary remarks and explanations.

We will follow here the vocabulary of the theory of independent functions[4]). Thus we will have to deal with series of the form

$$F(z) = \sum_{n=1}^{\infty} a_n(t) z^n; \tag{1}$$

the complex-valued and (L)-measurable functions $a_n(t)$ of the *stochastic variable* t $(0 \leqslant t \leqslant 1)$ are assumed to be *stochastically independent* as a whole (*en bloc*). It can be shown that, under such circumstances, the series (1) has a ***definite radius*** of convergence $r(F)$ $(0 \leqslant r \leqslant \infty)$, which means that for all values of t, a set of measure 0 if any excepted, $|z|=r$ is the circle of convergence of (1), r being a number and not a function of t. The proof is based on the fact that $1/r$ equals the

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(t)|}$$

and the lim sup of a sequence of independent functions is constant almost everywhere.

We will have also to consider series of the form

$$G(z) = \sum_{n=1}^{\infty} b_n(s,t) z^n; \tag{2}$$

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[3]) The conjecture has been formulated in 1947 by Prof. D. Blackwell of the Howard University, in a conversation with Prof. H. Steinhaus and communicated by the later to the author.

[4]) *Sur les fonctions indépendantes I-IX*: Studia Mathematica 6 (1936), p. 46-58, p. 59-66, p. 89-97; 7 (1938), p. 1-15, p.96-100; 9 (1940), p.121-132; 10 (1948), p. 1-20; 11 (1949), p. 133-144; 12 (1951), p. 102-107.

here the coefficients b_n are (L)-measurable functions of two stochastic variables s,t $(0\leqslant s\leqslant 1, 0\leqslant t\leqslant 1)$; assuming their independence it can be shown as in the case (1) the existence of a definite radius of convergence $r(G)$, the exceptional set of points (s,t) having a plane measure 0.

In both cases, (1) and (2), the series is called *singular* if its circle of convergence (defined as above) is a singular line for the function defined by the series, this being true for all points t, respectively for all points (s,t), a set of linear measure 0, respectively of plane measure 0, of such points, if any, being excepted.

Series of the form (1) or (2) will be called *random series*. A special case of a random series is an ordinary power series

(3) $$f(z)=\sum_{n=1}^{\infty} c_n z^n$$

with fix coefficients c_n. For clarity's sake we will denote the sums of random series by capital letters $F,G,\ldots$, the sums of ordinary series by small letters $f,g,\ldots$

The radius of convergence of any series S will be denoted by $r(S)$ putting $S=F,G,\ldots,f,g,\ldots$ *For almost all* t of the interval $(0,1)$ or *for allmost all* (s,t) of the square $(0,1)^2$ means that the property in question fails to be true only in a set of points t of linear measure 0, respectively in a set of points (s,t) of plane measure 0.

Theorem. *To every random series $F(z)$ an ordinary series $f_0(z)$ can be associated in such a manner that the random series*

(4) $$H(z)=F(z)+f_0(z)$$

has the following properties:

(i) $r(F+f)\leqslant r(H)$ *for every ordinary series* $f(z)$;

(ii) $H(z)$ *is singular*;

(iii) *if an ordinary series $f(z)$ satisfies the equation*

(5) $$r(F+f)=r(H),$$

the series $F(z)+f(z)$ is singular.

To prove the theorem we have to establish two lemmas of which it is an immediate consequence.

Lemma 1. *Let $F(z)$ be the random series (1) and $G(z)$ the random series defined by*

(6) $$G(z)=\sum_{n=1}^{\infty}\big(a_n(t)-a_n(s)\big)z^n \qquad (0\leqslant s\leqslant 1, 0\leqslant t\leqslant 1).$$

Then we will have for every ordinary series $f(z)$ the inequalities

1° $r(F+f)\leqslant r(G)$,

2° $r(G+f)\leqslant r(G)$,

and there will exist an ordinary series $f_0(z)$ satisfying the equation

3° $r(F+f_0)=r(G)$.

Proof of 1°. Let us write out $f(z)$ as in (3). It is then obvious that the series

$$\sum_{n=1}^{\infty}\big(a_n(t)+c_n\big)z^n$$

is convergent for almost all t of the interval $(0,1)$ for $|z|<r(F+f)$, and the series

$$\sum_{n=1}^{\infty}\big(a_n(s)+c_n\big)z^n$$

is convergent for almost all s of the interval $(0,1)$ for $|z|<r(F+f)$. Substraction gives the convergence of the series (6) for almost all points (s,t) of the square $(0,1)^2$ for $|z|<r(F+f)$, q. e. d.

Proof of 2°. The series

$$\sum_{n=1}^{\infty}\big(a_n(t)-a_n(s)+c_n\big)z^n$$

being convergent for almost all points (s,t) of the square $(0,1)^2$ for $|z|<r(G+f)$, there exists an s_0 $(0<s_0<1)$ such that the series

(7) $$\sum_{n=1}^{\infty}\big(a_n(t)-a_n(s_0)+c_n\big)z^n$$

is convergent for almost all t of $(0,1)$ for $|z|<r(G+f)$. Calling f_0 the ordinary series

$$\sum_{n=1}^{\infty}\big(c_n-a_n(s_0)\big)z^n,$$

the series (7) becomes $F+f_0$ and thus we get

(8) $$r(G+f)\leqslant r(F+f_0)\leqslant r(G),$$

the second inequality resulting from 1°.

Proof of 3°. Put $f\equiv 0$ in (8).

Lemma 2. *If a random series $F(z)$ has on its circle of convergence a point which is regular for almost all values of the stochastic variable, there exists an ordinary series $f_1(z)$ such that*

(9) $$r(F)<r(F+f_1).$$

Proof. Let us assume the regular point z_0 to be $r(F)$; this assumption does not diminish the generality, because $z_0=e^{i\vartheta_0}r(F)$ (with any real ϑ_0) is the general case and the series

$$F^*=\sum_{n=1}^{\infty} a_n(t)\,e^{in\vartheta_0}z \qquad (i=\sqrt{-1})$$

has independent coefficients, the same radius of convergence as $F(z)$, and $z=r(F)=r(F^*)$ is a regular point for $F^*(z)$. The assumption implies the existence of real numbers u,v such that

(10) $$0<u<r(F)<u+v,$$

and such that the series

(11) $$\sum_{m=0}^{\infty}\frac{F^{(m)}(t,u)}{m!}v^m$$

is convergent for almost all t of $(0,1)$. $F^{(m)}$ means here $d^mF(z)/dz^m$; we write $F(t,z)$ instead of $F(z)$ to recall the fact that the coefficients of $F(z)$ are functions of t.

The series (11) can be written as a double series:

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}[a_n(t)\,(u+v)^n]\binom{n}{m}\left(\frac{v}{u+v}\right)^m\left(\frac{u}{u+v}\right)^{n-m};$$

this transformation together with the definition

(12) $$A_{N,n}=\sum_{m=0}^{\min(N,n)}\binom{n}{m}\left(\frac{v}{u+v}\right)^m\left(\frac{u}{u+v}\right)^{n-m}$$

enables us to replace the convergence of (11) by the equivalent condition of the existence of the limit

(13) $$\lim_{N\to\infty}\sum_{n=0}^{\infty}A_{N,n}[a_n(t)(u+v)^n]$$

for almost all t of $(0,1)$.

Now we have to apply to (13) a theorem of J. Marcinkiewicz and A. Zygmund [5]) about series of independent functions: $\sum x_k(t)$ being such a series summable (T) almost everywhere, there exists a numerical sequence $\{\lambda_k\}$ which makes the series $\sum(x_k(t)+\lambda_k)$ convergent almost everywhere. Summability (T) means here the existence of the limit

(14) $$\lim_{i\to\infty}\sum_{k=1}^{\infty}a_{ik}x_k,$$

the matrix $T=\|a_{ik}\|$ being any matrix with the property

(15) $$\lim_{i\to\infty}a_{ik}=1 \qquad (k=1,2,3,\ldots).$$

To apply the theorem we have to verify the independence of the functions $a_n(t)(u+v)^n$, these functions playing the role of $x_n(t)$, and the property (15) for $a_{ik}=A_{i,k}$ as defined by (12). Both properties being evidently valid (for the second one notice that $A_{N,n}=1$ for $N\geqslant n$), the existence of the limit (13) yields a numerical sequence $\{\lambda_n\}$ which makes the series

(16) $$\sum_{n=0}^{\infty}[a_n(t)(u+v)^n+\lambda_n]$$

convergent for almost all t in $(0,1)$. If we define $f_1(z)$ by

$$f_1(z)=\sum_{n=0}^{\infty}\frac{\lambda_n}{(u+v)^n}z^n,$$

the convergence of (16) implies the convergence of the series $F+f_1$ for $|z|<u+v$. Thus we get

$$r(F+f_1)\geqslant(u+v)>r(F),$$

the second inequality resulting from (10). This implies (9), q. e. d.

Proof of the theorem. We choose $f_0(z)$ to satisfy 3° of lemma 1. We have to show that this choice gives to the random series $H(z)$ defined by (4) the properties (i), (ii) and (iii) spoken of in the text of the theorem. (i) follows from 1° and 3° of lemma 1. Suppose $H(z)$ not to be singular; lemma 2 gives then an ordinary series $f_1(z)$ with $r(H+f_1)>r(H)=r(F+f_0)=r(G)$, which leads to $r(F+f_0+f_1)>r(G)$

[5]) *Quelques théorèmes sur les fonctions indépendantes*, Studia Mathematica 7 (1938), p. 104-120; p. 116, théorème 7.

against 1° of lemma 1; this contradiction yields (ii). To prove (iii) let us suppose (5) and $F(z)+f(z)$ not singular. Lemma 2 assures the existence of $f_1(z)$ with $r(F+f)<r(F+f+f_1)$; comparing with (5) we get $r(F+g)>r(H)$, where g means the ordinary series $f+f_1$; the last inequality contradicts (i) and thus (iii) is established.

(Reçu par la Rédaction le 6. 10. 1951)

Sur la convergence des séries de puissances de l'opérateur différentiel

par

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1. Considérons la série

$$\gamma_0+\gamma_1 s\lambda+\gamma_2 s^2\lambda^2+\dots, \tag{1}$$

où γ_n sont des nombres complexes, s l'opérateur différentiel[1]) et λ une variable complexe.

Pour la convergence de cette série, on a le critère suivant:

S'il existe un nombre $\delta>1$, tel que

$$\limsup_{n\to\infty} (n^n)^\delta |\gamma_n| < \infty, \tag{2}$$

la suite (1) *converge pour tout λ complexe. Si*

$$\limsup_{n\to\infty} n^n |\gamma_n| > 0, \tag{3}$$

la suite (2) *diverge pour tout λ complexe non nul.*

Démonstration. Soit $1<1/a<\beta<\delta$. Posons pour t réel

$$f(t)=\int_J \exp(zt-z^a)\,dz,$$

où l'intégrale est prise le long de l'axe imaginaire J. On a

$$f^n(t)=\int_J z^n \exp(zt-z^a)\,dz \qquad (n=0,1,2,\dots), \tag{4}$$

car chacune des intégrales (4) converge uniformément pour tout t, ce qui résulte de l'inégalité

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[1]) Voir [1], p. 47.