et les conditions aux limites

(22) $$x(\lambda_1)=0, \qquad x(\lambda_2)=0.$$

La solution générale de (21) est

$$x(\lambda)=c_1e^{i\lambda}+c_2e^{-i\lambda},$$

où c_1 et c_2 sont des opérateurs arbitraires. Il s'ensuit que la solution satisfaisant aux conditions (22) a la forme

(23) $$x(\lambda)=f\sin(\lambda-\lambda_1),$$

où f est un opérateur arbitraire. Pour que la fonction (23) soit paramétrique, il faut et il suffit que f soit une fonction ordinaire de la variable t. C'est ce qui implique la forme (20) pour toute solution du problème initial, concernant l'équation (18).

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On the Paley-Wiener theorem

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§ 1. Introduction. By the well known and important theorem of Paley and Wiener[1]), every entire function $F(z)$ of exponential type which belongs to L_2 along an infinite axis can be represented as a Fourier integral. In this paper we shall formulate and prove an analoguous theorem for analytic functions which are considered in a half-plane only. From such a theorem we can easily obtain the theorem for entire functions (§ 3), but not conversely.

Plancherel and Pólya[2]) have shown that the condition L_2 can be replaced by L_1. However, they have given a new proof for both cases. In this paper, we shall prove our theorem by the hypothesis that $F(z)$ belongs to L_p $(1\leqslant p\leqslant 2)$ along the boundary of the considered half-plane.

The form of the Plancherel and Pólya theorem is slightly sharper than that of Paley and Wiener. This form and still sharper forms will be discussed in § 6.

The proof of Plancherel and Pólya is based on the properties of entire functions and cannot be applied to the half-plane. The proof given in the sequel is, in some points, analoguous to that of Paley and Wiener; it leads, however, to an independent and more elementary argument for the particular case $p=1$.

§ 2. Theorem. We suppose throughout this paper that $F(z)$ is an analytic function in the half-plane $\Re z>0$ and that

$$\lim_{x\to 0+} F(x+iy)=F(iy) \quad \text{for almost every real } y.$$

Theorem. *If $e^{-k|z|}F(z)$ is bounded in the half-plane $\Re z>0$ and $F(iy)\epsilon L_p(-\infty,\infty)$ $(1\leqslant p\leqslant 2)$, then $F(z)$ can be represented, for $\Re z>0$, as an absolutely convergent integral*

[1]) Paley-Wiener [4], p. 12-13.

[2]) Plancherel-Pólya [6].

$$F(z)=\int_{-k}^{\infty}e^{-zt}f(t)\,dt. \tag{1}$$

In the case $p=1$, the function $f(t)$ in (1) is continuous and bounded for $t\geqslant -k$ and is given in the form of the absolutely convergent integral

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iyt}F(iy)\,dy. \tag{2}$$

In the case $1<p\leqslant 2$, the function $f(t)$ in (1) belongs to $L_q\left(\frac{1}{p}+\frac{1}{q}=1\right)$ and is given as the limit in mean with exponent q

$$f(t)=\underset{\omega\to\infty}{\mathrm{l.i.m.}}\,\frac{1}{2\pi}\int_{-\omega}^{\omega}e^{iyt}F(iy)\,dy. \tag{3}$$

Moreover, the function $f(t)$ defined by (2) vanishes for $t\leqslant -k$ and that defined by (3) vanishes almost everywhere for $t\leqslant -k$.

§ 3. Case of entire functions. From the preceeding Theorem we can easily deduce an analoguous theorem for entire functions. In fact, if $F(z)$ is entire, $e^{-k|z|}F(z)$ is bounded (in the whole plane) and $F(iy)\,\epsilon\, L_p(-\infty,\infty)$ $(1\leqslant p\leqslant 2)$, we can obviously apply the Theorem. On the other hand, we can apply the same Theorem to the function $G(z)=F(-z)$. We see, in case $p=1$, that the function

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iyt}G(iy)\,dy \tag{4}$$

vanishes for $t\leqslant -k$. Replacing y by $-y$ in (4), we find that $g(t)=f(-t)$. Thus, $f(t)=0$ for $t\geqslant k$ and the integral (1) gets reduced to

$$F(z)=\int_{-k}^{k}e^{-zt}f(t)\,dt.$$

This equality holds for $\Re z>0$, but both its sides are entire functions and, therefore, it must hold in the whole plane.

An analoguous argument can be used in case $1<p\leqslant 2$.

§ 4. Two lemmas. In the proof of the theorem we shall need the following two lemmas:

Lemma 1. *Let θ and k be any positive numbers. If $e^{-k|z|}F(z)$ is bounded in $\Re z>0$ and*

$$\int_{y}^{y+\theta}F(i\eta)\,d\eta$$

is bounded in $-\infty<y<\infty$, then the function

$$\Phi(z)=\frac{1}{\theta i}\,e^{-kz}\int_{z}^{z+i\theta}F(\zeta)\,d\zeta$$

is continuous and bounded in $\Re z>0$.

Lemma 2. *Let a be any positive number. If $\Phi(z)$ is analytic in $\Re z>0$, continuous and bounded in $\Re z\geqslant 0$, then the function*

$$\varphi(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iyt}\frac{\Phi(iy)}{(1+aiy)^2}\,dy \tag{5}$$

is continuous and bounded in $-\infty<t<\infty$, and vanishing for $t\leqslant 0$; moreover, we have, in $\Re z>0$,

$$\frac{\Phi(z)}{(1+az)^2}=\int_{0}^{\infty}e^{-zt}\varphi(t)\,dt.$$

Proof of Lemma 1. The continuity of $\Phi(z)$ in $\Re z>0$ and that of $\Phi(iy)$ for real y are obvious. But $F(z)$ is bounded in any bounded region and this implies, by the Lebesgue theorem,

$$\lim_{x\to 0+}\int_{x+iy}^{x+i(y+\theta)}F(\zeta)\,d\zeta=\int_{iy}^{i(y+\theta)}F(\zeta)\,d\zeta$$

and, consequently,

$$\lim_{x\to 0+}\Phi(x+iy)=\Phi(iy).$$

This suffices to ensure the continuity of $\Phi(z)$ in the closed half-plane $\Re z\geqslant 0$.

Now, we have, for a positive number M,

$$|\Phi(z)|\leqslant\frac{M}{\theta}\left|\int_{z}^{z+i\theta}e^{k|z|}\,dz\right|=\frac{M}{\theta}\int_{y}^{y+\theta}e^{k|x+i\eta|}\,d\eta\qquad\text{for}\quad \Re z>0;$$

since in the last integral we have $|x+i\eta|\leqslant|x+iy|+\theta$, it follows that

$$|\Phi(z)|\leqslant Me^{k\theta}\cdot e^{k|z|}\qquad\text{for}\quad \Re z>0.$$

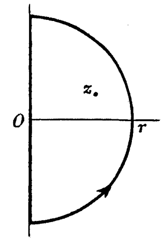
We have, further,

$$|\Phi(x)| \leqslant \frac{1}{\theta} e^{-kx} \int_0^\theta |F(x+i\eta)|\, d\eta \leqslant \frac{M}{\theta} e^{-kx} \int_0^\theta e^{k|x+i\eta|}\, d\eta \leqslant \frac{M}{\theta} \int_0^\theta e^{k\theta}\, d\eta = Me^{k\theta}$$

for positive x. Thus $\Phi(x)$ is bounded on the imaginary axis and on the real positive axis, and is of exponential type in the half-plane $\Re z > 0$. By the Phragmén-Lindelöf theorem, it must be bounded in the whole half-plane $\Re z \geqslant 0$.

Proof of Lemma 2. Let z be arbitrarily fixed in $\Re z > 0$. We have

(6) $$\frac{\Phi(z)}{(1+az)^2} = \frac{1}{2\pi i} \int_C \frac{\Phi(s)}{(1+as)^2} \cdot \frac{ds}{s-z}$$



where the contour C_r $(r > |z|)$ is composed of the semi-circle

(7) $$|s| = r, \qquad \Re s \geqslant 0$$

and of the segment of the imaginary axis embraced by this semi-circle. On the other hand, given any real u, we have

(8) $$0 = \frac{1}{2\pi i} \int_{C_r} \frac{\Phi(s)}{(1+as)^2} \, \frac{e^{-(s-z)u} - 1}{s-z}\, ds,$$

for the integrand is analytic inside the contour C_r and continuous on it. Adding (6) and (8) we get

$$\frac{\Phi(z)}{(1+az)^2} = \frac{1}{2\pi i} \int_{C_r} \frac{\Phi(s)}{(1+as)^2} e^{-(s-z)u} \frac{ds}{s-z}.$$

If $u \geqslant 0$, the part of that integral which belongs to the semi-circle (7) tends to 0, as $r \to \infty$, and so we get

(9) $$\frac{\Phi(z)}{(1+az)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(iy)}{(1+aiy)^2} \cdot \frac{e^{-(iy-z)u}}{z-iy}\, dy.$$

But

$$\frac{e^{-(s-z)u}}{z-s} = \int_{-u}^{\infty} e^{(s-z)t}\, dt \quad \text{for} \quad \Re s < \Re z;$$

substituting this into (9) and interchanging the order of integration we get

$$\frac{\Phi(z)}{(1+az)^2} = \int_{-u}^{\infty} e^{-zt}\, dt \, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyt} \frac{\Phi(iy)}{(1+aiy)^2}\, dy.$$

This formula is true for $\Re z > 0$ and any $u \geqslant 0$; but its left side does not depend of u, which implies that the interior integral, equal to the function $\varphi(t)$ in (5), must vanish almost everywhere for $t \leqslant 0$. On the other hand

$$|\varphi(t)| \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\Phi(iy)|}{1+a^2y^2}\, dy,$$

which proves the boundedness of $\varphi(t)$ and the uniform convergence of the integral in (5). Thus $\varphi(t)$ is continuous and our proof is complete.

§ 5. Proof of Theorem. Consider the function

(10) $$\Phi_\theta(z) = \frac{1}{\theta i} e^{-kz} \int_0^{z+i\theta} F(\zeta)\, d\zeta,$$

where θ is any given positive number. By the Hölder inequality, we have

$$|\Phi_\theta(iy)| \leqslant \frac{1}{\theta} \int_y^{y+\theta} |F(iy)|\, dy \leqslant \frac{1}{\theta} \left(\int_y^{y+\theta} d\eta \right)^{1/q} \cdot \left(\int_y^{y+\theta} |F(i\eta)|^p\, d\eta \right)^{1/p}$$

$$\leqslant \theta^{1/q-1} \left(\int_{-\infty}^{\infty} |F(i\eta)|^p\, d\eta \right)^{1/p}$$

for any real y and $1/p + 1/q = 1$. Thus $\Phi_\theta(z)$ is bounded on the imaginary axis. Since $e^{-k|z|}F(z)$ is bounded in $\Re z > 0$, we can apply Lemma 1. This asserts the continuity and boundedness of $\Phi_\theta(z)$ in $\Re z \geqslant 0$.

By Lemma 2 we have

(11) $$\frac{\Phi_\theta(z)}{(1+az)^2} = \int_0^\infty e^{-zt} \varphi_{a,\theta}(t)\, dt$$

where the function

$$\varphi_{a,\theta}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyt}\frac{\Phi_\theta(iy)}{(1+aiy)^2}\,dy,$$

is continuous and bounded in $-\infty<t<\infty$, and vanishing for $t\leqslant 0$. We shall show that, as $\theta\to 0$, the function $\varphi_{a,\theta}(t)$ tends to

$$\varphi_a(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iy(t-k)}\frac{F(iy)}{(1+aiy)^2}\,dy \tag{12}$$

uniformly in the interval $-\infty<t<\infty$. In fact, we have

$$|\varphi_{a,\theta}(t)-\varphi_a(t)|\leqslant\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1}{\theta}\int_y^{y+\theta}F(i\eta)\,d\eta-F(iy)\right|\frac{dy}{1+a^2y^2}$$

$$=\frac{1}{2\pi\theta}\int_{-\infty}^{\infty}\left|\int_0^{\theta}[F(iy+i\eta)-F(iy)]\,d\eta\right|\frac{dy}{1+a^2y^2} \tag{13}$$

$$\leqslant\frac{1}{2\pi\theta}\int_0^{\theta}\psi_a(\eta)\,d\eta,$$

where

$$\psi_a(\eta)=\int_{-\infty}^{\infty}|F(iy+i\eta)-F(iy)|\frac{dy}{1+a^2y^2}.$$

If $F(iy)$ belongs to L_1, we may write

$$\psi_a(\eta)\leqslant\int_{-\infty}^{\infty}|F(iy+i\eta)-F(iy)|\,dy; \tag{14}$$

if $F(iy)$ belongs to L_p $(1<p\leqslant 2)$, then by the Hölder inequality

$$\psi_a(\eta)\leqslant\left(\int_{-\infty}^{\infty}|F(iy+i\eta)-F(iy)|^p\,dy\right)^{1/p}\cdot\left(\int_{-\infty}^{\infty}\frac{dy}{(1+a^2y^2)^q}\right)^{1/q}. \tag{15}$$

Both (14) and (15) prove that

$$\lim_{\mu\to 0}\psi_a(\eta)=0.$$

Thus, by (13), $\varphi_{a,\theta}(t)$ tends to $\varphi_a(t)$ uniformly in $-\infty<t<\infty$. Moreover, $\varphi_a(t)$ is a continuous function, vanishing for $t\leqslant 0$.

Letting θ tend to 0, we obtain from (10) and (11)

$$\frac{e^{-kz}F(z)}{(1+az)^2}=\int_0^{\infty}e^{-zt}\varphi_a(t)\,dt$$

or, which is equivalent,

$$\frac{F(z)}{(1+az)^2}=\int_0^{\infty}e^{-zt}\varphi_a(t+k)\,dt. \tag{16}$$

Now we have to pass to the limit with a. If $F(iy)$ belongs to L_1, we get from (12)

$$|\varphi_a(t+k)|\leqslant\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(iy)|\,dy$$

and, by (2),

$$\lim_{a\to 0}\varphi_a(t+k)=f(t).$$

Thus, as $a\to 0$, we obtain from (16), by the Lebesgue theorem, the formula (1). Moreover, one sees from (2), that $f(t)$ is continuous.

If $F(iy)$ belongs to L_p $(1<p\leqslant 2)$, then, as $a\to 0$, the function

$$f_\omega(t)=\frac{1}{2\pi}\int_{-\omega}^{\omega}e^{iyt}F(iy)\,dy$$

converges, by a theorem of Titchmarsh[3]), in mean with exponent q to a function $f(t)$ such that

$$\int_{-\infty}^{\infty}|f(t)|^q\,dt\leqslant(2\pi)^{1-q}\left(\int_{-\infty}^{\infty}|F(iy)|^p\,dy\right)^{\frac{1}{p-1}}. \tag{17}$$

Similarly, by applying the same theorem of Titchmarsh to the difference

$$\frac{F(z)}{(1+az)^2}-F(z),$$

we are led to the inequality

$$\int_{-\infty}^{\infty}|\varphi_a(t+k)-f(t)|^q\,dt\leqslant(2\pi)^{1-q}\left(\int_{-\infty}^{\infty}\left|\frac{F(iy)}{(1+aiy)^2}-F(iy)\right|^p dy\right)^{\frac{1}{p-1}}. \tag{18}$$

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[3]) Titchmarsh [7], p. 96, Theorem 74.

But the last integral tends to 0, as $a\to 0$, which implies that $\varphi_a(t+k)$ converges in mean with exponent q to $f(t)$. Since $\varphi_a(t+k)$ vanishes for $t\leqslant -k$, the function $f(t)$ vanishes almost everywhere for $t\leqslant -k$.

Now, it is easy to see that, as $a\to 0$, the formula (16) takes the form (1) and that the integral in (1) is absolutely convergent for $\Re z>0$ (because f belongs to L_q).

§ 6. Sharper forms of Theorem. Plancherel and Pólya have given a slightly sharper form to the theorem of Paley and Wiener. Namely, suppose that

(α) $e^{-k|z|}F(z)$ is bounded in the whole plane of z,

(β) $F(iy)$ belongs to L_2.

Then the Paley-Wiener theorem asserts that $F(z)$ can be represented in the form

$$F(z)=\int_{-k}^{k} e^{-zt}f(t)\,dt. \tag{19}$$

Plancherel and Pólya have shown that if (α) and (β) hold and, moreover, if for some k' and k'' $(-k\leqslant -k'<k''\leqslant k)$

(γ) $e^{-k'x}F(x)$ and $e^{-k''x}F(-x)$ are bounded for $x>0$,

then the formula (19) can be improved by introducing narrower bounds of integration:

$$F(z)=\int_{-k'}^{k''} e^{-zt}f(t)\,dt. \tag{20}$$

It is easy to show that (20) follows, by (γ), from (19). In fact, when $|F(-x)|<Me^{k''x}$ for $x>0$, then it follows from (19) that, for $x>0$,

$$\Big|\int_{-k}^{k} e^{(t-k'')x}f(t)\,dt\Big|<M$$

or, which is equivalent,

$$\Big|\int_{-k-k''}^{k-k''} e^{tx}f(t+k'')\,dt\Big|<M.$$

Hence

$$\Big|\int_{0}^{k-k''} e^{tx}f(t+k'')\,dt\Big|<M_1=M+\int_{-k-k''}^{0}|f(t+k'')|\,dt,$$

and by an elementary theorem of Picone[4]) we have $f(t+k'')=0$ almost everywhere for $0\leqslant t\leqslant k-k''$ or $f(t)=0$ almost everywhere for $k''\leqslant t\leqslant k$.

Hence, the upper bound in the integral (19) is to be replaced by k''. Similarly, one can show that the effective bound in (19) is $-k'$.

If we use a stronger theorem than that of Picone, we can easily relax the condition (γ) and obtain in this way still sharper forms of the Paley-Wiener theorem. For instance, if $F(z)$ satisfies (α), (β) and

(δ) $e^{-k'x}F(x)$ is bounded for a sequence of positive numbers $x_1, x_2, \ldots$ such that $x_{n+1}-x_n>\delta>0$ and $\sum_{n=1}^{\infty} 1/x_n=\infty$,

then by a theorem of Levinson[5]), $e^{-k'x}F(x)$ will be bounded everywhere for $x>0$. Consequently, we can replace in (19) the lower bound of integration by $-k'$. We can proceed similarly with the upper bound.

The same argument holds, of course, in case of a half-plane, as in our Theorem.

[4]) Picone [5]; see also Mikusiński [2].

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[5]) Levinson [1], p. 241, Theorem VII; see also Mikusiński-Nardzewski [3].