The Fredholm Theory of Linear Equations in Banach spaces

by
T. LEŻAŃSKI (Warszawa).

In the first part of this paper\(^1\) we shall consider:
1. a fixed Banach space \( X \);
2. a fixed closed linear subspace \( E \) of the conjugate space \( X^* \) of all linear\(^2\) functionals on \( X \);
3. a fixed closed linear subspace \( K \) of the Banach space of all linear operations transforming \( E \) into \( E' \);
4. a fixed linear operation \( T \in \mathcal{L} \);
5. a fixed linear functional \( F \) defined on \( K \).

We shall use the following notations: If \( x \in X \) and \( \varphi \in \mathcal{L} \) then \( \varphi x \) is the value of the functional \( \varphi \) at the point \( x \). If \( K \) is a linear operation of \( \mathcal{L} \) into \( \mathcal{L}^* \) (or: into \( \mathcal{L} \)), then \( K\varphi \) is the functional associated with \( \varphi \) by \( K \) (i.e., the value of the mapping \( K \) at the point \( \varphi \)). Consequently \( K\varphi \) is the value of the functional \( K\varphi \) at the point \( x \).

Obviously, the expression \( K\varphi x \) can be interpreted as a bilinear functional on the Cartesian product \( \mathcal{L} \times X \). Conversely, each bilinear functional \( K\varphi \) on \( \mathcal{L} \times X \) can be interpreted as a linear operation \( K \) of \( \mathcal{L} \) into \( \mathcal{L}^* \) (in particular, into \( \mathcal{L} \)). Subsequently we shall always speak about linear operations of \( \mathcal{L} \) into \( \mathcal{L}^* \), although these operations will often be defined by expressions which are bilinear functionals on \( \mathcal{L} \times X \).

The superposition of two operations \( K_1, K_2 \in \mathcal{L} \) will be denoted by \( K_1 K_2 \). Of course, the expression \( K_1 K_2 \varphi \) should be read: \( (K_1 K_2)\varphi x \).

We shall suppose that the following conditions (K) and (F) are satisfied:

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\(^{2}\) The word “linear” always means “additive and continuous”.

\( \text{(K)} \) The identical mapping \( I \) of \( \mathcal{L} \) into \( \mathcal{L} \) belongs to \( \mathcal{L} \). If \( K_1, K_2 \in \mathcal{L} \), then \( K_1 K_2 \in \mathcal{L} \). If \( K \in \mathcal{L} \), and if \( x \in X \) and \( \varphi \in \mathcal{L} \) are fixed, then the linear operation \( M \),

\[ M\varphi x = K\varphi x_1 \varphi x_2 \]  

for \( y \in X \) and \( \varphi \in \mathcal{L} \),

also belongs to \( \mathcal{L} \), and

\( \text{(F)} \) \( F(M) = KT\varphi \).

The condition (F) yields that in some cases the values of the functional \( F \) are completely determined by \( T \) (see the example IV, p. 268). However, in general, \( F \) need not be uniquely determined by \( T \) (see the example A and the footnote\(^3\) on p. 269). Conversely, \( T \) is always completely determined by \( F \).

Under hypotheses (K) and (F), we shall examine the linear equation

\[ \varphi + KT\varphi = \psi \]

where \( K \in \mathcal{L} \), \( \psi \in \mathcal{L} \). The solution \( \varphi \) should also be in \( \mathcal{L} \).

We shall develop the theory of the equation (1) in a way completely analogous to the Fredholm theory of integral equations. For instance, we shall define an expression \( D \) determined by \( K, T \), and \( F \) only, such that the equation (1) has a (necessarily unique) solution for every \( \psi \in \mathcal{L} \) if and only if the number \( D \) is not equal to 0. If \( D = 0 \), then the homogeneous equation

\[ \varphi + KT\varphi = 0 \]

has non-trivial solutions. We shall define a sequence of multilinear functionals

\[ D_p \left( \varphi_1, \ldots, \varphi_p \right) \]

where \( \varphi_1, \ldots, \varphi_p \in \mathcal{L} \), \( x_1, \ldots, x_p \in X \), \( p = 1, 2, \ldots \), such that the (necessarily finite) dimension of the space of all solutions of (2) is the smallest integer \( p \) such that \( D_p \) is not identically equal to 0. We shall also determine under what conditions relating to \( \psi \) the equation (1) has a solution.

Clearly \( D \) should be called the determinant of the equation (1) by analogy with the finitely dimensional case. The functionals \( D_p \) are the analogue of the subdeterminants of the order \( p \) in the Fredholm theory of integral equations.
In the second part of this paper we shall specialize the spaces \( X \) and \( \mathcal{S} \). The following cases will be under consideration:

\[
X = L^p, \quad F, \quad L, \quad M, \quad V, \quad I, \quad m
\]

and

\[
\mathcal{S} = L^q, \quad F, \quad M, \quad L, \quad C, \quad m, \quad I \quad \quad (1/p + 1/q = 1)
\]

respectively.

In the cases of \( X = L^m \) and \( \mathcal{S} = m[I] \) the determinant \( D \) mentioned above is a generalization of Koch's determinant of an infinite system of linear equations with infinitely many variables (Koch's definitions\(^1\)) are introduced under more restrictive conditions. We shall also show, that in the cases mentioned above, the determinant \( D = D(I + T) \) of the equation

\[
(I + T)\varphi = \varphi_0
\]

is a multiplicative functional, i.e.

\[
D[(I + T)(I + T')] = D(I + T) \cdot D(I + T').
\]

This equation was earlier obtained by Koch under more restrictive conditions.

The infinite determinant \( D \) has also many other properties of the usual finite determinants. In particular, a formula analogous to the Cramer formula will be found for the solution of (3).

The existence of the functional \( F \) satisfying the condition (F) plays the fundamental part in our theory of linear equations. The functional \( F \) may be conceived as a generalization of the trace of a square matrix. In fact, in the case of \( X = L^m \), \( \mathcal{S} = m[I] \), linear operations \( K, T \) can be interpreted as certain infinite matrices, and \( F(K) \) is then the trace of the matrix \( KT \).

The restriction that we shall examine linear equations in conjugate spaces is not essential since each Banach space \( X \) can be interpreted as a closed subset of the space \( X^* \) conjugate to \( X \).

A similar Fredholm theory of linear equations in Banach spaces was developed by A. F. Ruston\(^4\). The theory of Ruston makes use of the notion of cross-spaces, which does not appear in my paper.

I. The general Fredholm theory.

In this part \( X, \mathcal{S}, R, T, F \) will have the meaning mentioned in the introduction. We suppose that conditions (K) and (F) are satisfied.

We adopt the following convenient notation. If

\[
B(\varphi_1, \ldots, \varphi_n; x_1, \ldots, x_n) = \mathcal{S} \quad \text{such that, for some fixed}
\]

\[
\varphi_1, \ldots, \varphi_n \in \mathcal{S}, \quad x_1, \ldots, x_n \in X
\]

the linear operation \( \mathcal{M} \),

\[
\mathcal{M}\varphi_1, \ldots, \varphi_n, x_1, \ldots, x_n \in \mathcal{S}
\]

belongs to \( \mathcal{S} \), then \( F_{\varphi_1, \ldots, \varphi_n}(B(\varphi_1, \ldots, \varphi_n; x_1, \ldots, x_n)) \) will denote the number

\[
F_{\mathcal{M}}[B(\varphi_1, \ldots, \varphi_n; x_1, \ldots, x_n)]
\]

is, in general, a function of variables

\[
\varphi_1, \ldots, \varphi_n \in \mathcal{S} \quad \text{and} \quad x_1, \ldots, x_n \in X
\]

however it does not depend on \( \varphi_n \) and \( x_n \) which are bound variables and can be replaced by other letters different from the remaining \( \varphi \) and \( x \) (compare e.g. the bound variable \( t \) in the expression \( \int f(t)dt \)).

Obviously

\[
(i) \quad |F_{\varphi_1, \ldots, \varphi_n}(B(\varphi_1, \ldots, \varphi_n; x_1, \ldots, x_n))| = |F(M)| < ||F|| \cdot ||M||
\]

\[
= |F| \cdot \sup_{1 \leq i \leq r} \sup_{1 \leq j \leq s} |B(\varphi_1, \ldots, \varphi_n; x_1, \ldots, x_n)|
\]

The condition (F) can now be formulated as follows:

\[
(F') \quad F_{\varphi_1, \ldots, \varphi_n}(K \varphi_1; \varphi_2) = K \mathcal{M}\varphi_1
\]

It follows from (F') that

\[
(ii) \quad \text{If} \quad \mathcal{K}, \mathcal{M} \in \mathcal{R}, \text{then for arbitrarily fixed} \quad \varphi \in \mathcal{S}, \quad x \in X, \text{the linear operation} \quad \mathcal{M},
\]

\[
\mathcal{M}\varphi = \mathcal{K}\varphi \cdot \mathcal{M}\varphi \quad \text{for} \quad \varphi \in \mathcal{S}, \quad y \in X,
\]

belongs to \( \mathcal{K} \) and the linear operation \( L = F(M), \text{ i.e.} \).
Let \( F_{p_1} \) be the solution of the equation \( L \psi = F_{p_1} \) for \( \psi \in \mathcal{S}, \, x \in X, \) satisfies the equation \( L = K TK x \). Consequently, \( L \in \mathcal{S} \), also.

Let \( \varphi \in K \). We have
\[
M_{\psi \psi} = K \varphi \cdot K \psi = K \varphi \cdot \varphi.
\]

Hence, for fixed \( x \) and \( \varphi, \) \( M \in \mathbb{R} \) by \( (K) \) and \( F(M) = K \varphi x \)
by \( (F) \), i.e.
\[
L_{\psi \psi} = K TK \psi = K TK \psi.
\]

We infer that \( L = K TK \psi \) on \( \mathcal{S} \) by \( (K) \).

(iii) If \( K \in \mathcal{S} \) and \( \pi_1, \pi_2, \ldots, \pi_p \) is a permutation of the sequence
\( 1, 2, \ldots, p \), then the expression (where \( q \leq p \))
\[
B_q = F_{\psi_{\pi_q}} F_{\psi_{\pi_{q-1}}} \cdots F_{\psi_{\pi_1}} \left( \prod_{j=1}^q K \varphi_j y_j \right)
\]
is well defined and independent of the ordering of the sequence of operators \( F_{\psi_{\pi_1}}, \ldots, F_{\psi_{\pi_p}} \).

Let \( K^{(m)} = (KT)(KT) \cdots (KT), \, K \in \mathcal{S}. \) We have \( K^{(0)} = K \) and by
\[
K^{(m)} \varphi x = F_{\psi_{\pi_1}} K^{(m)} \varphi x \cdot K^{(0)} \psi y
\]
if \( m = r + s + 1, \, r, s \geq 0. \)

First we shall prove by induction with respect to \( q = 0, \ldots, p \)
that \( B_q \) is well defined and
\[
B_q = a_{q-1} \prod_{j=1}^{p-q} K^{(m)} \psi_j y_j + f
\]
where \( \pi_1, \ldots, \pi_{p-q} \) is a permutation of the integers \( 1, \ldots, p, \) or
is a constant, and \( m_1, \ldots, m_{p-q} \) is a sequence of non-negative integers.

The formula \( (4) \) holds, of course, if \( q = 0. \) Suppose it is true
for \( q = 0 < q < p. \)

If \( s_1 = q + 1, \) the operation \( F_{\psi_{\pi_{q+1}}} \) is feasible and
\[
B_{q+1} = F_{\psi_{\pi_{q+1}}} \left( B_q \right) = a_{q+1} \prod_{j=1}^{p-q} K^{(m_j)} \psi_j y_j + f,
\]
where \( a_{q+1} = a_q F(K^{(m)}). \) Hence \( B_{q+1} \) is also of the form \( (4). \)

If \( q + 1 = s, \) \( r \neq 1, \) then the operation \( F_{\psi_{\pi_{q+1}}} \) is also feasible
and
\[
B_{q+1} = F_{\psi_{\pi_{q+1}}} \left( B_q \right) = a_{q+1} \prod_{j=1}^{p-q} K^{(m)} \psi_j y_j + f
\]
where \( a_{q+1} = a_q F(K^{(m)}). \) Hence \( B_{q+1} \) is also of the form \( (4). \)

Thus \( B_{q+1} \) is also of the form \( (4). \)

Now we shall prove that
\[
F_{\psi_{\pi_{q+1}}} F_{\psi_{\pi_{q+1}}} \left( B_q \right) = F_{\psi_{\pi_{q+1}}} F_{\psi_{\pi_{q+1}}} \left( B_q \right).
\]

The following seven cases should be considered:

(a) \( q + 1 = s, \) and \( q + 2 = s. \) Then both sides of \( (4) \) are equal to the number
\[
a_q F(K^{(m)}): F(K^{(m)}) \prod_{j=1}^{p-q} K^{(m)} \psi_j y_j + f.
\]

(b) \( q + 1 = s, \) and \( q + 2 = s, \) \( r > 2. \) Then both sides of \( (4) \) are equal to
\[
a_q F(K^{(m)}): K^{(m)} \psi_j y_j + f \prod_{j=1}^{p-q} K^{(m)} \psi_j y_j + f
\]
\[
= a_q F(K^{(m)}): K^{(m)} \psi_j y_j + f \prod_{j=1}^{p-q} K^{(m)} \psi_j y_j + f.
\]

(c) \( q + 1 = s, \) and \( q + 2 = s. \) Then both sides of \( (4) \) are equal to
\[
a_q F(K^{(m)}): K^{(m)} \psi_j y_j + f \prod_{j=1}^{p-q} K^{(m)} \psi_j y_j + f.
\]

(d) \( q + 1 = s, \) and \( q + 2 = s, \) \( r > 2. \) Then both sides of \( (4) \) are equal to
\[
a_q F(K^{(m)}): K^{(m)} \psi_j y_j + f \prod_{j=1}^{p-q} K^{(m)} \psi_j y_j + f.
\]

(e) \( q + 1 = s, \) \( r > 2, \) and \( q + 2 = s. \) The proof is the same as that of (d).

(f) \( q + 1 = s, \) \( r > 2, \) and \( q + 2 = s. \) The proof is the same as that of (b).
(g) \( q+1 = s, \quad q+2 = s, \quad r, t > 2. \) Then both sides of (4') are equal to
\[
\alpha K^{(m+1)} K_{n+1} y_{n+1} y_{n+2} + \sum_{r, t > 2} K^{(n)} y_{n+r} y_{n+t}.
\]

Notice moreover that

(iii') \( F_{rs} [K_{rs} x \cdot y_{rs} y_{r}] = F_{rs} [K_{rs} x \cdot y_{rs} y_{r}]. \)

(ii') \( F_{rs} [K_{rs} x \cdot K_{rs} y \cdot y_{r} y_{r}] = F_{rs} [K_{rs} x \cdot F_{rs} [K_{rs} y \cdot y_{r} y_{r}]]. \)

In fact both sides of (iii') are equal to \( KTKT_{rs} \) by (ii). Analogously both sides of (iii'') are equal to \( KTKT_{rs} \cdot KTKT_{rs}. \)

We shall now examine the equation (1') which, of course, can be written in the form

\[ F + A F = \psi_0 \quad \text{or} \quad (I + A) F = \psi_0, \]

where \( A = KTK \), \( K \in \mathcal{R} \), \( \psi_0 \in \mathcal{S}. \) The solution \( \varphi \) should also be in \( \mathcal{S}. \)

We introduce the following notations, analogous to those in the Fredholm theory of integral equations:

\[ F_{rs} \psi_{rs} x \psi_{rs} y_{rs} = \left[ F_{rs} x \psi_{rs} y_{rs} \right] \]

\[ K_{rs} \psi_{rs} x \psi_{rs} y_{rs} = \left[ K_{rs} x \psi_{rs} y_{rs} \right] \]

\[ a_0 = 1, \quad a_q = \frac{1}{q!} F_{rs} x \psi_{rs} y_{rs} \]

\[ A_q = A - A - A_{q-1}, \]

\[ K_a = \sum_{r, t > 2} K_{rs} x \psi_{rs} y_{rs}, \]

where \( q = 1, 2, \ldots \)

Clearly \( K_a \in \mathcal{R}. \) We shall prove that

\[ \frac{1}{q!} F_{rs} x \psi_{rs} y_{rs} \]

The proof is by induction with respect to \( q. \) The case \( q = 0 \) is obvious. Suppose (iv) is true for \( q - 1 \geq 0. \) We shall prove that (iv) is true for \( q. \)

\[ \prod_{r, t > 2} [K_{rs} x \psi_{rs} y_{rs}]. \]

We have

\[
F_{rs} \psi_{rs} x \psi_{rs} y_{rs} = K_{rs} x \cdot K_{rs} y \cdot y_{rs} y_{rs} + \sum_{q=1}^{s} (-1)^q K_{rs} x \cdot K_{rs} y \cdot y_{rs} y_{rs}.
\]

Hence, by the additivity, homogeneity and commutativity (see (iii)) of the operators \( F_{rs}, \) we obtain

\[
F_{rs} \psi_{rs} x \psi_{rs} y_{rs} = K_{rs} x \cdot F_{rs} \psi_{rs} x \psi_{rs} y_{rs} + \sum_{q=1}^{s} (-1)^q K_{rs} x \cdot F_{rs} \psi_{rs} x \psi_{rs} y_{rs}.
\]

Consequently, by the induction hypothesis,

\[
F_{rs} \psi_{rs} x \psi_{rs} y_{rs} = q! a_q K_{rs} x \cdot \sum_{q=1}^{s} (-1)^q K_{rs} x \cdot F_{rs} \psi_{rs} x \psi_{rs} y_{rs}.
\]

Now we prove

\[ \|K_{rs} \| \leq \frac{(q+1)^{s+1}}{q!} \|F\| \cdot \|K\|^{s+1}. \]

In fact, by (iv) and (i),
Fredholm theory of linear equations.

Now let

$$V_q(\varphi_1, \ldots, \varphi_p) = F_{n, q_1, \ldots, q_p} \left\{ x, \varphi_1, \ldots, \varphi_p \right\}.$$  

$V_q$ is a functional linear with respect to each variable $\varphi_1, \ldots, \varphi_p$; $x_1, \ldots, x_p$ running through $\mathcal{E}$ and $\mathcal{X}$ respectively. The norm of $V_q$ is

$$\|V_q\| = \sup_{t_1, \ldots, t_p} \left| V_q(\varphi_1, \ldots, \varphi_p) \right| \leq (p + q)^{\frac{1}{2}} \cdot \|F\| \cdot \|K\|^{1/2},$$

by an argument analogous to that in the proof of (v). Hence

$$\|V_q(\varphi_1, \ldots, \varphi_p)\| \leq (p + q)^{\frac{1}{2}} \cdot \|F\| \cdot \|K\|^{1/2} \cdot \|\varphi_1\| \cdot \|\varphi_2\| \cdot \ldots \cdot \|\varphi_p\|.$$  

This implies that

$$D_{n, q}(\varphi_1, \ldots, \varphi_p) = (-1)^p \sum_{t_1, \ldots, t_p} \|F\| \cdot \|K\|^{1/2} \cdot V_q(\varphi_1, \ldots, \varphi_p)$$

is an integral function of $\lambda$ and a functional linear with respect to each variable $\varphi_1, \ldots, \varphi_p \in \mathcal{E}$, $x_1, \ldots, x_p \in \mathcal{X}$ ($p > 0$).

(xii) For every $\lambda$, there is an integer $p \geq 0$ such that $D_{n, \lambda} \neq 0$ (i.e. $D_{n, \lambda}$ is not identically equal to 0).

Suppose the converse, i.e.

$$D_{n, \lambda} = 0 \quad (p = 0, 1, 2, \ldots)$$

for a number $\lambda$. Since

$$\frac{d^p D_{n, \lambda}}{dx^p} \bigg|_{x_1, \ldots, x_p} = \sum_{t_1, \ldots, t_p} \frac{2^p}{q_1!} V_q(\varphi_1, \ldots, \varphi_p),$$

we obtain that

$$\frac{d^p D_{n, \lambda}}{dx^p} \bigg|_{x_1, \ldots, x_p} = 0,$$

i.e. $D_{n, \lambda} = 0$ identically. This is impossible since $D_{n, \lambda} = a_q = 1$. 

We prove by an easy induction with respect to $q$ that (see (ii))

(vii) $A_q = K_q T_q.$

Hence, by (vi),

(viii) $\|A_q\| \leq \frac{(q + 1)^{1/2}}{q!} \cdot \|F\| \cdot \|K\|^{1/2} \cdot \|T\|.$

It follows from (vi) and (viii) that

(ix) The series

$$D_{n, \lambda} = \sum_{\lambda = 0}^{\infty} A_q \lambda^q$$

and

$$\sum_{\lambda = 0}^{\infty} \lambda^q A_q \lambda_q$$

are uniformly convergent in each interval $-r \leq \lambda \leq r$.  

Now let

$$V_q(\varphi_1, \ldots, \varphi_p) = F_{q_1, \ldots, q_p} \left\{ x, \varphi_1, \ldots, \varphi_p \right\}.$$  

$V_q$ is a functional linear with respect to each variable $\varphi_1, \ldots, \varphi_p$; $x_1, \ldots, x_p$ running through $\mathcal{E}$ and $\mathcal{X}$ respectively. The norm of $V_q$ is

$$\|V_q\| = \sup_{t_1, \ldots, t_p} \left| V_q(\varphi_1, \ldots, \varphi_p) \right| \leq (p + q)^{1/2} \cdot \|F\| \cdot \|K\|^{1/2},$$

by an argument analogous to that in the proof of (v). Hence

$$\|V_q(\varphi_1, \ldots, \varphi_p)\| \leq (p + q)^{1/2} \cdot \|F\| \cdot \|K\|^{1/2} \cdot \|\varphi_1\| \cdot \|\varphi_2\| \cdot \ldots \cdot \|\varphi_p\|.$$  

This implies that

$$D_{n, q}(\varphi_1, \ldots, \varphi_p) = (-1)^p \sum_{t_1, \ldots, t_p} \|F\| \cdot \|K\|^{1/2} \cdot V_q(\varphi_1, \ldots, \varphi_p)$$

is an integral function of $\lambda$ and a functional linear with respect to each variable $\varphi_1, \ldots, \varphi_p \in \mathcal{E}$, $x_1, \ldots, x_p \in \mathcal{X}$ ($p \geq 0$).
We shall write, for simplicity, \( D_p \) instead of \( D_{k,1} \), and \( D \) instead of \( D_0 = D_{1,1} \). Hence

\[
D = D_0 = \sum_{n=0}^{\infty} d_n t^n,
\]

\[
D_p \left( \frac{\varphi_1 \ldots \varphi_p}{x_1, \ldots, x_p} \right) = (-1)^{p} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\varphi_1 \ldots \varphi_p}{x_1, \ldots, x_p} \right)
\]

for \( p = 0, 1, 2, \ldots \).

We have

\[
D_{p+1} \left( \frac{\varphi_1 \varphi_2 \ldots \varphi_p}{x_1, x_2, \ldots, x_p} \right) = -K_{\varphi x} \cdot D_p \left( \frac{\varphi_1 \ldots \varphi_p}{x_1, \ldots, x_p} \right) - \sum_{n=1}^{p} (-1)^{n+1} K_{\varphi x} \cdot D_n \left( \frac{\varphi_1 \ldots \varphi_n}{x_1, \ldots, x_n} \right) - F_{\varphi, y} \left( \frac{\varphi_1 \varphi_2 \ldots \varphi_p}{x_1, x_2, \ldots, x_p} \right) \cdot K_{y y}.
\]

Hence

\[
D_{p+1} \left( \frac{\varphi_1 \varphi_2 \ldots \varphi_p}{x_1, x_2, \ldots, x_p} \right) = -K_{\varphi x} \cdot D_p \left( \frac{\varphi_1 \ldots \varphi_p}{x_1, \ldots, x_p} \right) - \sum_{n=1}^{p} (-1)^{n} K_{\varphi x} \cdot D_n \left( \frac{\varphi_1 \ldots \varphi_{n-1} \varphi_{n+1} \ldots \varphi_p}{x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_p} \right) - F_{\varphi, y} \left( \frac{\varphi_1 \varphi_2 \ldots \varphi_p}{y_1, y_2, \ldots, y_p} \right) \cdot K_{y y}.
\]

Expand the determinant

\[
K_{\varphi x} \cdot K_{\varphi y} \cdot K_{y y}
\]

on the first row:

\[
K_{\varphi x} \cdot K_{\varphi y} \cdot K_{y y} = K_{\varphi x} \cdot K_{\varphi y} \cdot K_{y y} - \sum_{n=1}^{p} (-1)^{n+1} K_{\varphi x} \cdot K_{\varphi y} \cdot K_{y y} - \sum_{n=1}^{p} (-1)^{n} K_{\varphi x} \cdot K_{\varphi y} \cdot K_{y y}.
\]

Apply the operator

\[
F_{\varphi, y_1} \ldots F_{\varphi, y_p}
\]

to both sides of the above equation. By the commutativity of \( F_{\varphi, y} \) (see (iii)) we obtain

\[
V_{\varphi} \left( \frac{\varphi_1 \ldots \varphi_p}{x_1, x_2, \ldots, x_p} \right) = K_{\varphi x} \cdot V_{\varphi} \left( \frac{\varphi_1 \ldots \varphi_p}{x_1, x_2, \ldots, x_p} \right) - \sum_{n=1}^{p} (-1)^{n+1} K_{\varphi x} \cdot V_{\varphi} \left( \frac{\varphi_1 \ldots \varphi_{n-1} \varphi_{n+1} \ldots \varphi_p}{x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_p} \right) - \sum_{n=1}^{p} (-1)^{n} K_{\varphi x} \cdot V_{\varphi} \left( \frac{\varphi_1 \varphi_2 \ldots \varphi_p}{x_1, x_2, \ldots, x_p} \right) \cdot K_{y y}.
\]

Replace \( \varphi_1, \ldots, \varphi_p \) by \( \psi, y \) respectively. Consequently

\[
V_{\psi} \left( \frac{\varphi_1 \ldots \varphi_p}{x_1, x_2, \ldots, x_p} \right) = K_{\psi x} \cdot V_{\psi} \left( \frac{\varphi_1 \ldots \varphi_p}{x_1, x_2, \ldots, x_p} \right) - \sum_{n=1}^{p} (-1)^{n+1} K_{\psi x} \cdot V_{\psi} \left( \frac{\varphi_1 \ldots \varphi_{n-1} \varphi_{n+1} \ldots \varphi_p}{x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_p} \right) - \sum_{n=1}^{p} (-1)^{n} K_{\psi x} \cdot V_{\psi} \left( \frac{\varphi_1 \varphi_2 \ldots \varphi_p}{x_1, x_2, \ldots, x_p} \right) \cdot K_{y y}.
\]

This implies immediately the equation (xii). The proof of (xiv) is analogous. We should now expand the determinant

\[
K_{\varphi x} \cdot K_{\varphi y} \cdot K_{y y}
\]

on the first column.

Replace now \( \varphi \) by \( \zeta \) in the equations (xii) and (xiv), multiply both sides of these equations by \( g_z \) and apply the operator \( F_{\varphi, z} \). We obtain by (iii) and (iii').
The equations (xv) and (xvi) can be written as follows (see (F)'):

\[(\text{xvii}) \quad U_p g = -\delta_p A g - U_p A g - \sum_{i=1}^{p} \Phi_i g_i g_i,\]

\[(\text{xviii}) \quad U_p g = -\delta_p A g - \delta_p A g - \sum_{i=1}^{p} \Phi_i g_i g_i.\]

The operation \(I + A\) is said to be of an order \(p\) if \(D_p \neq 0\) and \(D_r = 0\) for all \(r < p\) \((r \geq 0)\).

(xix) Suppose the order of \(I + A\) is \(p > 0\). Then \(\Phi_i g_i = -\delta_p \delta_p g_i\) and \(\Psi_i g_i = -\delta_p g_i\), where \(\delta_p\) is the Kronecker symbol.

In fact, from (xiv) where \(p\) is replaced by \(p-1\) we obtain

\[(9) \quad D_p \left[ g_1, \ldots, g_p \right] = \sum_{i=1}^{p} \Phi_i g_i \sum_{j=1}^{p} \Psi_i g_i g_j.\]

Replace now \(z_i\) by \(x_i\) and conversely. We obtain

\[(10) \quad D_p \left[ g_1, \ldots, g_p \right] = \sum_{i=1}^{p} \Phi_i g_i \sum_{j=1}^{p} \Psi_i g_i g_j.\]

i. e. by the definition of \(\Phi_i\) and \(g_i\) (see (F)'),

\[-\delta_p = \Phi_i g_i.\]

Replace \(z_i\) by \(x_i\) \((i \neq j)\) in the formula (10). The left side is equal to 0, and the other is equal to \(\Phi_i g_i\). This proves the first equation in (xix). The proof of the second is analogous (see (xiv)).

Subsequently we shall suppose that \(I + A\) is of the order \(p > 0\) and (in the case of \(p > 0\)) that

\[g_1, \ldots, g_p \in \mathcal{E}, \quad x_1, \ldots, x_p \in X\]

are so chosen that

\[\delta_p = D_p \left[ g_1, \ldots, g_p \right] \neq 0.\]

Under these hypotheses, it follows immediately from (xix) that
(xx) If \( I + A \) is of order \( p > 0 \), then the functionals \( g_1, \ldots, g_p \) are linearly independent. Analogously, the functionals \( \mathcal{Y}_1, \ldots, \mathcal{Y}_p \) are linearly independent.

**Theorem 1.** If the operation \( I + A \) has the order \( p > 0 \), then the sequence \( g_1, \ldots, g_p \) is the basis of the linear space of all \( \varphi \) which are solutions of the homogeneous equation

\[
\varphi + A\varphi = 0
\]

(i.e. \( \varphi \) satisfies (11) if and only if \( \varphi = \sum_{i=1}^{p} \alpha_i g_i \)).

If we replace \( x_i \) by \( y_i \) in (9) we obtain

\[
D_x \left( \varphi_1, \varphi_2, \ldots, \varphi_p \right) + P_{x_i} \left( K_{x_i,y_i} D_x \left( \varphi_1, \varphi_2, \ldots, \varphi_p \right) \right) = 0,
\]

i.e.

\[
g_i + P_{x_i} \left( K_{x_i,y_i} g_i \right) = 0.
\]

By (F) this equation can be written in the form

\[
g_i + K T g_i = 0.
\]

Thus \( g_i \) is a solution of (11). The proof that \( g_1, \ldots, g_p \) are also solutions of (11) is analogous.

Suppose now that \( \varphi \) is a solution of (11). It follows from (xvii) and (11) that

\[
U_p \varphi = \delta_p \varphi + U_p \varphi - \sum_{i=1}^{p} F_i \varphi \cdot g_i.
\]

Hence

\[
\delta_p \varphi + U_p \varphi = \sum_{i=1}^{p} F_i \varphi \cdot g_i,
\]

which completes the proof of Theorem 1.

(xxi) If the order of \( I + A \) is \( p > 0 \), and if the equation (1) has a solution \( \varphi \), then the functional

\[
\varphi_b = \varphi + \frac{1}{\delta_p} U_p \varphi
\]

is also a solution (1), i.e.

\[
\varphi_b + A\varphi_b = \varphi_b.
\]

Suppose \( \varphi + A\varphi = \varphi_b \), i.e. \( A\varphi = \varphi_b - \varphi \). Replace \( A\varphi \) by \( \varphi_b - \varphi \) in the equation (xviii). We obtain

\[
U_p \varphi = \delta_p (\varphi_b - \varphi) - U_p \varphi_b + U_p \varphi - \sum_{i=1}^{p} F_i \varphi \cdot g_i,
\]

i.e.

\[
\delta_p \varphi_b + U_p \varphi_b = \delta_p \varphi - \sum_{i=1}^{p} F_i \varphi \cdot g_i.
\]

Hence

\[
\varphi_b = \varphi - \sum_{i=1}^{p} \frac{\alpha_i}{\delta_p} g_i
\]

where \( \alpha_i = \frac{1}{\delta_p} F_i \varphi \).

Since \( \varphi \) is a solution of (1) and \( g_i \) are solutions of the homogeneous equation (11), \( \varphi_b \) is a solution of (1).

**Theorem 2.** Suppose the order of \( I + A \) is \( p > 0 \). The equation (1) has solutions if and only if \( \mathcal{Y}_1 \varphi_b = 0 \) for \( i=1, \ldots, p \).

By lemma (xxi), if (1) has solutions, then

\[
\varphi_b = \varphi + \frac{1}{\delta_p} U_p \varphi
\]

is also a solution of (1). Replacing \( \varphi \) by \( \varphi_b \) in (1) we obtain

\[
U_p \varphi_b + \delta_p A \varphi_b = A U_p \varphi_b = 0.
\]

By (xviii) the left side of the above equation is equal to

\[
\sum_{i=1}^{p} \mathcal{Y}_i \varphi_b \cdot g_i = 0.
\]

Hence

\[
\sum_{i=1}^{p} \mathcal{Y}_i \varphi_b \cdot g_i = 0.
\]

By (xxi) the functionals \( g_i \) are linearly independent. Hence

\[
\mathcal{Y}_i \varphi_b = 0 \quad \text{for} \quad i=1, \ldots, p.
\]

Conversely, if \( \mathcal{Y}_i \varphi_b = 0 \) for \( i=1, \ldots, p \), then (13) holds. Consequently (12) holds, i.e.

\[
\varphi_b = \varphi + \frac{1}{\delta_p} U_p \varphi
\]

is a solution of (1).

**Theorem 3.** The equation (1) is solvable for every \( \varphi \in E \) if and only if \( I + A \) has the order 0, i.e. if \( D \neq 0 \).
In fact, if \( D \neq 0 \), then the operation (see (7))
\[
I - \frac{1}{D} \sum_{q=1}^{\infty} A_q
\]
is converse to the operation \( I + A \) since, by (7),
\[
(I + A)
\left(I - \frac{1}{D} \sum_{q=1}^{\infty} A_q\right)
= I + A - \frac{1}{D} \left( \sum_{q=1}^{\infty} A_q(I + A) \right)
= I + A - \frac{1}{D} \left( \sum_{q=1}^{\infty} A_q + \sum_{q=1}^{\infty} (a_{q1} A - A_{q1}) \right)
= I + A - \frac{1}{D} DA = I.
\]

On the other hand, if \( I + A \) has the order \( p > 0 \), then, by Theorem 2, the equation \( \varphi + A \varphi = \varphi \) has no solution since \( \varphi_{\varphi q} \neq 0 \) by (xii).

Let \( \tilde{S} \) be the Banach space of all linear functionals on \( S \), and let \( \tilde{A} \) be the operation (of \( \tilde{S} \) into \( \tilde{S} \)) conjugate to \( A \), i.e. \( \varphi = \tilde{A} \varphi \) is equivalent to: \( \varphi(\varphi) = \varphi(\varphi) \) for each \( \varphi \in \tilde{S} \).

Theorem 4. Suppose the order of \( I + A \) is \( p > 0 \). The sequence \( \varphi_1, \ldots, \varphi_p \) is a basis for the linear space of all solutions of the homogeneous equation
\[
(14) \quad \varphi + \tilde{A} \varphi = 0 \quad (\varphi \in \tilde{S}).
\]

By the definition of \( \varphi_i \) and by the equation (see (F'))
\[
A \varphi = E \varphi \left[ K \psi \cdot \varphi \right]
\]
we obtain
\[
\varphi_i(\varphi) + \varphi_i(A(\varphi)) = E \varphi \left[ B \left[ K \psi \cdot \varphi \right] \right]
\]
\[
+ E \varphi \left[ B \left[ K \psi \cdot \varphi \right] \right]
\]
The expression in the outer brackets \( \left[ \right] \) is equal to 0 by (xii) (where \( p \) is replaced by \( p - 1 \)) since \( D_{p-1} = 0 \). Consequently

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\[
\mathcal{P}(q) + \mathcal{P}(\mathbb{A}(q)) = 0 \quad \text{for every } q \in \mathbb{S}, \text{ i.e. } \mathcal{P}(q) + \mathcal{A}(q) = 0 \quad \text{for } i = 1, \ldots, p.
\]
Thus the linearly independent (see (xx)) functionals \( \mathcal{P}_1, \ldots, \mathcal{P}_p \) are solutions of (14).

Suppose now that (14) holds, i.e. \( \mathcal{P} + \mathcal{A} \mathcal{P} = 0 \). By (xvii)
\[
\mathcal{P} U_q \mathcal{P} = - \delta_q \mathcal{P}(A(q)) \mathcal{P}(\mathbb{A}(q)) - \sum_{i=1}^{p} \mathcal{P}_i \mathcal{P}_i \cdot \mathcal{P}_q,
\]
Replacing \( \mathcal{P}(A(q)) \) by \( - \mathcal{P}(q) \) we obtain
\[
\mathcal{P} U_q \mathcal{P} = \delta_q \mathcal{P}_q + \sum_{i=1}^{p} \mathcal{P}_i \mathcal{P}_i \cdot \mathcal{P}_q,
\]
for every \( q \in \mathbb{S} \), i.e.
\[
\mathcal{P} = \frac{1}{\delta_q} \sum_{i=1}^{p} \mathcal{P}_i \mathcal{P}_i \cdot \mathcal{P}_q \quad q. \quad e. \quad d.
\]

For applications it is important to know that \( D \) depends continuously on \( F \) and \( K \). Let \( \mathbb{S} \) be the set of all functionals \( \varphi \) on \( \mathbb{R} \) such that there is a linear operation \( T \) on \( \mathbb{R} \) such that \( (F) \) holds. Clearly \( \mathbb{S} \) is a linear subspace of the normed space \( \mathbb{R} \) conjugate to \( \mathbb{R} \). The expression
\[
a_{q}(E^{01}, E^{02}, \ldots, E^{0n}, E^{11}, E^{12}, \ldots, E^{1n})
\]
\[
\frac{1}{q!} E^{01} \varphi_{1} \cdots E^{0n} \varphi_{n} \left[ K^{11} \varphi_{1} \cdots K^{1n} \varphi_{n} \right]
\]
is a 2k-linear operation on
\[
\mathbb{S} \times \mathbb{S} \times \cdots \times \mathbb{S} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}
\]
\[
\frac{q!}{2^{q}} E^{01} \varphi_{1} \cdots E^{0n} \varphi_{n} \left[ K^{11} \varphi_{1} \cdots K^{1n} \varphi_{n} \right]
\]
since
\[
|a_{q}(E^{01}, E^{02}, \ldots, E^{0n}, E^{11}, E^{12}, \ldots, E^{1n})| \leq \frac{q!}{2^{q}} \left| E^{01} \right| \cdots \left| E^{0n} \right| \cdots \left| E^{11} \right| \cdots \left| E^{1n} \right| \left| K^{11} \right| \cdots \left| K^{1n} \right|
\]
by an argumentation similar to that in the proof of (v) and (vi). Hence
\[
a_{q}(F, K) = a_{q}(F, \ldots, F, K, \ldots, K) = \frac{1}{q!} E_{q} \varphi_{1} \cdots E_{q} \varphi_{n} K \left( \varphi_{1} \cdots \varphi_{n} \right)
\]
for every \( q \in \mathbb{S} \).
is a continuous polynomial on $\mathfrak{F} \times \mathbb{R}$ and
\[
|a_q(F,K)| \leq \frac{e^2}{q^3} \frac{\|F\|^q}{\|K\|^q}.
\]

Therefore the series
\[
D(F,K) = \sum_{q \in \mathbb{N}} a_q(F,K) \quad (a_0(F,K) = 1)
\]
is uniformly convergent on each bounded subset of $\mathfrak{F} \times \mathbb{R}$, consequently it is a continuous function on $\mathfrak{F} \times \mathbb{R}$, i.e.:

(xxii) If $\|F^{(n)} - F\| \to 0$ and $\|K^{(n)} - K\| \to 0$, then
\[
D(F^{(n)}, K^{(n)}) \to D(F,K).
\]

II. Applications.

1. Consider first the case where $X$ is the space $L^2$ of all measurable functions $x(t)$ on $(0,1)$ with
\[
\|x\| = \left( \int_0^1 |x(t)|^2 dt \right)^{1/2} < \infty.
\]

$\mathcal{E}$ is the conjugate space $L^2(1/p + 1/q = 1, 1 < p, q < \infty)$. Clearly, each function $\varphi \in \mathcal{E}$ determines uniquely a functional on $X$ denoted by the same letter $\varphi$:
\[
\varphi x = \int_0^1 \varphi(t) x(t) dt.
\]

Each measurable function $K(t,s)$ defined on the square $0 < t, s < 1$, such that
\[
\frac{1}{4} \int_0^1 \int |K(t,s) x(t) y(s)| dt ds < \infty \quad \text{for} \quad x \in X, \varphi \in \mathcal{E},
\]
determines uniquely a linear operation of $\mathcal{E}$ into $\mathcal{E}$ denoted by the same letter $K$:
\[
K \varphi(t) = \int_0^1 K(t,s) \varphi(s) ds, \quad \text{i.e.} \quad \varphi x = \int_0^1 K(t,s) x(t) \varphi(s) ds dt.
\]

Let $\mathcal{R}_n$ be the class of all operations $K$ of the form (17) where $K(t,s)$ satisfies (16), and let $\mathcal{R}$ be the least closed linear set of li-

near operations of $\mathcal{E}$ into $\mathcal{E}$ such that $I \in \mathcal{R}$ and $\mathcal{R}_n \subset \mathcal{R}$. The linear space $\mathcal{R}$ satisfies the condition (K).

In fact, if $K_1, K_2 \in \mathcal{R}_n$, then $K = K_1 + K_2 \in \mathcal{R}_n$ since $K$ is determined by the function
\[
K(t,s) = \int_0^1 K_1(t,r) K_2(r,s) dr
\]
which satisfies the condition (16). Consequently, if $K_1, K_2 \in \mathcal{R}$, then $K_1 + K_2 \in \mathcal{R}$. If $K \in \mathcal{R}_n$, and if $x_0 \in X, \varphi_0 \in \mathcal{E}$, then
\[
M(x,y) = K x_0 \cdot \varphi_0 y = \int_0^1 M(t,x) \varphi_0(y(t)) dt ds
\]
where
\[
M(t,x) = \int_0^1 K(r,s) x_0(r) \varphi_0(s) dr,
\]

which proves that $M \in \mathcal{R}_n$. If $K = I$, we have
\[
M(x,y) = \int_0^1 x_0(t) \varphi_0(y(t)) dt ds
\]
where $M(t,x) = \varphi_0(x(t))$ which proves that $M \in \mathcal{R}_n$. By continuity, $M \in \mathcal{R}$ for any $K \in \mathcal{R}_n$, q.e.d.

Now let $E, S \in \mathcal{R}_n$ be two operations such that
\[
\int_0^1 \left( \frac{1}{4} \int |K(t,r)|^2 dr \right)^{1/2} < \infty,
\]

(21) \[
\int_0^1 \left( \frac{1}{4} \int |S(t,r)|^2 dr \right)^{1/2} < \infty,
\]

where $1/u + 1/v = 1, 1 < u, v < \infty^3$.

Let $T = E S \in \mathcal{R}_n$, i.e.
\[
T(t,s) = \int_0^1 R(t,r) S(r,s) dr.
\]

$^3$) The cases $u=1, v=\infty$ and $u=\infty, v=1$ are also admissible. Evidently the norm $(\int |\cdot|^v dr)^{1/v}$ should be replaced by $\sup |\cdot|$. 

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We define the functional $F$ on $\mathcal{R}_a$ by the formula
\begin{equation}
F(K) = \int \frac{1}{2} f(t, x) S(t, x) K(x, y) dt dx.
\end{equation}

$F$ is linear on $\mathcal{R}_a$ since
\begin{equation}
F(\lambda K + \mu R) = \lambda F(K) + \mu F(R).
\end{equation}

Now we extend $F$ to a linear functional on $\mathcal{R}$ in an arbitrary way. The extended functional $\mathcal{F}$ satisfies the condition (F). It suffices to verify the case of $K \in \mathcal{R}_a$ and $K = I$.

If $M$ is defined by (18), then
\begin{equation}
F(M) = \int \int T(s, t) M(t, s) dt ds = \int \frac{1}{2} f(t, x) S(t, x) K(x, y) dt dx = K T F Y.
\end{equation}

If $M$ is defined by (19), then
\begin{equation}
F(M) = \int \int T(s, t) M(t, s) dt ds = \int \frac{1}{2} f(t, x) S(t, x) K(x, y) dt dx = K T F Y.
\end{equation}

Consequently, the theory developed in the first part can be applied to the equation (1) where $K \in \mathcal{R}$ and $\varphi, \psi \in \mathcal{S}$ is $\mathcal{F}$. If $A = K T \in \mathcal{R}_a$, then (1) is the integral equation
\begin{equation}
\varphi(t) + \int A(t, s) \varphi(s) ds = \psi(t).
\end{equation}

Consider, in particular (the case $K = I$), the equation
\begin{equation}
\varphi + T \varphi = \psi,
\end{equation}
i. e. the integral equation
\begin{equation}
\varphi(t) + \int \frac{1}{2} f(t) \varphi(s) ds = \psi(t) \quad (\varphi, \psi \in \mathcal{F}).
\end{equation}

The determinant $D$ of the equation (23) coincides with the Fredholm determinant of the integral equation (26). In fact, we have (see the definition (7))
\begin{equation}
A_a = T, \quad A_n = A_{n-1} T A_n - 1,
\end{equation}
i. e.
\begin{equation}
A_a(t, s) = T(a, s),
\end{equation}

\begin{equation}
A_n(t, s) = - \int A_{n-1}(t, r) A_n(r, s) dr \quad (n = 1, 2, \ldots),
\end{equation}

where $A_n(t, s)$ denotes clearly the function determining the operation $A_n \in \mathcal{R}_a$. By (iv) and (vii),
\begin{equation}
a_n = 1,
\end{equation}

\begin{equation}
a_{n+1} = \frac{1}{n+1} F(K_n) = \frac{1}{n+1} \int T(s, t) K_n(t, s) dt ds
\end{equation}

\begin{equation}
= \frac{1}{n+1} \int A_{n}(s, s) ds \quad (n = 1, 2, \ldots).
\end{equation}

We may suppose that
\begin{equation}
F(I) = \int T(s, t) ds
\end{equation}
(if not, we can modify the function $T(s, t)$ on the diagonal of the unit square). Therefore
\begin{equation}
a_n = F(K_n) - F(I) = \int A_n(s, s) ds.
\end{equation}

The formulae (27), (28), (28') show that
\begin{equation}
D = \sum_{n=0}^\infty a_n.
\end{equation}
is the Fredholm determinant of (26), and
\[
\frac{1}{D} \sum_{\alpha=1}^{\infty} J_k(t,s)
\]
is Fredholm's resolvent kernel of equation (26).

The restriction that the kernel $T$ of (26) is a superposition of two other kernels is not essential since the examination of (26) can be reduced to the examination of the equation with the iterated kernel $TT$.

Analogous results can be obtained in the case where $X=F'$ and $\mathcal{E}=F$ $(t<p$, $q<\infty$, $1/p+1/q=1)$. The necessary modifications in the text are obvious.

II. Now let $X$ be the space $M$ of all bounded measurable functions $x(t)$ on the interval $(0,1)$ with the norm
\[
||x|| = \sup_{t} \text{ess sup} \, |x(t)|
\]
and let $\mathcal{E}$ be the space $L$ of all integrable functions $\varphi(s)$ on $(0,1)$ with the norm
\[
||\varphi|| = \int_{0}^{1} |\varphi(s)| \, ds.
\]

Clearly each $\varphi \in \mathcal{E}$ determines a functional on $X$ denoted by the same letter $\varphi$ (see the formula (15)).

Each measurable function $K(t,s)$ defined on the square $0 \leq t, s \leq 1$, such that
\[
\int_{0}^{1} \int_{0}^{1} |K(t,s)| \varphi(s) x(t) \, ds \, dt < \infty \quad \text{for each} \quad \varphi \in \mathcal{E}, x \in X,
\]
determines a linear operation of $\mathcal{E}$ into $\mathcal{E}$ denoted by the same letter $K$ and defined by the equation
\[
(30) \quad \varphi K x = \int_{0}^{1} \int_{0}^{1} K(t,s) \varphi(s) x(t) \, ds \, dt.
\]
The norm of $K$ is
\[
(31) \quad ||K|| = \sup_{\varphi} \int_{0}^{1} |K(t,s)| \, dt.
\]

Let $\mathcal{R}_K$ be the set of all linear operations $\mathcal{K}$ of the form (30), and let $\mathcal{R}$ be the least closed linear set of linear operations of $\mathcal{E}$ into $\mathcal{E}$ such that $I \in \mathcal{R}$ and $\mathcal{R}_K \subseteq \mathcal{R}$. $\mathcal{R}$ satisfies the condition (K).

The proof is the same as in I.

Let $T(t,s)$ be a fixed measurable function defined on the square $0 \leq t, s \leq 1$, such that
\[
\int_{0}^{1} \int_{0}^{1} |T(t,s)| \, dt \, ds < \infty.
\]
Clearly $T \in \mathcal{R}_K$. Let
\[
F(K) = \int_{0}^{1} \int_{0}^{1} T(t,s) K(s,t) \, ds \, dt \quad \text{for} \quad K \in \mathcal{R}_K.
\]
$F$ is a linear operation on $\mathcal{R}_K$. In fact
\[
|F(K)| \leq \int_{0}^{1} \int_{0}^{1} |T(t,s)| K(s,t) \, ds \, dt
\]
\[
\leq \int_{0}^{1} \left( \sup_{s} \int_{0}^{1} |T(t,s)| \, dt \right) \int_{0}^{1} |K(s,t)| \, ds \, dt
\]
\[
\leq \sup_{s} \int_{0}^{1} |K(s,t)| \, dt \cdot \int_{0}^{1} \sup_{s} \int_{0}^{1} |T(t,s)| \, dt \, ds = ||K|| \cdot \int_{0}^{1} \sup_{s} \int_{0}^{1} |T(t,s)| \, dt \, ds.
\]

Now we extend the functional $F$ to a linear functional over $\mathcal{R}$. The functional $F$ on $\mathcal{R}$ satisfies condition (F). The proof is the same as in I.

Consequently, the Fredholm theory from the first part of this paper can be applied to the equation
\[
\varphi + KT \varphi = \psi,
\]
where $K \in \mathcal{R}$. In particular, if $A = KT \in \mathcal{R}_K$, this equation is the integral equation
\[
\varphi(t) + \int_{0}^{1} A(t,s) \varphi(s) \, ds = \psi(t),
\]
It is easy to see that the case $X = L$, $S = M$ can be discussed in a completely analogous way to the case examined above: $X = M$, $S = L$.

III. Now let $X$ be the space $V$ of all functions $x(t)$ on $0,1)$ with finite total variation $\text{var} \, x(t)$. The norm in $V$ is

$$||x|| = \text{var} \, x(t).$$

Let $S$ be the space $C$ of all continuous functions $\varphi$ on $0,1)$ with the norm $||\varphi|| = \sup \varphi(s)$. Clearly each $\varphi \in C$ determines a linear functional on $X = V$, which will be denoted by the same letter $\varphi$:

$$\varphi x = \int_0^1 \varphi(s) \, dx(t).$$

All integrals in this section are of the Riemann-Stieltjes type.

Let $K(x,t)$ be a function on the square $0 \leq s \leq 1$ such that

(a) for each fixed $t$, the function $K(t,s)$ of one variable $s$ is of bounded variation, and

$$\sup_{t} \text{var} \, K(t,s) < \infty,$$

(b) for each fixed $s$, the function $K(t,s)$ of one variable $t$ is continuous.

The function $K(t,s)$ determines linear operation of $S$ into $S$ denoted by the same letter $K$:

$$K \varphi x = \int_0^1 \left[ \int_0^1 \varphi(s) \, dx(t) \right] \, dt = \int_0^1 \varphi(s) \, \left[ \int_0^1 K(t,s) \, dx(t) \right] \, dt.$$  \hspace{1cm} (32)

The norm of $K$ is

$$||K|| = \sup_{t,s} \text{var} \, K(t,s).$$  \hspace{1cm} (33)

Let $K_{0}$ be the class of all functions $K(t,s)$ satisfying (a) and (b), and let $K$ be the least closed linear set such that $I \in K$ and $K_{0} \subset K$. $K$ satisfies the condition $(K)$.

The superposition $K = K_{0}K_{0}$ ($K_{0}K_{0}K_{0}$) is determined by the function

$$K(t,s) = \int_0^1 K(t,r) \, dr \, K(r,s).$$

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If $K \in \mathbb{K}$ and $\varphi \in \mathbb{E}$, $x_{0} \in X$ are fixed, then

$$M \varphi y = K \varphi x, \varphi y = \int_0^1 \varphi(s) \, dx(t) \left[ \int_0^1 K(t,s) \, dx(t) \right],$$

where

$$M(t,s) = \varphi_{0}(t) \int_0^1 K(t,s) \, dx(t).$$

If $K = I$, then

$$M \varphi y = \varphi y, \varphi y = \int_0^1 \varphi(s) \, dx(t) \left[ \int_0^1 K(t,s) \, dx(t) \right],$$

where $M(t,s) = \varphi_{0}(t) \cdot x_{0}(s)$.

Consequently, $M \in \mathbb{K}$ for every $K \in \mathbb{K}$, c. c. e.

Let $T(t,s)$ be a fixed continuous function on the square $0 \leq s \leq t \leq 1$. This function determines an operation $T \in \mathbb{K}$ defined as follows:

$$T \varphi x = \int_0^1 \left[ \int_0^1 T(t,s) \, \varphi(s) \, dt \right] \, dx(t).$$

For each $K \in \mathbb{K}$ take

$$F(K) = \int_0^1 \left[ \int_0^1 T(t,s) \, K(t,s) \, dt \right] \, dx(t).$$

$F$ is a linear functional on $\mathbb{K}$ since

$$|F(K)| \leq \int_0^1 \left[ \int_0^1 |T(t,s)| \, dx(t) \right] \cdot \text{var} \, K(t,s) \, dt = ||K|| \cdot \sup_{t} |T(t,s)|.$$

The functional $F$ has the property $(F)$. In fact, if $M$ is defined by (34), then

$$F(M) = \int_0^1 \left[ \int_0^1 T(t,s) \, dx(t) \right] \, dx(t)$$

$$= \int_0^1 \left[ \int_0^1 \varphi_{0}(t) \, K(r,s) \, dx(t) \right] \, dx(r) = KT \varphi x_{0}.$$
Fredholm theory of linear equations.

The case \( K = I \) can be verified in an analogous way. Therefore we may apply the theory from the first part of this paper to the equation

\[
(1') \quad \varphi + A\varphi = \psi,
\]

which can be written in the form

\[
\varphi(t) = \sum_{s=1}^{t} \int_{s}^{t} \varphi(s) d_{s} A(t, s) = \psi(t).
\]

IV. We shall now apply the results of the first part of this paper to the case where \( X = \mathbb{I} \) or \( m \) and \( E = \mathbb{I} \) or \( I \) respectively.

We shall write explicitly all formulae and equations for the case where \( X = \mathbb{I} \). In square brackets \([\ ]\) we shall write analogous expressions for the case where \( X = m \).

Let \( X = \mathbb{I} \) \( [X = m] \) and let \( E = \mathbb{I} \) \( [E = I] \). Elements of \( X \) or \( E \) are sequences of real numbers. The \( n \)-th term of a sequence \( x \in X \) or \( \varphi \in E \) will always be denoted by \( x_{n} \) or \( \varphi_{n} \) respectively. We shall also write \( x = (x_{n}) \) or \( \varphi = (\varphi_{n}) \) respectively.

Infinite square matrices will be denoted by letters \( K, T, S, \ldots \); their elements will always be denoted by the same Greek letters with two indices: \( K = (k_{n}) \), \( T = (t_{n}) \), \( S = (s_{n}) \), etc. However, the unit matrix \( (\delta_{n}) \) (where \( \delta_{n} \) is the Kronecker symbol) will be denoted by \( I \).

We adopt the usual notations: \( S + T \) is the matrix \((s_{n} + t_{n})\), \( ST \) is the matrix

\[
\left( \sum_{k=1}^{\infty} s_{k} t_{n-k} \right).
\]

whenever all series defining the terms of \( ST \) are absolutely convergent.

Let \( H \) be the set of all matrices \( K = (k_{n}) \) such that

\[
\sum_{k=1}^{\infty} |k_{n}| < \infty \quad \text{for each} \quad \varphi \in E \quad \text{and} \quad x \in X.
\]

Each matrix \( K \in H \) determines uniquely a linear operation of \( E \) into \( E \) denoted by the same letter \( K \):

\[
(37) \quad \|K\| = \sup_{k} \left( \sum_{n=1}^{\infty} |k_{n}| \right) \quad \text{or} \quad \|K\| = \sup_{k} \left( \sum_{n=1}^{\infty} |k_{n}| \right).
\]

The superposition \( KT \) of two linear operations \( K \) and \( T \) is determined by the product of the matrices \( K, T \). The unit matrix \( I \) determines the identity mapping of \( E \) onto \( E \).

We shall interprete \( H \) as the Banach space of linear operations of \( E \) into \( E \). The norm of an operation \( K \in H \) is

\[
(38) \quad \|T\| = \sum_{n=1}^{\infty} \sup_{k} |t_{n}| < \infty \quad \text{or} \quad \|T\| = \sum_{n=1}^{\infty} \sup_{k} |t_{n}| < \infty.
\]

With respect to the norm \( \|\cdot\|^{*} \), \( H \) is a Banach algebra, since \( T, S \in H \) implies \( TS \in H \) and

\[
\|TS\|^{*} < \|T\|^{*} \cdot \|S\|^{*}.
\]

Clearly \( H \subset H \). Consequently, each \( T \) can be interpreted as a linear operation of \( E \) onto \( E \). Notice that

\[
(39) \quad \|T\|^{*} = \|T\|^{*}.
\]

Each \( T \in H \) determines also a linear functional \( F \) defined on \( E \) by the formula

\[
F(K) = \sum_{k=1}^{\infty} k_{n} \varphi_{n} \quad \text{for} \quad K = (k_{n}) \in H.
\]

This double series is absolutely convergent.

In the same way as in \( II \) we prove that \( |F(K)| \leq \|K\| \cdot \|\varphi\|^{*} \). Hence

\[
(40) \quad \|F\| \leq \|\varphi\|^{*}.
\]

Now take a fixed \( T \in H \) and consider the functional \( F \) determined by \( T \). The functional \( F \) fulfills the condition \( (F) \). The proof is the same as in case \( II \).

Consequently we can apply the results from the first part of this paper to the equation

\[
(1') \quad \varphi + A\varphi = \psi \quad \text{or} \quad (I + A)\varphi = \psi,
\]

where \( \varphi, \psi \in E \), \( A = KT \), \( K = (k_{n}) \in H \).
The equation (1') is, in fact, a system of infinitely many linear equations

\[ q_k + \sum_{l=1}^{\infty} a_{kl} p_l = y_k \quad (k = 1, 2, \ldots), \]

where \( q = (q_k), p = (p_l), \) and \( A = (a_{kl}) = \alpha(T) \in \mathbb{R}. \)

Since we shall examine in this section the equation (1') for different operations \( A, \) we shall write \( D(I + A) \) instead of \( D. \) By definition,

\[ D(I + A) = \sum_{k=1}^{\infty} \frac{1}{k!} F_{n+1} \cdots F_{n+1} \left[ K \left( y_1, \ldots, y_k \right) \right]. \]

Now

\[ F_{n+1} \cdots F_{n+1} \left[ K \left( y_1, \ldots, y_k \right) \right] = \sum_{\alpha_1, \ldots, \alpha_k = 1}^{n} \tau_{\alpha_1} \cdots \tau_{\alpha_k} \frac{\partial \alpha_{\alpha_1} \cdots \alpha_{\alpha_k}}{\partial y_1 \cdots y_k}, \]

where the last sign \( \Sigma \) is extended over all sequences of positive integers. The determinants under the last sign \( \Sigma \) are the principal subdeterminants \(^{14)} \) of the matrix \( A = (a_{kl}) \) of the order \( k. \) Consequently \(^{15)}, \)

\[ D(I + A) = \sum_{k=1}^{\infty} d_k, \]

where \( d_k \) is the sum of all principal subdeterminants of the order \( k \) of the matrix \( A \) if \( k > 0, \) and \( d_k = 1. \) By (ix), the series (41) is absolutely convergent.

Subsequently we shall examine the equations (1) only in the case where \( K = I, \) i.e., the equation

\[ q + T \varphi = \varphi \quad \text{or} \quad (I + T) \varphi = \varphi, \]

where \( T \in \mathbb{R}. \) In other words, we shall examine the following infinite system of linear equations:

\[ q_k + \sum_{l=1}^{\infty} \tau_{kl} p_l = y_k \quad \text{or} \quad \sum_{l=1}^{\infty} (\delta_{kl} + \tau_{kl}) p_l = y_k, \]

where \( k = 1, 2, \ldots, \) \( T = (\tau_{kl}) \in \mathbb{R}. \)

For each \( T \in \mathbb{R} \) let \( T_n = (\tau_{kl}^n) \) denote the matrix defined as follows:

\[ \tau_{kl}^n = \begin{cases} \tau_{kl} & \text{if } i \leq n \quad [k \leq n], \\ 0 & \text{if } i > n \quad [k > n]. \end{cases} \]

It is clear that \( T_n \in \mathbb{R} \) and \( \|T - T_n\| \to 0. \) The letters \( T^{(n)} \) and \( F \) will denote respectively the linear functions determined by \( T_n \) and \( T. \) By (38), \( \|T^{(n)} - F\| \to 0. \) Hence, by lemma (xxii),

\[ D(I + T_n) \to D(I + T). \]

We have

\[ D(I + T_n) = \sum_{k=1}^{\infty} d_k^n, \]

where \( d_k^n \) is the sum of all principal subdeterminants of the matrix \( T_n. \) Now each principal subdeterminant of \( T_n \) which is not equal to 0 is the principal subdeterminant of the finite matrix \( (\tau_{kl}) \) \((k, i = 1, \ldots, n). \) Hence, by the known properties of determinants, \( D(I + T_n) \) is equal to the determinant of the finite matrix \((\delta_{kl} + \tau_{kl})\), where \( k, i = 1, \ldots, n. \) Consequently:

**Theorem 5.** If \( T \in \mathbb{R}, \) then the sum \( d_k \) of all principal subdeterminants of the matrix \( T \) exists for each \( k, \) the series \( d_1 + d_2 + d_3 + \ldots \) is absolutely convergent, and

\[ D(I + T) = \sum_{k=1}^{\infty} d_k = \lim_{n \to \infty} \left[ \begin{array}{cccc} 1 + \tau_{11} & \tau_{12} & \ldots & \tau_{1n} \\ \tau_{21} & 1 + \tau_{22} & \ldots & \tau_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n1} & \tau_{n2} & \ldots & 1 + \tau_{nn} \end{array} \right]. \]

The number \( D(I + T) \) is called the determinant of the infinite matrix \( C = I + T. \) The determinant \( D(C) \) of an infinite matrix \( C \) is thus well defined if \( C = I + T. \)

Now let \( S \in \mathbb{R} \) be another matrix, and let \( S_n = (s_{kn}) \) be the matrix defined analogously to \( T_n, \) i.e.,

\[ s_{kn} = \begin{cases} s_{kl} & \text{if } i \leq n \quad [k \leq n], \\ 0 & \text{if } i > n \quad [k > n]. \end{cases} \]

\(^{14)} \) Principal subdeterminants (of an order \( k \)) of a matrix \( (a_{kl}) \) are the determinants of finite matrices \( (a_{kl}^p) \) \((p, i = 1, \ldots, k) \) where \( r_1, \ldots, r_p \) is a sequence of different positive integers.

\(^{15)} \) An analogous result holds in the case \( S = T, \) \( X = T', \) see p. 244.
We have \( \|S_\infty - S_0\| \to 0 \). Since \( \mathcal{E} \) is a Banach algebra, we obtain \( \|T_n S_n - T_0 S_0\| \to 0 \), and consequently
\[
(\sum T_n S_n + T_n S_0) - (T + TS) = 0.
\]

Hence, by (39) and (xxii),
\[
D((I+T)(I+S)) = D((I+T+S+TS)) = \lim_{n \to \infty} D((I+T_n S_n + T_0 S_0))
\]

Now, if \( i > n \), the \( i \)-th column (line) of the matrices \( T_n S_n \), \( T_n + S_n + T_0 S_0 \) contains exclusively the numbers 0. Consequently (see the proof of Theorem 5), \( D(I+T_n) \) is the determinant of the finite matrix \( (\delta_{ik} + \tau_{ik}) \), where \( k, i = 1, \ldots, n \), and \( D(I+S_n) \) is the determinant of the matrix \( (\delta_{ik} + \sigma_{ik}) \), where \( k, i = 1, \ldots, n \).

Analogously, \( D((I+T_n + S_n + T_0 S_0)) \) is equal to the determinant of the finite matrix \( (\delta_{ki} + \tau_{ki})(\delta_{ik} + \sigma_{ik}) \), where \( k, i = 1, \ldots, n \). Hence
\[
D((I+T_n + S_n + T_0 S_0)) = D((I+T_n)) \cdot D((I+S_n))
\]

and, consequently,
\[
\text{Theorem 6: } D((I+T)(I+S)) = D((I+T) \cdot D((I+S)) \text{ for arbitrary } T, S \in \mathcal{E}.
\]

In other words, if \( C_n + I \in \mathcal{E} \) and \( C_n - I \in \mathcal{E} \), then
\[
D(C_n C_l) = D(C_n) \cdot D(C_l).
\]

During the print of this paper R. Sikorski and the author have proved that Theorem 6 holds in arbitrary Banach space. Since the determinant of the equation \( (I+T)\gamma = \psi \) is uniquely determined by \( F \), but not by \( T \), the extension of Theorem 6 to the case of an arbitrary Banach space must be otherwise formulated.

The above theorem shows that the infinite determinant \( D(C) \) \( C - I \in \mathcal{E} \) has the multiplication property of usual determinants of a finite order. Obviously it has also other properties of finite determinants. For instance:

(xxii) Let \( C - I \in \mathcal{E} \), let \( a_e \in \mathcal{E} \) for \( i = 1, 2, 3 \), and let \( C_i \) be the matrix obtained from \( C \) by replacing the \( r \)-th column of \( C \) by the sequence \( a_e \). Since \( C_i - I \in \mathcal{E} \), the determinant \( D(C_i) \) exists \( (i = 1, 2, 3) \). If \( a_e = a_1 + a_2 \), then
\[
D(C) = D(C_1) + D(C_2).
\]

This additive property follows immediately from Theorem 5.

Analogously:

\[\text{(xxiv)}\]

Let \( C - I \in \mathcal{E} \), let \( a_e \in \mathcal{E} \) for \( i = 1, 2, 3 \), and let \( C_i \) be the matrix obtained from \( C \) by replacing the \( r \)-th column of \( C \) by the sequence \( a_e \). Since \( C_i - I \in \mathcal{E} \), the determinant \( D(C_i) \) exists \( (i = 1, 2, 3) \). If \( a_e = a_1 + a_2 \), then
\[
D(C) = D(C_1) + D(C_2).
\]

The formula (46) is the expansion of \( C \) on the \( r \)-th column. The equation (45) follows from (xxiii). (47) follows from (45), (39) and (xxii) since \( \|C_n - I\| \to 0 \).

The analogous statement is true for the expansion on the \( t \)-th row. The sequence \( a \) should belong to \( I \) \( (a_e) \).

Let \( C = (\gamma_h) \) be such that
\[
\lim_{n \to \infty} \gamma_n = 0 \quad \text{for } i = 1, 2, \ldots \quad \text{and } C - I \in \mathcal{E} \quad \text{[C - I \in \mathcal{E}].}
\]

Let \( \rho_{k, h} \) denote the determinant of the matrix obtained from the matrix \( C \) by replacing the term \( \gamma_h \) by the number 1, and all other terms in the \( k \)-th line and in the \( t \)-th column by 0. Let \( M = (\rho_{k, h}) \). We shall prove that
\[
M \cdot C = D(C) \cdot I.
\]

Set \( a = (\gamma_1, \gamma_2, \ldots) \) in lemma (xxv). We obtain
\[
C' = \sum_{k} \rho_{k, h} \gamma_k.
\]

On the other hand, \( C' = C \) if \( a = \gamma \); hence
\[
\sum_{k} \rho_{k, h} \gamma_k = D(C) \quad \text{if } a = \gamma.
\]

\[1] \) \( \xi_0 \) is the space of all sequences convergent to 0. It is not known whether \( \xi_0 \) can be replaced here by \( m \). Consequently, it is not known whether the condition \( \lim_{n \to \infty} \gamma_n = 0 \) in (47) is necessary.
Sur un type de conditions mixtes pour les équations aux dérivées partielles

J. G. MIKUSINSKI (Wrocław).

§ 1. Introduction.

Considérons l'équation différentielle

\[ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\nu} \frac{\partial^{\nu+n}}{\partial x^{\nu} \partial \lambda^{n}} \varphi(t, \lambda) = \varphi(t, \lambda) \]

dont les coefficients \( a_{\nu} \) sont constants (réels ou complexes). Dans un travail antérieur [1], j'ai discuté la méthode opérationnelle de résolution de cette équation et le problème d'unicité, lorsqu'elles

\[ D: \quad 0 < t < \infty, \quad \lambda_{1} < \lambda < \lambda_{2}. \]

Ce type de conditions intervient, par exemple, dans le problème de la propagation de la chaleur, lorsque la température est connue sur les deux extrémités d'une barre, dans certains problèmes de la ligne électrique et dans beaucoup d'autres problèmes,