

Mean ergodic theorem in locally convex linear topological spaces

by

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KOSAKU YOSIDA has proved the following generalization of the mean ergodic theorem of J. v. NEUMANN:

Let T be a linear transformation which maps a Banach space into itself. Let us further assume that

1° the sequence $\{T^n(x)\}$ ($n=1,2,\dots$) is bounded for any element x and

2° the transformation T is weakly completely continuous.

Under these assumptions, the sequence

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i(x) \right\}$$

converges strongly to a point y , and $T(y)=y$.

Using the same method as in the paper [2] we can prove the above theorem for locally convex linear topological spaces.

Let \mathfrak{X} be a locally convex linear topological space¹⁾ and let \mathfrak{X} be the space of all linear functionals defined in \mathfrak{X} .

Definition 1. A sequence of elements $\{x_n\}$ is said to be *weakly convergent to an element x* if for any $X \in \mathfrak{X}$ the sequence $\{X(x_n)\}$ converges to $X(x)$.

Definition 2. A transformation T is said to be *weakly completely continuous* if there exists a neighbourhood O of zero such that the image $T(O)$ is sequentially weakly compact, i. e. any infinite part contains a sequence weakly convergent to some element of the space.

It is known (see [2]) that the space \mathfrak{X} is isomorphic with a certain (B_α^*) -space in which is defined a class of pseudonorms $|x|_\alpha$, where

$\vartheta \in \Phi$. The system of neighbourhoods of zero consists of the sets of all elements $x \in \mathfrak{X}$ such that $|x|_{\alpha_i} < \varepsilon$ ($i=1,2,\dots,k$), $\varepsilon > 0$. (Φ is an abstract set and $\bar{\Phi} = \kappa_\alpha$).

The sequence $\{x_n\}$ is said to be *convergent to zero* if $|x_n|_\alpha \rightarrow 0$ for any $\vartheta \in \Phi$.

Definition 3. A set $ZC\mathfrak{X}$ is said to be *bounded* if for any sequence $\{x_n\} \subset ZC\mathfrak{X}$ and for every arbitrary sequence of numbers λ_n convergent to zero we have $\lambda_n x_n \rightarrow 0$, when $n \rightarrow \infty$.

Let T be a linear²⁾ weakly completely continuous transformation which maps the neighbourhood of zero O into a sequentially weakly compact set. Then there exists a neighbourhood of zero V such that $V \subset O$, where V is defined by the inequalities $|x|_{\alpha_i} < \varepsilon$ ($i=1,2,\dots,k$). We define the pseudonorm $|x| = \sup_i |x|_{\alpha_i}$. Then the image of any set bounded in this pseudonorm is sequentially weakly compact and therefore bounded in the sense of the definition 3. Hence it appears the following

Lemma. If T is a linear weakly completely continuous transformation, then for every ϑ there exists a constant M_ϑ such that $|T(x)|_\vartheta \leq M_\vartheta |x|$.

The proof is the same as in [2] (lemma 1).

Theorem. Let T be a linear transformation which maps a locally convex linear topological space \mathfrak{X} into itself. Let us further assume that

1° the sequence $\{T^n(x)\}$ ($n=1,2,\dots$) is bounded for any $x \in \mathfrak{X}$,

2° the transformation T is weakly completely continuous.

Under these assumptions, the sequence

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i(x) \right\}$$

converges for any $x \in \mathfrak{X}$ to a point y , and $T(y)=y$.

Proof. We divide the space \mathfrak{X} into classes and we say that x_1 and x_2 belong to the same class \mathfrak{x} , if $|x_1 - x_2| = 0$. The set of all elements $x \in \mathfrak{X}$ such that $|x| = 0$ constitutes the zero class. We denote by \mathfrak{X}^* the obtained quotient space which is a Banach space with the norm $|x| = |x|$.

²⁾ We consider the continuity in the sense of Heine, i. e. $x_n \rightarrow 0$ implies that $T(x_n)$ converges to zero.

¹⁾ Concerning the definition see [2].

The transformation $y = T(x)$ defines a transformation $\mathfrak{v} = \mathfrak{T}(x)$ in the space \mathfrak{X}^* , where $x \in \mathfrak{X}, y \in \mathfrak{Y}$. It follows from the lemma that \mathfrak{T} is a linear transformation. It is obvious that \mathfrak{T} is a weakly completely continuous transformation.

We denote by \mathfrak{X}' the completion of the space \mathfrak{X} . The transformation \mathfrak{T} can be extended on the whole space \mathfrak{X}' . We shall show that the range of this extension is contained in \mathfrak{X}^* . Let x be an arbitrary element of \mathfrak{X}' . There exists a sequence $\{x_n\} \subset \mathfrak{X}$ such that $x_n \rightarrow x$. Since the sequence $\{x_n\}$ is convergent and therefore bounded, the sequence $\mathfrak{T}(x_n)$ is weakly compact in \mathfrak{X}^* and convergent to $\mathfrak{v} \in \mathfrak{X}^*$, hence $\mathfrak{v} = \mathfrak{T}(x)$. From the condition 1° it follows that the sequence $\{\mathfrak{T}^n(x)\}$ is bounded for any $x \in \mathfrak{X}'$. Since the range of the transformation \mathfrak{T} is contained in \mathfrak{X}^* , the sequence $\{\mathfrak{T}^n(x)\}$ is bounded for any $x \in \mathfrak{X}'$, and the theorem of Kosaku Yosida is valid for the transformation \mathfrak{T} .

Consider now the sequence

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i(x) \right\}.$$

The sequence

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \mathfrak{T}^i(x) \right\}, \quad \text{where } x \in \mathfrak{X},$$

converges to some element $\mathfrak{v} \in \mathfrak{X}^*$, and $\mathfrak{v} = \mathfrak{T}(\mathfrak{v}) \in \mathfrak{X}^*$. On account of the lemma and since

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} T^i(x) - \bar{y} \right| = \left| \frac{1}{n} \sum_{i=0}^{n-1} \mathfrak{T}^i(x) - \mathfrak{v} \right|, \quad \text{where } \bar{y} \in \mathfrak{Y},$$

the sequence

$$\left\{ T \left(\frac{1}{n} \sum_{i=0}^{n-1} T^i(x) \right) \right\} = \left\{ \frac{1}{n} \sum_{i=1}^n T^i(x) \right\}$$

converges to $y = T(\bar{y})$. Since

$$\frac{1}{n} \sum_{i=0}^{n-1} T^i(x) = \frac{1}{n} x + \sum_{i=1}^n T^i(x) - \frac{1}{n} T^n(x),$$

the sequences $\{x/n\}$ and $\{T^n(x)/n\}$ converge to 0, the sequence

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i(x) \right\}$$

converges to y . Then the sequence

$$\left\{ \frac{1}{n} \sum_{i=1}^n T^i(x) \right\}$$

converges to $T(y)$. Since

$$\frac{1}{n} \sum_{i=1}^n T^i(x) = \frac{1}{n} \sum_{i=0}^{n-1} T^i(x) + \frac{1}{n} T^n(x) - \frac{1}{n} x,$$

we have $T(y) = y$.

References.

- [1] Kosaku Yosida, *Mean ergodic theorem in Banach spaces*, Proc. Imp. Acad. Japan 14 (1938), p. 292-294.
 [2] M. Altman, *On linear functional equations in locally convex li near topological spaces*, Studia Mathematica, this volume.

(Reçu par la Rédaction le 16. 6. 1952)