Remarks on the Poisson Stochastic Process (II)*)

by

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I denote by Ω_0 the set of all integral valued functions $\omega(t)$ defined for $t \geqslant 0$ which are continuous on the right, non-decreasing, and such that $\omega(0)=0$.

For every $\Omega \subset \Omega_0$ I denote by $\boldsymbol{B}_\Omega$ the smallest σ-field of sets containing as elements all the sets of the form $\underset{\omega}{E}[\omega \in \Omega; \omega(t)<y]$ and by μ a probability measure in $\boldsymbol{B}_\Omega$[1]).

For every half-open interval $B=(u,v\rangle$ I denote by $\iota_B(\omega)$ the increment $\omega(v)-\omega(u)$. Obviously for every B $\iota_B(\omega)$ is a random variable, *i. e.* a real function measurable with respect to $\boldsymbol{B}_\Omega$. It is well known that some qualitative properties of $\iota_B(\omega)$ determine its distribution function. In particular, if the considered process is a *homogeneous differential* one, *i. e.* if for any intervals $B, B_1, B_2, \ldots$

(h) *the distribution function of* ι_B *depends only of the number* $|B|$[2]),

(i) *the random variables* $\iota_{B_1}, \iota_{B_2}, \ldots, \iota_{B_n}$ *are independent (in the stochastic sense) whenever* $B_1, B_2, \ldots, B_n$ *are disjoint,*

and, if, moreover, every function $\omega \in \Omega$ has only jumps equal to 1, then ι_B has the classical Poisson distribution with the mean value proportional to $|B|$[3]). The form of the distribution function of ι_B is also known without any assumptions on jumps[4]).

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*) Presented to the Polish Mathematical Society, Wrocław Section, on March 14, 1952.

[1]) Cf. *e. g.* Doob [1], and Florek, Marczewski and Ryll-Nardzewski [2].

[2]) I denote by $|B|$ the length or, more generally, the Lebesgue measure of B.

[3]) Cf. Florek, Marczewski and Ryll-Nardzewski [2].

[4]) Cf. *e. g* Jánossy, Rényi and Aczél [5], § 2, p. 213-217.

The purpose of this paper is to investigate *the increments* ι_B *not only for intervals but, more generally, for arbitrary Borel sets.*

Stricly speaking, I denote by $\iota_B(\omega)$ the sum of all positive jumps $\omega(t)-\omega(t-0)$ for $t \in B$. Obviously for every set $B \subset (0,+\infty)$ the set S of all $t \in B$ with $\omega(t)-\omega(t-0)>0$ is at most denumerable. If B is bounded, then S is finite. Consequently, for every B, $\iota_B(\omega)$ is a non-negative integer or $+\infty$, and, for bounded B, $\iota_B(\omega)$ is finite. In the case of an interval B this definition of ι_B is compatible with the preceding one.

I shall prove that the conditions (h) and (i) for intervals imply the same conditions for Borel sets. More precisely, I shall prove the following

Theorem. *Let* $(\Omega, \boldsymbol{B}_\Omega, \mu)$ *be a stochastic process with* $\Omega \subset \Omega_0$. *If the conditions* (h) *and* (i) *are fulfilled by any intervals, then*

I. *For every Borel set B the increment* ι_B *is a random variable and fulfills the condition* (h).

II. *If the Borel sets* B_j *converge in measure (i. e. in the Lebesgue measure) to* B, *then* ι_{B_j} *converges in probability (i. e. in the measure* μ) *to* ι_B.

III. *The condition* (i) *is fulfilled by any Borel sets* $B_1, B_2, \ldots B_n$.

In the sequel the letter ω denotes always the elements of Ω. Put $T=(0,t\rangle$ (where $0 \leqslant t \leqslant +\infty$), and

$$(1)\qquad \begin{aligned} P_k(t) &= \mu \underset{\omega}{E}[\iota_T(\omega)=k], \\ R_k(t) &= \mu \underset{\omega}{E}[\iota_T(\omega)>k] = 1-[P_0(t)+\ldots+P_k(t)], \end{aligned} \qquad \text{for } k=0,1,2,\ldots$$

Obviously

$$(2)\qquad P_0(0)=1, \quad P_k(0)=0 \quad \text{for} \quad k=1,2,\ldots,+\infty.$$

If the process is degenerate, *i. e.* if $P_0(t)=1$ for all $t \geqslant 0$, then Theorem is trivially true. Consequently we may suppose that it is not the case.

In what follows we use only the following properties of $P_k(t)$:

$$(3)\qquad P_{+\infty}(+\infty)=1, \quad P_k(+\infty)=0 \quad \text{for finite } k,$$

$$(4)\qquad \lim_{t \to t_0} P_k(t) = P_k(t_0) \quad \text{for finite } k, \text{ and } 0 \leqslant t_0 \leqslant +\infty;$$

they are simple consequences of the analytical form of these functions[5]).

Part I of the theorem implies the

Corollary. *Under the assumptions of Theorem*

$$\mu \underset{\omega}{E}[\iota_B(\omega)=k]=P_k(|B|)$$

for any Borel set $B\subset(0,+\infty)$. *If, moreover, the process is non-degenerate and* $|B|=+\infty$, *then*

$$\mu \underset{\omega}{E}[\iota_B(\omega)=+\infty]=1.$$

Proof of I.

Lemma. *If* $\boldsymbol{A}$ *is a class of Borel subsets of* $(0,+\infty)$ *such that*

(a) *the class* $\boldsymbol{E}$ *of all sets of the form*

$$(a_1,b_1\rangle+(a_2,b_2\rangle+\ldots+(a_n,b_n\rangle \qquad (a_j\geqslant 0,\ b_j\geqslant 0)$$

is contained in $\boldsymbol{A}$,

(b) *if* $B_j\in\boldsymbol{A}$, $B_1\subset B_2\subset\ldots$, *then* $B_1+B_2+\ldots\in\boldsymbol{A}$,

(c) *if* $B_j\in\boldsymbol{A}$, $B_1\supset B_2\supset\ldots$, *and* B_1 *is bounded, then* $B_1B_2\ldots\in\boldsymbol{A}$,

then every Borel subset of $(0,+\infty))$ *belongs to* $\boldsymbol{A}$.

Let us denote respectively by $\boldsymbol{A}_t$ and $\boldsymbol{E}_t$ (where $0<t<+\infty$) the class of all sets $B\in\boldsymbol{A}$ or $B\in\boldsymbol{E}$ such that $B\subset(0,t\rangle$. The class $\boldsymbol{E}_t$ is a field of subsets of $(0,t\rangle$. The class $\boldsymbol{A}_t$ has obviously the following properties:

(a') $\boldsymbol{E}_t\subset\boldsymbol{A}_t$,

(b') if $B_j\in\boldsymbol{A}_t$, $B_1\subset B_2\subset\ldots$, then $B_1+B_2+\ldots\in\boldsymbol{A}_t$,

(c') if $B_j\in\boldsymbol{A}_t$, $B_1\supset B_2\supset\ldots$, then $B_1B_2\ldots\in\boldsymbol{A}_t$.

It follows from a known theorem[6]) that $\boldsymbol{A}_t$ is the class of all Borel subsets of $(0,t\rangle$, which implies, in view of (b), that $\boldsymbol{A}$ is the class of all Borel subsets of $(0,+\infty)$.

The lemma is thus proved.

Let us denote by D^t the distribution function of ι_T, where $T=(0,t\rangle$, and by $\boldsymbol{A}$ the class of all Borel sets B such that

$$(5)\qquad \begin{cases} \iota_B \text{ is a random variable,} \\ \text{the distribution function of } \iota_B \text{ is } D^{|B|}, \end{cases}$$

[5]) See footnotes [3]) and [4]).

[6]) See Halmos [3], p. 27, Theorem B.

or, in other terms, such that

$$(6)\qquad \begin{aligned} &\underset{\omega}{E}[\iota_B(\omega)>k]\in\boldsymbol{B}_\Omega, \\ &\mu\underset{\omega}{E}[\iota_B(\omega)>k]=R_k(|B|), \end{aligned} \qquad \text{for}\quad k=0,1,2,\ldots$$

In virtue of Lemma it suffices to prove that $\boldsymbol{A}$ satisfies the conditions (a), (b) and (c).

$\boldsymbol{A}$ satisfies (a). Since the condition (h) is fulfilled by every interval $I=(u,v\rangle$, we have $I\in\boldsymbol{A}$. Let be $B=I_1+I_2+\ldots+I_n$, where I_j are disjoint intervals with $|I_j|=d_j$. Obviously

$$\iota_B(\omega)=\iota_{I_1}(\omega)+\iota_{I_2}(\omega)+\ldots+\iota_{I_n}(\omega),$$

whence ι_B is a random variable.

Put

$$J_1=(0,d_1\rangle,$$
$$J_j=(d_1+\ldots+d_{j-1},d_1+\ldots+d_j\rangle \quad \text{for}\quad j=2,3,\ldots,n,$$
$$J=J_1+J_2+\ldots+J_n,$$
$$d=|J|=|B|.$$

It follows from the conditions (h) and (i) for intervals and from the equality

$$\iota_J(\omega)=\iota_{J_1}(\omega)+\iota_{J_2}(\omega)+\ldots+\iota_{J_n}(\omega)$$

that ι_J and ι_B have the same distribution function, namely, that of the sum of independent random variables with distribution functions $D^{d_1},D^{d_2},\ldots,D^{d_n}$. Since the distribution function of ι_J is D^d, that of ι_B is also D^d. Thus, for every $B\in\boldsymbol{E}$ the conditions (5) are fulfilled.

$\boldsymbol{A}$ satisfies (b). Let $B_1\subset B_2\subset\ldots$ and let us suppose that $B_j\in\boldsymbol{A}$ or, in other words, that B_j satisfy (h). Put $B=B_1+B_2+\ldots$ and, for a fixed k,

$$\Xi_0=\underset{\omega}{E}[\iota_B(\omega)>k], \qquad \Xi_j=\underset{\omega}{E}[\iota_{B_j}(\omega)>k].$$

Obviously

$$(7)\qquad \Xi_1\subset\Xi_2\subset\ldots\subset\Xi_0.$$

Now, I am going to prove that

$$(8)\qquad \Xi_0=\Xi_1+\Xi_2+\ldots$$

In view of (7) it suffices to prove that if $\omega \in \Xi_0$, then there is an n such that $\omega \in \Xi_n$. If $\omega \in \Xi_0$, then there is a finite sequence $S=(t_1,t_2,\ldots,t_{r_m})$ of numbers belonging to B and such that

$$\sum_{j=1}^{m}[\omega(t_j)-\omega(t_j-0)]>k. \tag{9}$$

Consequently, in view of the definition of B, there is an integer n such that $S\subset B_n$, whence by (9) we obtain $\omega\in\Xi_n$ and thus the formula (8) is proved.

Obviously $\Xi_0 \in \boldsymbol{B}_\Omega$ and the relations (7) and (8) imply

$$\mu(\Xi_0)=\lim_{j\to\infty}\mu(\Xi_j),$$

and since by hypothesis

$$\mu(\Xi_j)=R_k(|B_j|),$$

we obtain in view of (1) and (4)

$$\mu(\Xi)=R_k(|B|).$$

Consequently $B\in\boldsymbol{A}$, q. e. d.

$\boldsymbol{A}$ satisfies (c). Let $B_1\supset B_2\supset\ldots$, where B_j are bounded sets belonging to $\boldsymbol{A}$, *i. e.* satisfying (6). Put $B=B_1B_2\ldots$ and, for a fixed k,

$$\Xi^0=\underset{\omega}{E}[\iota_B(\omega)>k],\qquad \Xi^j=\underset{\omega}{E}[\iota_{B_j}(\omega)>k].$$

Obviously

$$\Xi^1\supset\Xi^2\supset\ldots\supset\Xi^0. \tag{10}$$

Now, I am going to prove that

$$\Xi^0=\Xi^1\cdot\Xi^2\cdot\ldots \tag{11}$$

In view of (10) it suffices to prove that, if $\omega\in\Xi^j$ for $j=1,2,\ldots$, then $\omega\in\Xi^0$. Let us denote by S_j the set of all $t\in B_j$ such that $\omega(t_j)-\omega(t_j-0)>0$. Since B_j are bounded, the sets S_j are finite, and we have $S_1\supset S_2\supset\ldots$ Consequently there is an integer n such that $S_n=S_{n+1}=S_{n+2}=\ldots$, whence $S_n\subset B$. Since

$$\iota_B(\omega)\geqslant\sum_{t\in S_n}[\omega(t)-\omega(t-0)]=\iota_{B_n}(\omega)>k,$$

we obtain $\omega\in\Xi_0$ and thus the formula (11) is proved.

Analogously to the proof of the condition (b) we prove $B\in\boldsymbol{A}$ by the aid of (10), (11), (1) and (4).

Proposition I is thus proved.

Proof of II.

Let us suppose that the Borel sets $B_j\subset(0,+\infty)$ converge in measure to B and denote by Q_j the symmetric difference $B\dot{-}B_j$. Then we have $|Q_j|\to 0$.

Since

$$\underset{\omega}{E}[\iota_{B_j}(\omega)\neq\iota_B(\omega)]\subset\underset{\omega}{E}[\iota_{Q_j}\neq 0],$$

we have

$$\mu\underset{\omega}{E}[\iota_{B_j}(\omega)\neq\iota_B(\omega)]\leqslant\mu\underset{\omega}{E}[\iota_{Q_j}(\omega)\neq 0]=1-P_0(|Q_j|),$$

whence by (4)

$$\lim_{j\to\infty}\mu\underset{\omega}{E}[\iota_{B_j}(\omega)\neq\iota_B(\omega)]=0.$$

Consequently ι_{B_j} converges in probability to ι_B, q. e. d.

Proof of III.

Let $B_1,B_2,\ldots,B_k$ be disjoint Borel subsets of $(0,+\infty)$. We may without loss of generality assume that $|B_j|<\infty$ $(j=1,2,\ldots,k)$. Thus, there exist k infinite sequences of sets $E_1^j,E_2^j,\ldots,E_k^j$ $(j=1,2,\ldots)$ belonging to $\boldsymbol{E}$ (cf. the definition, p. 133), such that

1° $E_l^jE_m^j=0$ for $l\neq m$,

and

2° for each $i=1,2,\ldots,k$ the sequence $E_i^1,E_i^2,\ldots$ converges in measure to B_i.

It follows easily from the condition (i) for intervals that $\iota_{E_i^1},\iota_{E_i^2},\ldots,\iota_{E_i^k}$ are independent. In view of II, the random variables $\iota_{E_i^1},\iota_{E_i^2},\ldots$ tend in probability to ι_{B_i}. Since the passage to limit in probability preserves independence[7]), the random variables $\iota_{B_1},\ldots,\iota_{B_2},\iota_{B_k}$ are independent, q. e. d.

Theorem is thus proved.

The problem arises whether the condition (i) for intervals implies (i) for any Borel sets, without the assumption (h) of the homogeneity. This problem has been solved quite recently by C. Ryll-Nardzewski, who has also applied the method of in-

[7]) See Hartman and Marczewski [4], p. 130, Theorem 4.

crements in Borel sets for the complete discussion of non-homogeneous Poisson processes [8]). These results of Ryll-Nardzewski will be published in a forthcoming paper.

References.

[1] J. L. Doob, *Stochastic processes depending on a continuous parameter*, Transactions of the American Mathematical Society 42 (1937), p. 107-140.

[2] K. Florek, E. Marczewski and C. Ryll-Nardzewski, *Remarks on the Poisson stochastic processes (I)*, Studia Mathematica 13 (1952), p. 123-130.

[3] P. R. Halmos, *Measure Theory*, New York 1950.

[4] S. Hartman and E. Marczewski, *On the convergence in measure*, Acta Scientiarum Mathematicarum (Szeged) 12 A (1950), p. 125-131.

[5] L. Jánossy, A. Rényi and J. Aczél, *On composed Poisson distributions I*, Acta Mathematica Academiae Scientiarum Hungaricae 1 (1950), p. 209-224.

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[8]) (Added in proof). Increments in Borel sets were considered recently by H. Cramér, *A contribution to the Theory of Stochastic Processes*, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability 1950, Berkeley-Los Angeles 1951, p. 329-340, especially p. 331.

(Reçu par la Rédaction le 20. 5. 1952)