Remarks on the Poisson Stochastic Process (II)*

by

E. MARCZEWSKI (Wrocław).

I denote by $\Omega$, the set of all integral valued functions $\omega(t)$ defined for $t \geq 0$ which are continuous on the right, non-decreasing, and such that $\omega(0) = 0$.

For every $D \subseteq \Omega$, I denote by $B_D$ the smallest $\sigma$-field of sets containing as elements all the sets of the form $\{\omega \in D : \omega(t) < y\}$ and by $\mu$ a probability measure in $B_D$.

For every half-open interval $B = (u, v]$ I denote by $\lambda_B(\omega)$ the increment $\omega(v) - \omega(u)$. Obviously for every $B$, $\lambda_B(\omega)$ is a random variable, i.e. a real function measurable with respect to $B_D$. It is well known that some qualitative properties of $\lambda_B(\omega)$ determine its distribution function. In particular, if the considered process is a homogeneous differential one, i.e. if for any intervals $B, B_1, B_2, \ldots$ the distribution function of $\lambda_B$ depends only on the number $|B|$.

(i) The random variables $\lambda_{B_1}, \lambda_{B_2}, \ldots, \lambda_{B_n}$ are independent (in the stochastic sense) whenever $B_1, B_2, \ldots, B_n$ are disjoint, and, if, moreover, every function $\omega \in \Omega$ has only jumps equal to 1, then $\lambda_B$ has the classical Poisson distribution with the mean value proportional to $|B|^\mu$. The form of the distribution function of $\lambda_B$ is also known without any assumptions on jumps.*

The purpose of this paper is to investigate the increments $\lambda_B$ not only for intervals but, more generally, for arbitrary Borel sets.

Strictly speaking, I denote by $\lambda_B(\omega)$ the sum of all positive jumps $\omega(t) - \omega(t-0)$ for $t \in B$. Obviously for every set $B \subseteq (0, +\infty)$ the set $S$ of all $t \in B$ with $\omega(t) - \omega(t-0) > 0$ is at most denumerable. If $B$ is bounded, then $S$ is finite. Consequently, for every $B$, $\lambda_B(\omega)$ is a non-negative integer or $+\infty$ and, for bounded $B$, $\lambda_B(\omega)$ is finite.

In the case of an interval $B$ this definition of $\lambda_B$ is compatible with the preceding one.

I shall prove that the conditions (h) and (i) for intervals imply the same conditions for Borel sets. More precisely, I shall prove the following

Theorem. Let $(\Omega, B_D, \mu)$ be a stochastic process with $D \subseteq \Omega$.

If the conditions (h) and (i) are fulfilled by any intervals, then

1. For every Borel set $B$ the increment $\lambda_B$ is a random variable and fulfills the condition (h).

II. If the Borel sets $B_t$ converge in measure (i.e. in the Lebesgue measure) to $B$, then $\lambda_B$ converges in probability (i.e. in the measure $\mu$) to $\lambda_B$.

III. The condition (i) is fulfilled by any Borel sets $B_1, B_2, \ldots, B_n$.

In the sequel the letter $\omega$ denotes always the elements of $\Omega$.

Put $T = (0, t)$ (where $0 < t \leq +\infty$), and

\begin{align*}
\lambda_B(t) &= \mu E[\lambda_B(\omega) - t], \\
R_B(t) &= \mu E[\lambda_B(\omega) > k] = 1 - \mu \sum_{k=0}^{+\infty} P_B(t) = 1 - [P_B(t) + \cdots + P_B(0)],
\end{align*}

for $k = 0, 1, 2, \ldots$

Obviously

\begin{align*}
P_B(0) &= 1, \\
P_B(t) &= 0 \quad \text{for } k = 1, 2, \ldots, +\infty.
\end{align*}

If the process is degenerate, i.e. if $P_B(t) = 1$ for all $t \geq 0$, then Theorem is trivially true. Consequently we may suppose that it is not the case.

In what follows we use only the following properties of $P_B(t)$:

\begin{align*}
P_B(+\infty) &= 1, \\
P_B(t) &= 0 \quad \text{for finite } k, \\
\lim_{t \to +\infty} P_B(t) &= P_B(t) \quad \text{for finite } k, \\
0 &\leq t \leq +\infty;
\end{align*}

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1) Cf. e.g. Doob [1], and Florek, Marczewski and Ryll-Nardzewski [2].

2) I denote by $|B|$ the length of $B$, more generally, the Lebesgue measure of $B$.

3) Cf. Florek, Marczewski and Ryll-Nardzewski [3].

4) Cf. e.g. Jánossy, Rényi and Anzé [8], § 2, p. 213-217.

9*
they are simple consequences of the analytical form of these functions).

Part I of the theorem implies the

Corollary. Under the assumptions of Theorem

\[ \mu E[\tau_p(w) = k] = P_k(B_1) \]

for any Borel set \( B \subset (0, +\infty) \). If, moreover, the process is non-degenerate and \(|B| = +\infty\), then

\[ \mu E[\tau_p(w) = +\infty] = 1. \]

Proof of Lemma. If \( A \) is a class of Borel subsets of \( (0, +\infty) \) such that

(a) the class \( E \) of all sets of the form

\[ \{a_1b_1 + a_2b_2 + \ldots + a_nb_n \} \quad (a_j \geq 0, b_j > 0) \]

is contained in \( A \),

(b) if \( B_i \in A \), \( B_1 \subset B_2 \subset \ldots \), then \( B_1 + B_2 + \ldots \in A \),

(c) if \( B \in A \), \( B \supset B \), \( \ldots \), then \( B_1, B_2, \ldots \in A \),

then every Borel subset of \( (0, +\infty) \) belongs to \( A \).

Let us denote respectively by \( A_i \) and \( E_i \) (where \( 0 < t < +\infty \)) the class of all sets \( B \in A \) or \( B \in E \) such that \( B \in (0, t) \). The class \( E_1 \) is a field of subsets of \( (0, t) \). The class \( A_1 \) has obviously the following properties:

(a') \( E_1 \subset A_1 \),

(b') if \( B_i \in A_i \), \( B_1 \subset B_2 \subset \ldots \), then \( B_1 + B_2 + \ldots \in A_i \),

(c') if \( B_i \in A_i \), \( B_i \supset B_i \), \( \ldots \), then \( B_1, B_2, \ldots \in A_i \).

It follows from a known theorem\(^2\) that \( A_1 \) is the class of all Borel subsets of \( (0, t) \), which implies, in view of (b), that \( A \) is the class of all Borel subsets of \( (0, +\infty) \).

The lemma is thus proved.

Let us denote by \( D^k \) the distribution function of \( \tau_p \), where \( T = (0, t) \), and by \( A \) the class of all Borel sets \( B \) such that

\[ E[\tau_p(w) > k] = P_k(B_i), \]

where \( \tau_p \) is a random variable, \( E[\tau_p] = D^k \); \( E[\tau_p(w) > k] = P_k(B_i) \).

In virtue of Lemma it suffices to prove that \( A \) satisfies the conditions (a), (b) and (c).

A satisfies (a). Since the condition (b) is fulfilled by every interval \( I = (a, b) \), we have \( I \in A \). Let be \( B = I_1 + I_2 + \ldots + I_n \), where \( I_j \) are disjoint intervals with \( |I_j| = d_j \). Obviously

\[ \tau_p(w) = \tau_{I_1}(w) + \tau_{I_2}(w) + \ldots + \tau_{I_n}(w), \]

whence \( \tau_p \) is a random variable.

Put

\[ J_j = (d_1 + \ldots + d_{j-1}, d_j + \ldots + d_n) \quad \text{for} \quad j = 2, 3, \ldots, n, \]

\[ J_j = J_1 + J_2 + \ldots + J_n, \]

\[ d = |J_1| = |B|. \]

It follows from the conditions (b) and (i) for intervals and from the equality

\[ \tau_p(w) = \tau_{J_1}(w) + \tau_{J_2}(w) + \ldots + \tau_{J_n}(w) \]

that \( \tau_{J} \) and \( \tau_p \) have the same distribution function, namely, that of the sum of independent random variables with distribution functions \( D^k, D^{k-1}, \ldots, D^0 \). Since the distribution function of \( \tau_p \) is \( D^k \), that of \( \tau_p \) is also \( D^k \). Thus, for every \( B \in E \), the conditions (3) are fulfilled.

A satisfies (b). Let \( B_1 \subset B_2 \subset \ldots \) and let us suppose that \( B_2 \in A \), in other words, that \( B_2 \) satisfy \( (b) \). Put \( B = B_1 + B_2 + \ldots \) and, for a fixed \( k \),

\[ E[B[\tau_p(w) > k]] = E[\tau_p(w) > k]. \]

Obviously

\[ E[B] = E[\tau_p(w) > k]. \]

Now, I am going to prove that

\[ E[B] = E_1 + E_2 + \ldots. \]

\(^2\) See Note 7 and 10.

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\(^\dagger\) See Halmos [3], p. 27, Theorem B.
In view of (7) it suffices to prove that if \( \omega \in \mathcal{E}_n \), then there is an \( n \) such that \( \omega \in \mathcal{E}_n \). If \( \omega \in \mathcal{E}_n \), then there is a finite sequence \( S=(t_1, t_2, \ldots, t_n) \) of numbers belonging to \( B \) and such that

\[
\sum_{j=1}^{n} \left[ \omega(t_j) - \omega(t_{j-1}) \right] > k.
\]

Consequently, in view of the definition of \( B \), there is an integer \( n \) such that \( S \subseteq B_n \), whence by (9) we obtain \( \omega \in \mathcal{E}_n \), and thus the formula (8) is proved.

Obviously \( \mathcal{E}_n \subseteq B_n \) and the relations (7) and (8) imply

\[
\mu(\mathcal{E}_0) = \lim_{n \to \infty} \mu(\mathcal{E}_n),
\]

and since by hypothesis

\[
\mu(\mathcal{E}) = R_B(\mathcal{E}),
\]

we obtain in view of (1) and (4)

\[
\mu(\mathcal{E}) = R_B(\mathcal{E}).
\]

Consequently \( A \subseteq A \), q. e. d.

A satisfies (c). Let \( B_1 \cup B_2 \cup \cdots \), where \( B_i \) are bounded sets belonging to \( A \), i. e. satisfying (6). Put \( B = B_1 \cup B_2 \cup \cdots \) and, for a fixed \( k \),

\[
\mathcal{E} = \mathcal{E}(\tau_B(\omega) > k), \quad \mathcal{E}' = \mathcal{E}(\tau_B(\omega) \geq k).
\]

Obviously

\[
\mathcal{E} \supseteq \mathcal{E}^0 \supseteq \mathcal{E}^1 \supseteq \cdots \subseteq \mathcal{E}.
\]

Now, I am going to prove that

\[
\mathcal{E} = \mathcal{E}^0 \cup \mathcal{E}^1 \cup \cdots.
\]

In view of (10) it suffices to prove that, if \( \omega \in \mathcal{E}^0 \) for \( j=1,2,\ldots \), then \( \omega \in \mathcal{E} \). Let us denote by \( S_j \) the set of all \( t \in B_j \) such that \( \omega(t) - \omega(t-1) > 0 \). Since \( B_j \) are bounded, the sets \( S_j \) are finite, and we have \( S_1 \supseteq S_2 \supseteq \cdots \) Consequently there is an integer \( n \) such that \( S_n = S_{n+1} = S_{n+2} = \cdots \), whence \( S_n \subseteq B \). Since

\[
\tau_B(\omega) \geq \sum_{j=n}^{\infty} \left[ \omega(t) - \omega(t-1) \right] = \tau_B(\omega) > k,
\]

we obtain \( \omega \in \mathcal{E}_n \) and thus the formula (11) is proved.

Analogously to the proof of the condition (b) we prove \( B \subseteq A \) by the aid of (10), (11), (1) and (4).

Proposition I is thus proved.

Proof of III.

Let \( B_1, B_2, \ldots, B_k \) be disjoint Borel subsets of \( (0, +\infty) \). We may without loss of generality assume that \( |B_j| < \infty \) \( (j=1,2,\ldots, k) \).

Thus, there exist \( k \) infinite sequences of sets \( E_1^1, E_1^2, \ldots, E_1^k \)

\[
(j=1,2,\ldots) \text{ belonging to } E (\text{cf. the definition, p. 133)},
\]

such that

\[
1^o \ E_i^{m-1} = 0 \text{ for } l \neq m,
\]

and

\[
2^o \text{ for each } j=1,2,\ldots, k \text{ the sequence } E_1^1, E_1^2, \ldots \text{ converges in measure to } B_1.
\]

It follows easily from the condition (i) for intervals that \( \tau_{E_j}, \tau_{E_j^2}, \ldots, \tau_{E_j^k} \) are independent. In view of II, the random variables

\[
\tau_{E_1}, \tau_{E_2}, \ldots \text{ tend in probability to } \tau_{E_1}.
\]

Since the passage to limit in probability preserves independence\(^1\), the random variables

\[
\tau_{E_1}, \tau_{E_2}, \ldots \text{ are independent, q. e. d.}
\]

Theorem is thus proved.

The problem arises whether the condition (i) for intervals implies (i) for any Borel sets, without the assumption (h) of the homogeneity. This problem has been solved quite recently by C. BULL-MAREZENSKI, who has also applied the method of in-

\(^1\) See Hartman and Marczewski [4], p. 130, Theorem 4.
ements in Borel sets for the complete discussion of non-homogeneous Poisson processes. These results of Byll-Nardzewski will be published in a forthcoming paper.

References.


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