Remarks on the Poisson Stochastic Process (I)

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We adopt in this paper*) the known measure theoretic treatment of probability[1]). In particular we treat a stochastic process as a functional space Ω with a probability-measure. Thus, in the classical Poisson process, each $\omega\epsilon\Omega$ is a function, the value $\omega(t)$ of which is *e. g.* the number of calls in a telephone exchange during the half-open time-intervall $(0,t\rangle$.

Strictly speaking we denote by Ω any set of real valued functions $\omega(t)$ defined for $t\geqslant 0$, by $\boldsymbol{B}_\Omega$ the smallest *σ-field* (*i. e.* a countably additive and complementative class of sets) which contains as elements all the sets of the form

$$A(t,y)=\underset{\omega}{E}[\omega\epsilon\Omega;\omega(t)<y],$$

and by μ a *probability measure* (*i. e.* a non-negative and countably additive set function with $\mu(\Omega)=1$) in $\boldsymbol{B}_\Omega$. The triple $(\Omega,\boldsymbol{B}_\Omega,\mu)$ is called a *stochastic process*[2]).

In particular we denote by Ω_0 the set of all integral valued functions $\omega(t)$ defined for $t\geqslant 0$, which are continuous on the right, non decreasing, and such that $\omega(0)=0$. Finally, we denote by Ω_1 the set of all functions $\omega\epsilon\Omega_0$ possessing only jumps equal to 1.

An important class of stochastic processes is that of *homogeneous differential* ones, *i. e.* fulfilling the following conditions:

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1) Cf. Kolmogoroff [8] and Halmos [5], Chapter IX, p. 184-215.

2) Cf. Doob [2].

(h) $\mu\underset{\omega}{E}[\omega(t+\tau)-\omega(t)<y]$ does not depend on t (the homogeneity),

(i) $$\mu\underset{\omega}{E}[\omega(u_1)-\omega(t_1)<y_1;\ldots;\omega(u_n)-\omega(t_n)<y_n]$$
$$=\mu\underset{\omega}{E}[\omega(u_1)-\omega(t_1)<y_1]\cdot\ldots\cdot\mu\underset{\omega}{E}[\omega(u_n)-\omega(t_n)<y_n]$$

for $0\leqslant t_1<u_1\leqslant t_2<\ldots\leqslant t_n<u_n$ (the independence of increments in non-overlapping intervals).

In the sequel we shall use instead of the condition (i) a formally weaker one:

(j) $$\mu\underset{\omega}{E}[\omega(u)-\omega(t)<y;\omega(v)-\omega(u)<z]$$
$$=\mu\underset{\omega}{E}[\omega(u)-\omega(t)<y]\cdot\mu\underset{\omega}{E}[\omega(v)-\omega(u)<z]$$

for $0\leqslant t<u<v$ (the independence of increments in two contiguous intervals).

A process is called *degenerate* if there is an $\omega_0\epsilon\Omega$ such that $\mu(B)=1$ for each $B\epsilon\boldsymbol{B}_\Omega$ containing ω_0.

Let us suppose $\Omega\subset\Omega_0$ and put

$$P_k(t)=\mu\underset{\omega}{E}[\omega\epsilon\Omega;\omega(t)=k]\qquad(k=0,1,2,\ldots).$$

It is well known that, if (h), (j),

(n) $$\lim_{t\to 0}\frac{1-P_0(t)}{t}=a$$

and

(o) $$\lim_{t\to 0}\frac{1-P_0(t)-P_1(t)}{t}=0,$$

then the random variable $\omega(t)$ has the Poisson distribution with the mean value at[3]), *i. e.*

(p) $$P_k(t)=e^{-at}\frac{(at)^k}{k!}\qquad(k=0,1,2,\ldots).$$

3) Cf. Khintchine [7], p. 19 and 20, Feller [3], p. 405, and [4], p. 364-367. The application of the Poisson distribution in this process goes back to Bortkiewicz [1], § 8, p. 16-19. A generalization for the non-homogeneous case is contained in a recent paper by Rényi [12]. Cf. also Lévy [11], chapter VII.

(For $a=0$ the condition (p) takes the form $P_0(t)=1$, $P_k(t)=0$ for $t\geqslant 0$, $k=1,2,\ldots$, and the process is degenerate).

In recent works the condition (n) is omitted; it tourns out namely that, *under the assumptions* (h) *and* (j), *if* (o), *then there is a number* $a\geqslant 0$ *such that* (p) (*and conversely*)[4].

The purpose of the § 1 of this paper is to prove directly that the condition (o) may be replaced by

(q) $\mu(\Omega-\Omega_1)=0$

or, in other words, that (q) *if and only if* (p) *for a number* $a\geqslant 0$ (Theorem 2).

Obviously the condition (q) is fulfilled in the important case of $\Omega\subset\Omega_1$, *i. e.* if all $\omega\in\Omega$ have only jumps equal to 1.

It seems that condition (q) has a more expressive probabilistic sense than the analytic condition (o). In the case of telephone calls the condition (q) says that two simultaneous calls are almost impossible.

In § 2 we prove in a very simple way that *every probability-measure in* Ω_0 *or* Ω_1 *vanishing for all one point sets is point-isomorphic to the Lebesgue measure* (Theorem 3).

§ 1. The Poisson distribution. A known reasoning[4]) gives directly the following

Theorem 1. *If* $(\Omega,\boldsymbol{B}_\Omega,\mu)$ *is a stochastic process with* $\Omega\subset\Omega_0$, *and fulfills* (h) *and* (j), *then there are two non-negative numbers* a *and* b *such that*

(1) $$P_0(t)=e^{-at},\qquad P_1(t)=bte^{-at}.$$

We shall prove the

Theorem 2. *If* $(\Omega,\boldsymbol{B}_\Omega,\mu)$ *is a stochastic process with* $\Omega\subset\Omega_0$, *and fulfills* (h) *and* (j), *then* (p) *for a number* $a\geqslant 0$ *if and only if* (q).

1. (q)→(p). In view of Theorem 1 we have (1), and we shall prove that

(2) $$a=b.$$

Let us establish a number $T>0$ and put

$$\Xi=\underset{\omega}{E}[\omega\in\Omega;\omega(T)>1],$$

[4]) Feller [3], p. 365, footnote 5, and Jánossy, Rényi and Aczél [6], § 1, p. 211-213.

$$\Xi_n=\underset{\omega}{E}\left[\omega\in\Xi;\omega\left(\frac{j}{2^n}T\right)-\omega\left(\frac{j-1}{2^n}T\right)\leqslant 1\quad\text{for } j=1,2,\ldots,2^n\right].$$

Obviously $\lim\limits_n\Xi_n=\Xi\Omega_1$, whence, in view of (q),

(3) $$\lim_n\mu(\Xi_n)=\mu(\Xi)=1-P_0(T)-P_1(T).$$

Next, putting

$$\Xi_n^j=\underset{\omega}{E}\left[\omega\in\Omega;\omega\left(\frac{j-1}{2^n}T\right)=1;\ \omega\left(\frac{j}{2^n}T\right)=2\right],$$

we obtain

(4) $$\Xi_n\subset\sum_{j=1}^{2^n}\Xi_n^j\subset\Xi\quad\text{and}\quad\Xi_n^i\cdot\Xi_n^j=0\quad\text{for}\quad i\neq j.$$

By virtue of (j) and (h) we have

$$\mu(\Xi_n^j)=\mu\underset{\omega}{E}\left[\omega\left(\frac{j-1}{2^n}T\right)-\omega(0)=1\right]\cdot\mu\underset{\omega}{E}\left[\omega\left(\frac{j}{2^n}T\right)-\omega\left(\frac{j-1}{2^n}T\right)=1\right]$$
$$=P_1\left(\frac{j-1}{2^n}T\right)P_1\left(\frac{1}{2^n}T\right),$$

whence, following (3) and (4),

(5) $$\lim_n P_1\left(\frac{1}{2^n}T\right)\sum_{j=1}^{2^n}P_1\left(\frac{j-1}{2^n}T\right)=1-P_0(T)-P_1(T).$$

Since, by (1)

$$\lim_n\frac{2^n}{T}P_1\left(\frac{1}{2^n}T\right)=b$$

and since obviously

$$\lim_n\frac{T}{2^n}\sum_{j=1}^{2^n}P_1\left(\frac{j-1}{2^n}T\right)=\int_0^T P_1(t)\,dt=\int_0^T bte^{-at}\,dt,$$

we deduce from (5)

$$b^2\int_0^T te^{-at}\,dt=1-e^{-aT}-bte^{-aT},$$

and by differentiation

$$b^2te^{-at}=ae^{-at}-b^{-at}+abte^{-at}.$$

Hence, putting $t=0$ we obtain (2).

The formula (p) is thus proved for $k=0$ and $k=1$. By a known reasoning we prove successively the same formula for $k=2,3,\ldots$

2. (p)→(q)[5]. The formula (p) for $k=0$ and $k=1$ implies directly the equality (o).

Let us establish a number $T>0$ and put

$$\Delta=\underset{\omega}{E}[\omega\in\Omega;\ \text{there is } t<T \text{ with } \omega(t)-\omega(t-0)\geqslant 2],$$

$$\Delta_n^j=\underset{\omega}{E}\left[\omega\in\Omega;\ \omega\left(\frac{j}{n}T\right)-\omega\left(\frac{j-1}{n}T\right)\geqslant 2\right].$$

Obviously $\Delta\subset\Delta_n^1+\Delta_n^2+\ldots+\Delta_n^n$ for $n=1,2,\ldots$ When $n\to\infty$

$$\mu(\Delta_n^1)+\mu(\Delta_n^2)+\ldots+\mu(\Delta_n^n)=n\left[1-P_0\left(\frac{1}{n}\right)-P_1\left(\frac{1}{n}\right)\right]\to 0,$$

following (o). Consequently $\mu(\Delta)=0$, whence, since T was chosen arbitrarily and μ is countably additive, we obtain the equality (q).

Theorem 2 is thus proved.

Let us remark that it is possible to deduce the same result from the general form of the distribution function for all homogeneous differential processes with $\Omega\subset\Omega_0$[6].

§ 2. The point isomorphism with the Lebesgue measure. Let R denote the set of all rational non-negative numbers. For each set Ω of real functions defined for $t\geqslant 0$ we denote by $\Omega|R$ the set of all partial functions $\omega|R$, where $\omega\in\Omega$.

Let us treat the space $\mathcal{E}^R$ of all real functions of a rational variable $r\geqslant 0$ as the denumerable Cartesian power of the set of real numbers, and consequently as a complete and separable metric space[7].

It is easy to see that

(a) If, for $\omega_1,\omega_2\in\Omega_0$, $\omega_1|R=\omega_2|R$, then $\omega_1=\omega_2$.

(b) For every $\Omega\subset\Omega_0$ and every decreasing sequence $r_n\in R$ tending to t

$$A(t,y)=\sum_{m=1}^{\infty}\prod_{n=m}^{\infty}A(r_n,y).$$

[5]) Cf. Lévy [11], p. 173.

[6]) See *e. g.* Jánossy, Rényi and Aczél [6], § 2, p. 213-217, in particular the remark of Kolmogoroff, formulated in the same paper, p. 216.

[7]) See *e. g.* Kuratowski [9], p. 231 and 313.

Next we shall prove

(c) $\Omega_0|R$ and $\Omega_1|R$ are Borel subsets of $\mathcal{E}^R$.

The set I of all $\omega\in\mathcal{E}^R$ with non-negative integral values is obviously closed in $\mathcal{E}^R$. Obviously $\omega\in\Omega_0|R$ if and only if $\omega\in I$, and ω is non-decreasing and continuous on the right in the set R. Consequently

$$\Omega_0|R=I\cdot\underset{\omega}{E}\Big\{\prod_{r_1<r_2}[\omega(r_1)\leqslant\omega(r_2)]\cdot\prod_{r_1}\sum_{r_2}(r_1<r_2)\big[\prod_{r_3}(r_1<r_3<r_2)\big(\omega(r_3)=\omega(r_1)\big)\big]\Big\},$$

where Σ and Π are quantifiers. It follows that $\Omega_0|R$ is an $F_{\sigma\delta}$-set in $\mathcal{E}^R$.

Similarly

$$\Omega_1|R=(\Omega_0|R)\cdot\underset{\omega}{E}\big[\prod_{r_1}\sum_{r_2}(r_2<r_1)(\omega(r_1)-\omega(r_2)\leqslant 1)\big],$$

which implies that $\Omega_1|R$ is also an $F_{\sigma\delta}$-set in $\mathcal{E}^R$.

Now we shall prove the following

LEMMA 1. *If Ω is a non-denumerable set of functions of a non-negative variable t, such that*

(α) *if $\omega_1|R=\omega_2|R$, where $\omega_1,\omega_2\in\Omega$, then $\omega_1=\omega_2$,*

(β) *for each $t\geqslant 0$ the sets $A(t,y)$ belong to the smallest σ-field containing all the sets $A(r,z)$, where $r\in R$,*

(γ) *$\Omega|R$ is a Borel subset of $\mathcal{E}^R$,*

then the field $\boldsymbol{B}_\Omega$ is point-isomorphic with the field of all Borel subsets of the unit interval.

PROOF. Let us associate with every function $\omega\in\Omega$ the function $\omega|R$. In view of (α) we obtain in this way a ~~one-one~~ mapping h_1 of Ω onto $\Omega'=\Omega|R$.

In view of (β) this mapping transforms the class $\boldsymbol{B}_\Omega$ onto the class $\boldsymbol{B}'$ of all Borel subsets of Ω'.

Finally, in view of (γ), there is a measurable (B) one-one mapping h_2 of Ω' onto the unit interval[8]). It transforms the class $\boldsymbol{B}'$ onto the field $\boldsymbol{B}$ of all Borel subsets of the unit interval.

[8]) Every non-denumerable Borel subset of a separable and complete metric space is the image of the unit interval by a measurable (B) one-one mapping (Kuratowski [9], p. 358, Theorem 2).

The mapping $x=h_2(h_1(\omega))$ is a point-isomorphism of $\boldsymbol{B}_\Omega$ and $\boldsymbol{B}$.

Lemma 2. *If $(\Omega,\boldsymbol{B}_\Omega,\mu)$ is a stochastic process satisfying the conditions* (α), (β) *and* (γ), *and such that μ vanishes for every one-point set, then μ is point-isomorphic with the Lebesgue measure in the field of all Borel subsets in the unit interval.*

Proof. The condition (α) implies that $\omega\epsilon\boldsymbol{B}_\Omega$ for every $\omega\epsilon\Omega$. The Lemma 2 follows from Lemma 1 and from the fact that every probability measure in the field $\boldsymbol{B}$, which vanishes for all one point sets is point-isomorphic with the Lebesgue measure in $\boldsymbol{B}$ [9]).

The propositions (a), (b), (c) and Lemma 2 imply directly the

Theorem 3. *Every stochastic process $(\Omega,\boldsymbol{B}_\Omega,\mu)$ with $\Omega=\Omega_0$ or $\Omega=\Omega_1$, and such that μ vanishes for each one point set, is point-isomorphic with the Lebesgue measure in the field of Borel subsets of the unit interval* [10]).

In particular the hypotheses of Theorem 3 are fulfilled by every non-degenerate process of the form $(\Omega_0,\boldsymbol{B}_{\Omega_0},\mu)$ or $(\Omega_1\boldsymbol{B}_{\Omega_1},\mu)$, satisfying the conditions (h) and (j). In fact, in view of Theorem 1 we have the equalities (1). If $a=0$, then $P_0(t)=1$ and the process is degenerate, and if $a>0$ it is easy to prove that $\mu((\omega))=0$ for every $\omega\epsilon\Omega$.

Lemma 2 implies an analogous theorem for the space Ω_c of all continuous real functions of a non-negative variable (and so particularly in the case of the Brownian motion [11])) and for many other spaces considered in the theory of stochastic processes.

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[9]) See *e. g.* Marczewski [10], p. 57.

[10]) For the case of processes having the Poisson distribution cf. Wiener [14], p. 51.

[11]) Wiener's proof of the existence of the measure for Brownian motions furnishes at the same time the construction of such an isomorphism (Wiener [13], p. 216). Obviously our theorems do not imply the existence of a measure for the considered stochastic processes. For the existence proofs, see Doob [2], in particular p. 120-123.

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