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Extremal perturbations of semi-Fredholm operators

by

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Abstract. Let T be a bounded operator on an infinite-dimensional Banach space X and Ω a compact subset of the semi-Fredholm domain of T. We construct a finite rank perturbation F such that $\min[\dim N(T+F-\lambda), \operatorname{codim} R(T+F-\lambda)] = 0$ for all $\lambda \in \Omega$, and which is extremal in the sense that $F^2 = 0$ and $\operatorname{rank} F = \max\{\min[\dim N(T-\lambda), \operatorname{codim} R(T-\lambda)] : \lambda \in \Omega\}$.

0. Introduction. Let X and Y be complex Banach spaces and $B(X,Y)$ the space of bounded operators from X to Y. An operator $T \in B(X,Y)$ is called *semi-Fredholm* if its range $R(T)$ is closed and its minimum index is finite. That is,

$$\text{min.ind}(T) = \min[\dim N(T), \operatorname{codim} R(T)] < \infty.$$

(Here $N(T)$ denotes the kernel of T.) In this case the index of T is well defined as

$$\operatorname{ind}(T) = \dim N(T) - \operatorname{codim} R(T).$$

For two operators $S, T \in B(X,Y)$ the *semi-Fredholm domain* is the set

$$\varrho_{\text{S-F}}(T:S) = \{\lambda \in \mathbb{C} : T - \lambda S \text{ is semi-Fredholm}\}.$$

It is well known that $\varrho_{\text{S-F}}(T:S)$ is open and that on its connected components the mapping $\lambda \to \operatorname{ind}(T-\lambda S)$ is constant. The mapping $\lambda \to \text{min.ind}(T-\lambda S)$, however, is constant on each connected component of $\varrho_{\text{S-F}}(T:S)$ except for a discrete subset where its value jumps up (see [Ka66], Chap. IV, §5). Those exceptional points are called *Kato's jumps* or *jumps of* $\lambda \to \text{min.ind}(T-\lambda S)$. They are precisely the points of discontinuity of the mapping $\lambda \to \gamma(T-\lambda S)$ in $\varrho_{\text{S-F}}(T:S)$; here $\gamma(T-\lambda S) = \inf\{\|(T-\lambda S)x\| : \operatorname{dist}[x, N(T-\lambda S)] = 1\}$ denotes the minimum modulus of $T-\lambda S$ (see [Ka58], Thm. 3 in §6 and Thm. 4 in §7).

An analytic, $B(X,Y)$-valued function $\lambda \to A(\lambda)$ is called *uniformly regular* on the open set $D \subseteq \mathbb{C}$ if $\lambda \to \gamma(A(\lambda))$ is strictly positive and continuous

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on D. In particular, the function $\lambda \to T-\lambda S$ is uniformly regular on the open set $D \subseteq \varrho_{\text{S-F}}(T:S)$ if and only if there are no jumps of $\lambda \to \text{min.ind}(T-\lambda S)$ in D. (Recall that $\gamma(T)>0$ if and only if $R(T)$ is closed.)

Now let $T \in B(X) = B(X,X)$ and let Ω be a compact subset of $\varrho_{\text{S-F}}(T) = \varrho_{\text{S-F}}(T:I)$. Then (see [Ze92]) there exists a finite rank operator $F \in B(X)$ such that

$$(*) \qquad \text{min.ind}(T+F-\lambda) = 0 \quad \text{for all } \lambda \in \Omega.$$

There have been a number of efforts to profit from the freedom which one has in the choice of F, in order to give additional extremal properties to F:

$$\text{(L-W)} \qquad [T,F]^2 = (FT-TF)^2 = 0,$$

$$\text{(I)} \qquad \text{rank}\, F = \max\{\text{min.ind}(T-\lambda) : \lambda \in \Omega\},$$

$$\text{(M-S)} \qquad F^2 = 0.$$

If F has property (L-W), it can be considered as extremal in view of property $(*)$, because in general it cannot be chosen commuting with T (see [LaWe82]). The authors of [LaWe82] further showed that if Ω consists of one Fredholm point of T, then there exists F with $(*)$ and (L-W). This was extended by Ó Searcóid, Boulmaarouf, and by Mbekhta (see [Ó S88], [Bo88], [Bo90] and [Mb93]). Finally, Zemánek proved in [Ze92] that for every compact set $\Omega \subseteq \varrho_{\text{S-F}}(T)$ there is an operator F satisfying $(*)$ and (L-W).

Condition (I) expresses that rank F is as small as possible in view of property $(*)$. It was considered first by Islamov for finite-dimensional X and for arbitrary proper (not necessarily compact) subsets Ω of $\mathbb{C}$ (see [Is87]). Förster and Jahn found F with property (I) in the case where Ω is a compact subset of a connected component of the Fredholm domain of T (see [FöJa92]).

Condition (M-S) was first considered by Markus and Sementsul in [MaSe78]. For infinite-dimensional X and a Fredholm operator T with ind(T) $= 0$ they constructed F with $T+F$ invertible and with properties (M-S) and (I). Note that this is not possible in a finite-dimensional space X if for example $T=0$.

All the three conditions can be satisfied simultaneously for a compact set $\Omega \subseteq \varrho_{\text{S-F}}(T)$ if $\lambda \to T-\lambda$ is uniformly regular on an open neighborhood D of Ω. This was shown in [Ze92]. However, it is not always true in the non-uniformly regular case. Förster and Krause showed (see [FöKr96]): For each Banach space there is a Fredholm operator T such that there is no finite rank operator F with min.ind$(T+F)=0$ which satisfies (L-W) and one of the conditions (I) or (M-S). But for the remaining combination "(I) and (M-S)" we have: If X is infinite-dimensional and if Ω is a compact subset of a connected component of $\varrho_{\text{S-F}}(T)$, then there is $F \in B(X)$ with $(*)$, (I) and (M-S).

Our main issue here is that in the above result we can remove the restriction that Ω has to be contained in a connected component (see Theorem 2.1). This is new even for the condition (I) or (M-S) alone. Theorem 2.1 rounds off the results obtained so far: For every compact subset Ω of the semi-Fredholm domain $\varrho_{\text{S-F}}(T)$, the properties (I) or (L-W) or (M-S) can always be satisfied, except that (M-S) requires infinite-dimensions. The only combination of several of these properties is possible with (I) and (M-S) in the infinite-dimensional case.

Our proof is different from the methods used in [FöKr96] which do not work in the more general case. We construct the range and kernel of the extremal perturbation by an inductive process which allows us to take care of more than one connected component of $\varrho_{\text{S-F}}(T)$ at the same time.

The paper is organized as follows: The first section concerns complementing spaces for the kernels and ranges of a given family of semi-Fredholm operators. We prove in Lemma 1.1 that the relation "$R(A(\lambda)) \cap V = \{0\}$ for all $\lambda \in \Omega$" is stable under small perturbations of the finite-dimensional space $V \subseteq Y$ if the $B(X,Y)$-valued function $A(\cdot)$ is uniformly regular on a neighborhood of the compact set Ω.

The main result of the first section (see Theorems 1.4 and 1.5) is that in Theorem 1 of [Ze92] we can drop the uniform regularity assumption if we consider a family of semi-Fredholm operators; note that Theorem 1 of [Ze92] was an important tool when constructing F with properties (L-W), (I) and (M-S) in the uniformly regular case. Roughly speaking, we prove Theorem 1.4 by first complementing the ranges of the uniformly regular part of the semi-Fredholm family using Theorem 1 of [Ze92]. The complement obtained already has rather nice properties. It remains to manipulate it by a suitable small perturbation, while Lemma 1.1 ensures that its nice properties are preserved. Finally, Theorem 1.5 follows by induction.

In the second section the extremal perturbation is constructed (see Theorem 2.1). We use our results on complementing subspaces to give an inductive construction of two spaces which by Proposition 2.1 of [FöKr96] are suitable to be the kernel and range of an operator with the desired properties.

1. Construction of complementing subspaces.

The following lemma will be an important tool when in the proof of Theorem 1.4 we pass to the non-uniformly regular case.

LEMMA 1.1. *Let X and Y be Banach spaces, $D \subseteq \mathbb{C}$ an open set and $\lambda \to A(\lambda) \in B(X,Y)$ uniformly regular on D. Let $v_1,\ldots,v_s,u_1,\ldots,u_r$ be linearly independent vectors in Y, where $r \le s$. If Ω is a compact subset of*

D and

$$R(A(\lambda)) \cap \operatorname{span}[v_1, \ldots, v_s] = \{0\} \quad \textit{for all } \lambda \in \Omega,$$

then there exists $\alpha_0 > 0$ *such that for all* $\alpha \in \mathbb{C}$ *with* $|\alpha| \le \alpha_0$ *we have*

$$R(A(\lambda)) \cap \operatorname{span}[v_1 + \alpha u_1, \ldots, v_r + \alpha u_r, v_{r+1}, \ldots, v_s] = \{0\} \quad \textit{for all } \lambda \in \Omega.$$

Proof. Assume there is no such α_0. Then there exist sequences $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}}$ in Ω with $\lim_n \alpha_n = 0$ and

$$R(A(\lambda_n)) \cap V_n \neq \{0\} \quad \text{for all } n \in \mathbb{N},$$

where $V_n = \operatorname{span}[v_1 + \alpha_n u_1, \ldots, v_r + \alpha_n u_r, v_{r+1}, \ldots, v_s]$. For each $n \in \mathbb{N}$ choose $x_n \in X$ with $A(\lambda_n)x_n \in V_n$, $\operatorname{dist}[x_n, N(A(\lambda_n))] \ge 1/2$ and $\|x_n\| = 1$; this is possible, because for fixed $n \in \mathbb{N}$ there is first $z_n \in X$ with $0 \neq A(\lambda_n)z_n \in V_n$ and $m \in N(A(\lambda_n))$ with $0 < d := \operatorname{dist}[z_n, N(A(\lambda_n))] \le \|z_n - m\| \le 2d$. For $x_n := \|z_n - m\|^{-1}(z_n - m)$ we have

$$\operatorname{dist}[x_n, N(A(\lambda_n))] = \|z_n - m\|^{-1} \operatorname{dist}[z_n - m, N(A(\lambda_n))] \ge (2d)^{-1} d = 1/2$$

and $A(\lambda_n)x_n = \|z_n - m\|^{-1} A(\lambda_n) z_n \in V_n$. Since we have $\sup_n \|A(\lambda_n)\| \le \sup_{\lambda\in\Omega} \|A(\lambda)\| < \infty$, the sequence $(A(\lambda_n)x_n)_n$ is bounded in the finite-dimensional space $\operatorname{span}[v_1, \ldots, v_s, u_1, \ldots, u_r]$. Moreover, Ω is compact, hence $(A(\lambda_n)x_n)_n$ as well as $(\lambda_n)_{n\in\mathbb{N}}$ have some accumulation points. We assume without loss of generality that $(A(\lambda_n)x_n)_n$ and $(\lambda_n)_{n\in\mathbb{N}}$ are convergent and that $\lim_n \lambda_n = \lambda_0$ for some $\lambda_0 \in \Omega$. Let

$$A(\lambda_n)x_n = \sum_{i=1}^{r} \beta_i^{(n)}(v_i + \alpha_n u_i) + \sum_{i=r+1}^{s} \beta_i^{(n)} v_i \quad \text{for all } n \in \mathbb{N}.$$

Since $v_1, \ldots, v_s, u_1, \ldots, u_r$ are linearly independent, there are $\beta_i \in \mathbb{C}$ with $\lim_n \beta_i^{(n)} = \beta_i$ for $i = 1, \ldots, s$. This implies

$$\lim_n A(\lambda_n)x_n = \sum_{i=1}^{s} \beta_i v_i =: v \in \operatorname{span}[v_1, \ldots, v_s],$$

because $\lim_n \alpha_n = 0$. Since $(x_n)_n$ is bounded and $\lim_n \lambda_n = \lambda_0$, we have $\lim_n A(\lambda_0)x_n = v$. This yields $v \in R(A(\lambda_0)) \cap \operatorname{span}[v_1, \ldots, v_s] = \{0\}$. On the other hand, for large $n \in \mathbb{N}$ we have

$$\|A(\lambda_n)x_n\| \ge \gamma(A(\lambda_n)) \operatorname{dist}[x_n, N(A(\lambda_n))] \ge \tfrac{1}{2}\gamma(A(\lambda_0))\tfrac{1}{2} > 0,$$

because $\gamma(A(\cdot))$ is continuous on D. Hence $\|v\| = \lim_n \|A(\lambda_n)x_n\| > 0$, and we got a contradiction. ∎

If Ω is a finite set, the above result holds without any regularity assumptions. This follows by induction from Lemma 1.2, but is not needed here.

LEMMA 1.2. *Let* X *be a vector space, let* $m \in X$ *and let* $R_i \subseteq X$ *be subspaces with* $m \notin R_i$ *for* $i = 1, \ldots, r$. *If* $z \in X$, *then there are at most* r *complex numbers* β *with* $z + \beta m \in R_1 \cup \ldots \cup R_r$.

Proof. Assume the assertion is not true. Then for at least one of the spaces R_i, say R_1, there exist two different complex numbers β and γ such that $z+\beta m$ and $z+\gamma m$ are in R_1. But then $(\beta-\gamma)m = z+\beta m-(z+\gamma m) \in R_1$ and, consequently, $m \in R_1$, which is a contradiction. ∎

Our next aim is to prove Theorem 1.5. Its result is partly contained in Theorem 1 of [Ze92] if a uniformly regular family is considered. Although this is not explicitly mentioned, the argument in the proof of Theorem 1 in [Ze92] and in the following remark goes through for operators $T, S \in B(X,Y)$ also if $X \neq Y$. To see this, note that by Theorem 2 of [Sl86] the following is ensured:

If the $B(X,Y)$*-valued function* $\lambda \to A(\lambda)$ *is uniformly regular on the open, bounded and connected set* $D \subseteq \mathbb{C}$ *and if* $\lambda_0 \in D$, *then there exists an analytic function* f *with* $f(\lambda) \in N(A(\lambda))$ *for all* $\lambda \in D$ *and with* $f(\lambda_0)$ *prescribed.*

From this expanded version of Theorem 1 of [Ze92] we draw the following corollary.

COROLLARY 1.3. *Let* X *and* Y *be Banach spaces and* $T, S \in B(X,Y)$. *Let* $\lambda \to T - \lambda S$ *be uniformly regular on an open neighborhood of a compact set* $\Omega \subseteq \mathbb{C}$. *If* $\operatorname{codim} R(T - \lambda S) \ge 1$ *for all* $\lambda \in \Omega$, *then there exists* $y \in Y$ *such that*

$$y \notin R(T - \lambda S) \quad \textit{for all } \lambda \in \Omega.$$

Now we prove Theorem 1.5 in the special case $n = 1$. The proof of Theorem 1.5 itself then goes by induction.

THEOREM 1.4. *Let* X *and* Y *be Banach spaces, let* $T, S \in B(X,Y)$ *and let* $\Omega \subseteq \varrho_{\text{S-F}}(T : S)$ *be compact.*

(a) *If* $Z_i \subseteq Y$ *are subspaces with* $\operatorname{codim} Z_i \ge 1$ *for* $i = 1, \ldots, r$, *then there exists a subspace* $V \subseteq Y$ *with* $\dim V = 1$ *such that*

$$Z_i \cap V = \{0\} \quad \textit{for } i = 1, \ldots, r, \textit{ and}$$
$$R(T - \lambda S) \cap V = \{0\} \quad \textit{for all } \lambda \in \Omega \textit{ with } \operatorname{codim} R(T - \lambda S) \ge 1.$$

(b) *If* $Z_i \subseteq X$ *are subspaces with* $\dim Z_i \ge 1$ *for* $i = 1, \ldots, r$, *then there exists a closed subspace* $W \subseteq X$ *with* $\operatorname{codim} W = 1$ *such that*

$$Z_i + W = X \quad \textit{for } i = 1, \ldots, r, \textit{ and}$$
$$N(T - \lambda S) + W = X \quad \textit{for all } \lambda \in \Omega \textit{ with } \dim N(T - \lambda S) \ge 1.$$

Proof. (a) The jumps of $\lambda \to \text{min.ind}(T-\lambda S)$ form a discrete subset of $\varrho_{\text{S-F}}(T:S)$. Since Ω is compact, the set of jumps in Ω is finite. Hence if we iterate Kato's decomposition theorem (see Thm. 4 of §7 in [Ka58]), we obtain the "Kato decomposition corresponding to the finite number of jumps in Ω". That is,

$$T = T_0 \oplus T_1, \quad S = S_0 \oplus S_1$$

with respect to some decompositions $X = X_0 \oplus X_1$ and $Y = Y_0 \oplus Y_1$, where X_0 and Y_0 are closed and $\dim X_1 = \dim Y_1 < \infty$. In particular, $\varrho_{\text{S-F}}(T:S) = \varrho_{\text{S-F}}(T_0:S_0)$. Here $\lambda \to \text{min.ind}(T_0-\lambda S_0)$ has no jumps in Ω. Further, $\sigma(T_1:S_1) := \{\lambda \in \mathbb{C} : T_1 - \lambda S_1 \text{ is not invertible}\}$ consists of exactly the jumps of $\lambda \to \text{min.ind}(T-\lambda S)$ in Ω. In particular, $\sigma(T_1:S_1)$ is a finite set.

From the punctured neighborhood theorem (see [Ka66], Chap. IV, §5) it follows that

$$\Omega_1 := \{\lambda \in \Omega : \text{codim}\, R(T_0 - \lambda S_0) \geq 1\}$$

is compact. Since $\lambda \to \text{min.ind}(T_0 - \lambda S_0)$ has no jumps in Ω, we know that $\lambda \to T_0 - \lambda S_0$ is uniformly regular on an open neighborhood of Ω. By Corollary 1.3 we find $v_0 \in Y_0$ with

$$(1) \qquad v_0 \notin R(T_0 - \lambda S_0) \quad \text{for all } \lambda \in \Omega_1.$$

By [FöJa92], Lemma 2.4, there exists $v_1 \in Y_1$ with

$$(2) \qquad v_1 \notin R(T_1 - \lambda S_1) \quad \text{for all } \lambda \in \sigma(T_1:S_1).$$

Again by [FöJa92], Lemma 2.4, there are $m_0 \in Y_0$ and $m_1 \in Y_1$ such that

$$(3) \qquad m_0 + m_1 \notin Z_i \quad \text{for } i = 1, \ldots, r.$$

Applying Lemma 1.1 and using (1), we find $\alpha_0 > 0$ such that

$$(4) \qquad v_0 + \alpha m_0 \notin R(T_0 - \lambda S_0) \quad \text{for all } \lambda \in \Omega_1 \text{ if } |\alpha| \leq \alpha_0.$$

According to Lemma 1.2, by (2) and (3), we can choose $0 \leq \alpha_1 < \alpha_0$ such that

$$(5) \qquad v_1 + \alpha_1 m_1 \notin R(T_1 - \lambda S_1) \quad \text{for all } \lambda \in \sigma(T_1:S_1)$$

and

$$(6) \qquad (v_0 + v_1) + \alpha_1(m_0 + m_1) \notin Z_1 \cup \ldots \cup Z_r.$$

Let

$$v := (v_0 + v_1) + \alpha_1(m_0 + m_1) \quad \text{and} \quad V := \text{span}[v].$$

By (6) we already know that $v \notin Z_1 \cup \ldots \cup Z_r$. Hence it is only left to show that $v \notin R(T-\lambda S)$ if $\lambda \in \Omega$ with $\text{codim}\, R(T-\lambda S) \geq 1$. Pick such a λ. If $\text{codim}\, R(T_0 - \lambda S_0) \geq 1$, then $\lambda \in \Omega_1$. Since $0 \leq \alpha_1 < \alpha_0$, we know from (4) that $v_0 + \alpha_1 m_0 \notin R(T_0 - \lambda S_0)$ and hence $v \notin R(T-\lambda S)$. If on the other hand $\text{codim}\, R(T_0 - \lambda S_0) = 0$, then we must have $\text{codim}\, R(T_1 - \lambda S_1) \geq 1$, which means $\lambda \in \sigma(T_1:S_1)$. By (5) we have $v_1 + \alpha_1 m_1 \notin R(T_1 - \lambda S_1)$ and it follows that $v \notin R(T-\lambda S)$.

(b) follows by duality from (a). ∎

THEOREM 1.5. *Let X and Y be Banach spaces and let $T, S \in B(X,Y)$. Let $\Omega \subseteq \varrho_{\text{S-F}}(T:S)$ be compact and let $n \in \mathbb{N}$.*

(a) *Let $Z \subseteq Y$ and $M \subseteq Y$ be closed subspaces with* $\text{codim}\, Z \geq n \geq \dim M$, *and let* $\text{codim}\, R(T-\lambda S) \geq n$ *for all $\lambda \in \Omega$. Assume that*

$$Z \cap M = \{0\}, \quad \text{and} \quad R(T-\lambda S) \cap M = \{0\} \quad \text{for all } \lambda \in \Omega.$$

Then there exists a subspace $V \subseteq Y$ with $\dim V = n$, $M \subseteq V$ such that

$$Z \cap V = \{0\}, \quad \text{and} \quad R(T-\lambda S) \cap V = \{0\} \quad \text{for all } \lambda \in \Omega.$$

(b) *Let $Z \subseteq X$ and $M \subseteq X$ be closed subspaces with* $\dim Z \geq n \geq \text{codim}\, M$, *and let* $\dim N(T-\lambda S) \geq n$ *for all $\lambda \in \Omega$. Assume that*

$$Z + M = X, \quad \text{and} \quad N(T-\lambda S) + M = X \quad \text{for all } \lambda \in \Omega.$$

Then there exists a closed subspace $W \subseteq X$ with $\text{codim}\, W = n$, *$W \subseteq M$ such that*

$$Z + W = X, \quad \text{and} \quad N(T-\lambda S) + W = X \quad \text{for all } \lambda \in \Omega.$$

Proof. (a) Let $d := n - \dim M$. We construct a sequence of subspaces $M = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_d \subseteq Y$ with $\dim V_i = \dim M + i$ such that

$$Z \cap V_i = \{0\}, \quad \text{and} \quad R(T-\lambda S) \cap V_i = \{0\} \quad \text{for all } \lambda \in \Omega.$$

Then the assertion is true for $V := V_d$.

We proceed by induction. Set $V_0 := M$. Now suppose V_i as above is already constructed for some $0 \leq i < d$. Let Q be a bounded projection of Y along V_i. Then Q, considered as an operator from Y to $R(Q)$, is Fredholm. Hence for $\lambda \in \Omega$ we know that $Q(T-\lambda S)$ is a semi-Fredholm operator from X to $R(Q)$. That is,

$$(1) \qquad \Omega \subseteq \varrho_{\text{S-F}}(QT:QS) \quad \text{with } QT, QS \in B(X, R(Q)).$$

Further, we have

$$(2) \qquad \text{codim}_{R(Q)}\, Q[Z] \geq 1.$$

Indeed, assuming $Q[Z] = R(Q)$ we obtain $Y = R(Q) + N(Q) = Q[Z] + N(Q) \subseteq Z + N(Q)$. Hence $\text{codim}\, Z \leq \dim N(Q) = \dim V_i = i + \dim M < d + \dim M = n$. We got a contradiction.

Replacing Z by $R(T-\lambda S)$ in the above argument we also obtain

$$(3) \qquad \text{codim}_{R(Q)}\, R(Q(T-\lambda S)) \geq 1 \quad \text{for all } \lambda \in \Omega.$$

By (1)–(3) we are just in the situation of Theorem 1.4(a), where Y, T, S and Z are replaced by $R(Q)$, QT, QS and $Q[Z]$, respectively. Therefore there is a vector $r \in R(Q)$ such that

$$r \notin Q[Z], \quad \text{and} \quad r \notin R(Q(T-\lambda S)) \quad \text{for all } \lambda \in \Omega.$$

Note that, in particular, $r \notin N(Q) = V_i$.

Set $V_{i+1} = V_i \oplus \operatorname{span}[r]$. Obviously $\dim V_{i+1} = \dim M + i + 1$. We show

$$Z \cap V_{i+1} = \{0\}.$$

Let $z \in Z \cap V_{i+1}$. Since $V_i = N(Q)$ and $r \in R(Q)$, we get $Qz \in \operatorname{span}[r]$. But then $Qz = 0$ because of $r \notin Q[Z]$. Hence by the induction hypothesis we obtain $z \in Z \cap N(Q) = Z \cap V_i = \{0\}$.

It is left to show that $R(T-\lambda S) \cap V_{i+1} = \{0\}$ for all $\lambda \in \Omega$. But this can be done by replacing Z by $R(T-\lambda S)$ in the above argument. Thus V_{i+1} is constructed.

(b) follows by duality from (a).

2. Construction of the extremal perturbation

THEOREM 2.1. *Let X be a Banach space with $\dim X = \infty$. Let $T \in B(X)$ and let Ω be a non-empty compact subset of $\varrho_{\text{S-F}}(T)$. Then there exists a finite rank operator $F \in B(X)$ with*

$$\operatorname{min.ind}(T+F-\lambda) = 0 \quad \textit{for all } \lambda \in \Omega, \quad F^2 = 0 \quad \textit{and} \quad \dim R(F) = \text{Max}.$$

Here $\text{Max} := \max\{\operatorname{min.ind}(T-\lambda) : \lambda \in \Omega\}$.

Proof. First we construct the kernel of our operator F. Namely, we show that there exists a subspace $W \subseteq X$ such that

$$\text{(0)} \quad \begin{aligned} &W \text{ is closed with } \operatorname{codim} W = \text{Max}, \\ &W \cap N(T-\lambda) = \{0\} \text{ for all } \lambda \in \Omega \text{ with } \dim N(T-\lambda) \le \text{Max, and} \\ &W + N(T-\lambda_0) = X = W + R(T-\lambda_0) \text{ for some } \lambda_0 \in \Omega. \end{aligned}$$

For this purpose we first choose $\lambda_0 \in \Omega$ with $\operatorname{min.ind}(T-\lambda_0) = \text{Max}$.

Next we construct a sequence of subspaces $W_{\text{Max}} \subseteq \ldots \subseteq W_1 \subseteq W_0 = X$ such that for $i = 0, 1, \ldots, \text{Max}$,

$$\text{(1)} \quad \begin{aligned} &W_i \text{ is closed with } \operatorname{codim} W_i = i, \ W_i + R(T-\lambda_0) = X, \text{ and} \\ &W_i + N(T-\lambda) = X \text{ for all } \lambda \in \Omega \text{ with } \dim N(T-\lambda) \ge i. \end{aligned}$$

Finally, for $W := W_{\text{Max}}$ we show the properties stated under (0).

Construction of the spaces W_i: Set $W_0 = X$. Now suppose W_i as above is constructed for some i with $\text{Max} > i \ge 0$. According to the punctured neighborhood theorem (see [Ka66], Chap. IV, §5), the set

$$\Omega_{i+1} := \{\lambda \in \Omega : \dim N(T-\lambda) \ge i+1\}$$

is a compact subset of $\varrho_{\text{S-F}}(T)$. Now the induction hypothesis shows that W_i and $R(T-\lambda_0)$ are closed subspaces of X with $\dim R(T-\lambda_0) \ge i+1 \ge \operatorname{codim} W_i$ and

$$R(T-\lambda_0) + W_i = X = N(T-\lambda) + W_i \quad \text{for all } \lambda \in \Omega_{i+1}.$$

(Here we need $\dim X = \infty$ to ensure that the semi-Fredholm operator $T-\lambda_0$ is not of finite rank and hence $\dim R(T-\lambda_0) \ge i+1$.) Applying Theorem 1.5(b) to $Z = R(T-\lambda_0)$, $M = W_i$ and $n = i+1$, we obtain a closed subspace $W_{i+1} \subseteq W_i$ with $\operatorname{codim} W_{i+1} = i+1$ and

$$R(T-\lambda_0) + W_{i+1} = X = N(T-\lambda) + W_{i+1} \quad \text{for all } \lambda \in \Omega_{i+1}.$$

Thus W_{i+1} is constructed.

Now $W = W_{\text{Max}}$ satisfies (0): From (1) it follows directly that W is a closed subspace with $\operatorname{codim} W = \text{Max}$ and $W + R(T-\lambda_0) = X$. Further, $W + N(T-\lambda_0) = W_{\text{Max}} + N(T-\lambda_0) = X$, because $\dim N(T-\lambda_0) \ge \operatorname{min.ind}(T-\lambda_0) = \text{Max}$. It is left to show that

$$W \cap N(T-\lambda) = \{0\} \quad \text{for all } \lambda \in \Omega \text{ with } \dim N(T-\lambda) \le \text{Max}.$$

Pick such a λ and let $i := \dim N(T-\lambda)$. Then (1) yields $W_i + N(T-\lambda) = X$. Since $\operatorname{codim} W_i = i = \dim N(T-\lambda)$, we have $W_i \cap N(T-\lambda) = \{0\}$. But $W = W_{\text{Max}} \subseteq W_i$ and, consequently, $W \cap N(T-\lambda) = \{0\}$. This shows that W has properties (0).

In particular, (0) implies $X = W + R(T-\lambda_0) = W + (T-\lambda_0)[W]$. Therefore, from $\operatorname{codim} W = \text{Max}$ it follows that there is a subspace

$$\text{(2)} \qquad V \subseteq W \quad \text{with} \quad \dim V = \text{Max and } X = W \oplus (T-\lambda_0)[V].$$

But then we even have

$$\text{(3)} \qquad X = W \oplus (T-\lambda)[V] \quad \text{for all } \lambda \in \mathbb{C}.$$

Indeed, for arbitrary $\lambda \in \mathbb{C}$ we deduce from (2) that

$$X = W \oplus (T-\lambda_0)[V] \subseteq W + (T-\lambda)[V] + V \subseteq W + (T-\lambda)[V].$$

But $\dim(T-\lambda)[V] \le \dim V = \text{Max} = \operatorname{codim} W$. This proves (3).

Now set

$$U = \{w \in W : Tw \in W\} = T^{-1}(W) \cap W.$$

Clearly U is a closed subspace of W. We have

$$\text{(4)} \qquad U + V = W \quad \text{and} \quad \operatorname{codim} U < \infty.$$

Indeed, for $w \in W$ we see from (3) that $Tw = w' + Tv$ for some $w' \in W$ and $v \in V$. Since $V \subseteq W$, this yields $w - v \in W$ with $T(w-v) = w' \in W$. Hence $w - v \in U$, which proves $U + V = W$. Now $\operatorname{codim} U < \infty$ follows from $\dim V = \operatorname{codim} W < \infty$.

We are going to construct the range of our operator F. This will be a space

(5a) $$L \subseteq W \quad \text{with } \dim L = \mathrm{Max}$$

such that

(5b) $(T-\lambda)[W] \cap L = \{0\}$ for all $\lambda \in \Omega$ with $\mathrm{ind}(T-\lambda) \le 0$,

(5c) $(T-\lambda)[W] + L = X$ for all $\lambda \in \Omega$ with $\mathrm{ind}(T-\lambda) \ge 0$.

For this purpose let $T_U, S_U : U \to W$ be the restrictions of T and Id_X to U, respectively. Then T_U and S_U are well defined, because $T[U] \subseteq W$ and $U \subseteq W$. Clearly $T_U, S_U \in B(U, W)$. Now by (4), U is closed with finite codimension. Hence if $\lambda \in \varrho_{\text{S-F}}(T)$, then $T_U - \lambda S_U$ has closed range by Lemma 333 of [Ka58] and the minimum index of $T_U - \lambda S_U$ remains finite. This shows

$$\Omega \subseteq \varrho_{\text{S-F}}(T) \subseteq \varrho_{\text{S-F}}(T_U : S_U).$$

By Theorem 1.5(a) there is a sequence of subspaces $\{0\} = L_0 \subseteq L_1 \subseteq \ldots \subseteq L_{\mathrm{Max}} \subseteq W$ such that for $i = 1, \ldots, \mathrm{Max}$ we have

(6) $\dim L_i = i$, and $R(T_U - \lambda S_U) \cap L_i = \{0\}$ for all $\lambda \in \Omega$ with $\mathrm{codim}_W R(T_U - \lambda S_U) \ge i$.

Now $L = L_{\mathrm{Max}}$ has the properties stated under (5): Obviously we have $L \subseteq W$ with $\dim L = \mathrm{Max}$. Hence (5a) holds. In order to verify properties (5b) and (5c) we make the following two observations (7) and (8):

(7) $$\mathrm{codim}_W (T-\lambda)[U] = \mathrm{codim}\,(T-\lambda)[W] \quad \text{for all } \lambda \in \mathbb{C}.$$

Indeed, from (3) we have $X = W \oplus (T-\lambda)[V]$. Hence $\mathrm{codim}_W(T-\lambda)[U] = \mathrm{codim}\{(T-\lambda)[U] \oplus (T-\lambda)[V]\}$. But $U + V = W$ by (4). This proves (7). Next,

(8) $\mathrm{codim}\,(T-\lambda)[W] = \mathrm{Max} - \mathrm{ind}(T-\lambda)$ if $\lambda \in \Omega$ with $\dim N(T-\lambda) \le \mathrm{Max}$.

Indeed, let P be a projection of X on W. Since W is closed with $\mathrm{codim}\, W < \infty$, we see that P is Fredholm with $\mathrm{ind}(P) = 0$. If $\lambda \in \Omega$ with $\dim N(T-\lambda) \le \mathrm{Max}$, then it follows from (0) that $R(P) \cap N(T-\lambda) = W \cap N(T-\lambda) = \{0\}$. This implies $N((T-\lambda)P) = N(P)$. Now (8) follows from

$$\begin{aligned}\mathrm{ind}(T-\lambda) &= \mathrm{ind}[(T-\lambda)P] = \dim N[(T-\lambda)P] - \mathrm{codim}\, R[(T-\lambda)P] \\ &= \mathrm{Max} - \mathrm{codim}\,(T-\lambda)[W].\end{aligned}$$

To prove (5b) let $\lambda \in \Omega$ with $\mathrm{ind}(T-\lambda) \le 0$. We have to show that $(T-\lambda)[W] \cap L = \{0\}$. Now $\dim N(T-\lambda) \le \mathrm{Max}$. Hence (7) and (8) yield

$$\mathrm{codim}_W (T-\lambda)[U] = \mathrm{codim}\,(T-\lambda)[W] = \mathrm{Max} - \mathrm{ind}(T-\lambda) \ge \mathrm{Max}.$$

We recall that by definition $R(T_U - \lambda S_U) = (T-\lambda)[U]$. Hence (6) implies

$$(T-\lambda)[U] \cap L = R(T_U - \lambda S_U) \cap L_{\mathrm{Max}} = \{0\}.$$

Now $(T-\lambda)[U] \oplus L$ is a subspace of W by definition of U and L. Further, W is disjoint from $(T-\lambda)[V]$ by (3). Hence

$$\{(T-\lambda)[U] \oplus L\} \cap (T-\lambda)[V] \subseteq W \cap (T-\lambda)[V] = \{0\}.$$

Since $U + V = W$ by (4), this yields

$$(T-\lambda)[W] \cap L = \{(T-\lambda)[U] \oplus (T-\lambda)[V]\} \cap L = \{0\}.$$

This proves (5b).

To prove (5c) let $\lambda \in \Omega$ with $\mathrm{ind}(T-\lambda) \ge 0$. We have to show that $(T-\lambda)[W] + L = X$. If $\dim N(T-\lambda) \le \mathrm{Max}$, then by (7) and (8) we get

$$\mathrm{codim}_W (T-\lambda)[U] = \mathrm{Max} - \mathrm{ind}(T-\lambda) \le \mathrm{Max}.$$

If on the other hand $\dim N(T-\lambda) \ge \mathrm{Max}$, then by (1) and $W = W_{\mathrm{Max}}$ we have $W + N(T-\lambda) = X$. By (7) we get

$$\begin{aligned}\mathrm{codim}_W (T-\lambda)[U] &= \mathrm{codim}\,(T-\lambda)[W] = \mathrm{codim}\, R(T-\lambda) \\ &= \mathrm{min.ind}(T-\lambda) \le \mathrm{Max}.\end{aligned}$$

Hence in any case

$$i := \mathrm{codim}_W (T-\lambda)[U] \le \mathrm{Max}.$$

By definition, $R(T_U - \lambda S_U) = (T-\lambda)[U]$. Hence from (6) it follows that $(T-\lambda)[U] \cap L_i = R(T_U - \lambda S_U) \cap L_i = \{0\}$. Since we have $\dim L_i = i = \mathrm{codim}_W (T-\lambda)[U]$ this means

$$(T-\lambda)[U] \oplus L_i = W.$$

Now it follows from (3), from $U + V = W$ and from $L_i \subseteq L$ that

$$X = W + (T-\lambda)[V] = [(T-\lambda)[U] + L_i] + (T-\lambda)[V] \subseteq (T-\lambda)[W] + L.$$

This proves (5c), and hence the space L has properties (5).

Since W is closed with $\mathrm{codim}\, W = \mathrm{Max} = \dim L$, we can choose an operator $F \in B(X)$ with kernel $N(F) = W$ and range $R(F) = L$. Then we know from (0) and (5b) that for all $\lambda \in \Omega$ with $\mathrm{ind}(T-\lambda) \le 0$,

$$\begin{aligned}&N(T-\lambda) \cap N(F) = N(T-\lambda) \cap W = \{0\} \quad \text{and} \\ &(T-\lambda)[N(F)] \cap R(F) = (T-\lambda)[W] \cap L = \{0\}.\end{aligned}$$

From (5c) it follows that for all $\lambda \in \Omega$ with $\mathrm{ind}(T-\lambda) \ge 0$,

$$\begin{aligned}&R(T-\lambda) + R(F) = R(T-\lambda) + L = X \quad \text{and} \\ &N(F) + (T-\lambda)^{-1}(R(F)) = W + (T-\lambda)^{-1}(L) = X.\end{aligned}$$

The sets $\{\lambda \in \Omega : \mathrm{ind}(T-\lambda) \le 0\}$ and $\{\lambda \in \Omega : \mathrm{ind}(T-\lambda) \ge 0\}$ are compact, because the index is locally constant and because Ω is compact.

According to Proposition 2.1 of [FöKr96], there exists a large number $\mu \in \mathbb{C}$ such that $\min.\mathrm{ind}(T + \mu F - \lambda) = 0$ for all $\lambda \in \Omega$. We have $(\mu F)^2 = 0$, because $R(\mu F) = L \subseteq W = N(\mu F)$. Finally, $\dim R(\mu F) = \dim L = \mathrm{Max}$. Hence the operator μF has the desired properties. ■

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On regularization in superreflexive Banach spaces by infimal convolution formulas

by

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Abstract. We present here a new method for approximating functions defined on superreflexive Banach spaces by differentiable functions with α-Hölder derivatives (for some $0 < \alpha \le 1$). The smooth approximation is given by means of an explicit formula enjoying good properties from the minimization point of view. For instance, for any function f which is bounded below and uniformly continuous on bounded sets this formula gives a sequence of Δ-convex $C^{1,\alpha}$ functions converging to f uniformly on bounded sets and preserving the infimum and the set of minimizers of f. The techniques we develop are based on the use of *extended inf-convolution* formulas and convexity properties such as the preservation of smoothness for the convex envelope of certain differentiable functions.

0. Introduction and preliminaries. This paper introduces an explicit regularization procedure for functions defined on superreflexive Banach spaces. For any function f bounded below and l.s.c. (resp. uniformly continuous on bounded sets) on a superreflexive Banach space X we give by means of a "standard" formula a sequence of $C^{1,\alpha}$-smooth functions converging pointwise (resp. uniformly on bounded sets) to f (where $0 < \alpha \le 1$ only depends on X). Under some additional conditions, the convergence of the sequence of approximate functions is uniform on the whole space X. Moreover, the approximate functions preserve the infimum and the set of minimizers of f. These features cannot be easily obtained from regularization methods like the *smooth partition of unity* techniques (for a detailed study of this topic we refer to Chapter VIII.3 of [DGZ], the references therein and [Fr]) or other results that only ensure the existence of smooth approximants (for instance, see [DFH]).

In Hilbert spaces, our work is closely linked with the *Lasry–Lions approximation method* (introduced in [LL] and subsequently studied by several

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