STUDIA MATHEMATICA 129 (3) (1998)

A noncommutative limit theorem for homogeneous correlations

by

ROMUALD LENCZEWSKI (Wrocław)

Abstract. We state and prove a noncommutative limit theorem for correlations which are homogeneous with respect to order-preserving injections. The most interesting examples of central limit theorems in quantum probability (for commuting, anticommuting, and free independence and also various q-qclt's), as well as the limit theorems for the Poisson law and the free Poisson law are special cases of the theorem. In particular, the theorem contains the q-central limit theorem for non-identically distributed variables, derived in our previous work in the context of q-bialgebras and quantum groups. More importantly, new examples of limit theorems of q-Poisson type are derived for both the infinite tensor product algebra and the reduced free product, leading to new q-laws. In the first case the limit as $q \to 1$ is studied in more detail and a connection with partial Bell polynomials is established.

1. Introduction. Noncommutative analogs of the central limit theorem have been investigated by many authors. The existence of a relatively large number of those theorems may be attributed to the fact that there is no single definition of independence in quantum probability. Thus even in the case of identically distributed independent random variables we obtain different noncommutative versions of the same classical theorem (we restrict our attention to the method of moments). They are related to various kinds of independence like commuting [C-H, G-W], anticommuting [H, W], free [V, S], or some form of q-independence [B-S, S-W, Sch]. Essentially, most of those theorems are based on the same combinatorial argument that leads to the vanishing of non-pair partitions and, perhaps, some other pair partitions like, for example, the crossing ones as in the case of free independence.

However, it is also interesting to study the case of non-identically distributed quantum variables. Moreover, it turns out that such variables appear naturally in mathematical physics. Namely, if we consider the Jimbo–Drinfeld quantum groups $U_q(g)$, then the coproduct of the so-called non-group-like generators may be viewed as a sum of non-identically distributed

1991 *Mathematics Subject Classification*: Primary 60F05.

q-independent random variables (see [L1]–[L3]). The simplest example is furnished by the two-level system represented by the fundamental representation of $U_q(su(2))$ and the vacuum state [L-P]. In the limit we obtain the algebra of the q-harmonic oscillator in the vacuum state and the limit law is q-Gaussian. One would like to include such examples in a more general type of limit theorem. The central limit theorem for weakly dependent maps studied in [A-L], although quite general (it allows for non-identically distributed variables), does not cover the above-mentioned examples due to an assumption on the variance of the sums (it must be of order equal to the number of terms). This assumption is crucial to the combinatorics, and therefore, to all further results.

Moreover, it would be interesting to prove a limit theorem of which not only central limit theorems, but also those leading to other laws, especially those of Poisson type, would be special cases. Therefore, we state and prove a limit theorem assuming a homogeneity condition for correlations. This theorem includes the most important examples of quantum central limit theorems (for commuting, anticommuting and free independences) as well as their various q-analogs as special cases, but also the limit theorems for the Poisson law, the free Poisson law and certain q-laws obtained from convergence of q-Poisson type.

The mathematical setup is as follows. We have a 'small' *-algebra $\mathcal{C}$ and a 'large' *-algebra $\widehat{\mathcal{C}}$ with a state $\widehat{\phi}$. Two standard examples of $(\widehat{\mathcal{C}},\widehat{\phi})$ come to mind immediately: the infinite tensor product $(\mathcal{C}^{\otimes\infty},\phi^{\otimes\infty})$ and the reduced free product $(\mathcal{C}^{\star\infty},\phi^{\star\infty})$. Let $j_{k,N}$, where $N\in\mathbb{N}$, $0\le k\le N$, be a triangular array of *-homomorphisms (or, more generally, *-injections) from $\mathcal{C}$ into $\widehat{\mathcal{C}}$. Consider the sums of random variables

$$S_N(v)=\sum_{k=1}^{N} j_{k,N}(v)$$

where v is a generator of $\mathcal{C}$, and calculate the correlations of such sums for different generators $v_1,\ldots,v_p$ in the state $\widehat{\phi}$. Using appropriate 'rescaling' $(v\to v^N)$, one calculates the limits of the 'rescaled' correlations as $N\to\infty$. In order to do that one assumes: (i) some kind of invariance condition for the correlations

$$\widehat{\phi}(j_{i_1,N}(v_1)\ldots j_{i_p,N}(v_p))$$

with respect to the indices $i_1,\ldots,i_p$; (ii) a condition on the growth rate of the normalized correlations of this type. Note also that with each tuple of indices we can associate a partition of the set $\{1,\ldots,p\}$ in a natural way.

The first step would be to cover the case of identically distributed variables. The main two assumptions could be stated as follows:

(I) the correlations are invariant under order-preserving injections,
(P) the correlations decay polynomially or faster

(for details, see Section 3). Here, the order of the polynomial is equal to the number r of blocks in the (ordered) partition associated with the correlation. More precisely, (P) stands for the condition that the correlations decay as N^{-r} or faster. Clearly, if a correlation vanishes even before taking the limit, then it also satisfies (P). It is not hard to show that conditions (I)–(P) ensure the existence of the limits and thus lead to a quite general limit theorem. Other assumptions are characteristic of specific types of limit theorems, leading to various laws (Gaussian, Wigner, Poisson, free Poisson, etc.). Essentially, one can say that a nonvanishing contribution comes only from those correlations which decay as N^{-r}, whereas those which decay faster, disappear.

The second step would be to include the convolution q-analog of the central limit theorem studied previously for the Jimbo–Drinfeld quantum groups $U_q(g)$. In that case, it is natural to assume that the variables are not identically distributed. That is why (I) and (P) will be replaced by the following conditions:

(H) the correlations are homogeneous with respect to order-preserving injections,
(E) the correlations decay exponentially or faster.

The precise definitions are given in Section 4. In order to include the case $q=1$, or conditions (I)–(P), we can replace (E) by

(PE) the correlations decay polynomially or faster for $q=1$ and exponentially or faster for $q\neq 1$.

In general, one can say that our objective is to state and prove a noncommutative limit theorem that would be general enough to include: (i) the known examples of independence, and (ii) different kinds of limit theorems, i.e. weak laws of large numbers, central and higher order limit theorems, limit theorems of Poisson type, etc. Loosely speaking, going from [S] through [S-W] to this paper, the main assumption on the correlations changes as follows: invariance w.r.t. permutations → invariance w.r.t. order-preserving injections → homogeneity w.r.t. order-preserving injections. This is the main reason why this paper is more general; however, we should also add that its scope is fairly large because of (i)–(ii) above (cf. [S] for free independence, and [S-W] for central limit theorems). Moreover, it leads to new interesting examples of limit theorems, which may be viewed as q-analogs of Poisson convergence, singles out new limit laws and provides a nice connection with combinatorics and partial Bell polynomials.

In Section 2 we give definitions and derive certain combinatorial formulas. In Section 3 we state and prove a limit theorem for the correlations which are invariant with respect to order-preserving injections and decay polynomially (Theorem 3.1). In Section 4 we state and prove a limit theorem for the correlations which are homogeneous with respect to order-preserving injections and decay exponentially (Theorem 4.2). To combine those two theorems into one, one needs a new formulation, which is given in Section 5, where we also discuss the central limit theorem. In Section 6 we consider special cases of our main theorem, namely those of convergence of Poisson type for both the infinite tensor product and the reduced free product. In Section 7 we study the case of the infinite tensor product and the corresponding new limit law in more detail. A connection with partial Bell polynomials for $q \to 1$ is established.

We assume throughout this work that $q > 0$ without further mention. The case $q < 0$ is almost equivalent and can be easily treated along the same lines.

2. Definitions, combinatorics and q-formulas. Let $\widehat{\mathcal{C}}$ be a *-algebra and $\mathcal{C}_{i,N}$ its *-subalgebras, where $1 \le i \le N$ and $N \in \mathbb{N}$. In other words, we have a triangular array of subalgebras. Moreover, we assume that there exist **-injections* $j_{i,N}$ of a *-algebra $\mathcal{C}$ into $\widehat{\mathcal{C}}$ such that $j_{i,N}(\mathcal{C}) = \mathcal{C}_{i,N}$. Further, let $\widehat{\phi}$ be a *state* on $\widehat{\mathcal{C}}$. It is convenient to think of $\widehat{\mathcal{C}}$ as some large algebra, such as the tensor algebra or the reduced free product of Voiculescu. This is the general framework for this paper. On the algebraic level, it is enough to assume that we work with *algebras*, *injections* and *functionals* instead of *-algebras, *-injections (or even, *-homomorphic embeddings) and states. Especially, we will speak of injections most of the time, but whenever we deal with states we will tacitly assume that they are *-injections.

The central notion in our approach is that of an *ordered partition* S, by which we mean an ordered tuple $(S_1, \ldots, S_r)$ of disjoint subsets (called *blocks*) of an index set I, with union I. By the *signature* of S we mean the tuple $(\gamma_1, \ldots, \gamma_r)$, where γ_i is the cardinality of S_i. The set of ordered partitions of $I = \{1, \ldots, p\}$ will be denoted by $\mathcal{P}^{\mathrm{ord}}\{1, \ldots, p\}$, in contrast to $\mathcal{P}\{1, \ldots, p\}$, which will denote the usual (unordered) partitions. Among the latter we distinguish the noncrossing partitions denoted by $\mathcal{P}^{\mathrm{nc}}\{1, \ldots, p\}$ (see, for instance, [S]). Note that to each unordered partition $\{S_1, \ldots, S_r\}$ there correspond $r!$ different ordered partitions. We denote by S^{rev} the ordered partition with reversed order of blocks, i.e. $S^{\mathrm{rev}} = (S_r, \ldots, S_1)$.

If $\mathcal{C}$ is a *-algebra generated by $\mathcal{G}_+ \cup \mathcal{G}_-$ with $\mathcal{G}_- = \mathcal{G}_+^*$, we use the notation

$$S_j^+ = \{k \in S_j \mid v_k \in \mathcal{G}_+\}, \quad S_j^- = \{k \in S_j \mid v_k \in \mathcal{G}_-\},$$

i.e. the blocks associated with the generators from $\mathcal{G}_+$ and $\mathcal{G}_-$, respectively. We also introduce the set of ordered inversions

$$W_S = \{(i,j) \mid i \in S_k,\ j \in S_m,\ i < j \text{ and } k > m\}$$

as well as W_{S^+} and W_{S^-}, with S_k, S_m replaced by S_k^+, S_m^+, or S_k^-, S_m^-, respectively (these are inversions associated with the generators from $\mathcal{G}_+$ and $\mathcal{G}_-$, respectively).

Consider p order-preserving injections corresponding to the tuple of indices $(i_1, \ldots, i_p)$ and fixed N and let $I_0 = \{i_1, \ldots, i_p\} = \{k_1, \ldots, k_r\}$, where $k_1 < \ldots < k_r$. The set I_0 defines an ordered partition $S = (S_1, \ldots, S_r)$ of $\{1, \ldots, p\}$ in a natural way, i.e. we have $S_m = \{j \mid i_j = k_m\}$ (note that in S the order of sets is relevant). Conversely, to each ordered partition $S = (S_1, \ldots, S_r)$ there corresponds the *minimal tuple* $(i_1^*, \ldots, i_r^*)$ such that $i_j^* = m$ iff $j \in S_m$. Actually, this correspondence is one-to-one. We choose $*$ instead of the superscript S for notational convenience. Finally, an *order-preserving injection* T from $\{1, \ldots, M\}$ into $\{1, \ldots, N\}$, where $M \le N$, is an injection for which $T(i) < T(j)$ iff $i < j$.

Let us briefly illustrate the above notions. For example, let $(i_1, \ldots, i_5) = (4, 2, 2, 4, 6)$. Hence, $I_0 = \{2, 4, 6\}$ and the ordered indices are $k_1 = 2$, $k_2 = 4$ and $k_3 = 6$. The ordered partition defined by I_0 is given by $S = (S_1, S_2, S_3)$, where $S_1 = \{2, 3\}$, $S_2 = \{1, 4\}$ and $S_3 = \{6\}$. The minimal tuple associated with this partition is $(2, 1, 1, 2, 3)$. One can see that the tuple $(4, 2, 2, 4, 6)$ can be obtained from the minimal tuple associated with the same partition S by the order-preserving injection T, where $T(2) = 4$, $T(1) = 2$ and $T(3) = 6$.

Using this language, we say that the tuple $(i_1, \ldots, i_p)$ *has a singleton* if the partition associated with it has at least one set of cardinality 1. We also say that a correlation has a singleton if the tuple of indices of its injections has a singleton. In turn, the correlation is of *second order* if the partition associated with the set of those indices is a pair partition.

Finally, in the limit theorems we consider sums of the type

$$S_N(v) = \sum_{k=1}^{N} j_{k,N}(v),$$

i.e. the sums of the images of the generator v under the consecutive injections $j_{k,N}$. The generators will be denoted by $v_1, \ldots, v_p$, which after N-dependent 'rescaling' become $v_1^N, \ldots, v_p^N$. If no confusion arises (in one-dimensional cases), we write $S_N(v^N) = S_N$.

The rest of this section is devoted to q-expressions and formulas. The latter (Propositions 2.1, 2.2 and Lemma 2.3) are rather technical, but their proofs are given for the reader's convenience.

For $q \neq 1$ and $\delta \neq 0$ let

$$[N]_{\delta,\sigma} = \frac{q^{\sigma N} - q^{(\sigma+\delta)N}}{q^{\sigma} - q^{(\sigma+\delta)}}.$$

We define $[N]_{-2\alpha,\alpha} = [N]_{\alpha}$. For convenience, we use another notation if $\sigma = 0$, namely $[[x]]_q = (q^x - 1)/(q-1)$. We introduce the multivariate q-expressions

$$\langle x_1, \ldots, x_r\rangle_q = \frac{1}{(q^{x_1}-1)(q^{x_1+x_2}-1)\ldots(q^{x_1+\ldots+x_r}-1)},$$

$$[x_1,\ldots,x_r]_q = \frac{1}{[[x_1]]_q[[x_1+x_2]]_q\ldots[[x_1+\ldots+x_r]]_q},$$

$$[[x_1,\ldots,x_r]]_q = \frac{[[x_1]]_q[[x_2]]_q\ldots[[x_r]]_q}{[[x_1]]_q[[x_1+x_2]]_q\ldots[[x_1+\ldots+x_r]]_q}.$$

We also write

$$[x_1,\ldots,x_r]_1 = [x_1,\ldots,x_r], \qquad [[x_1,\ldots,x_r]]_1 = [[x_1,\ldots,x_r]].$$

Further, $\widehat{S}$ denotes the *symmetrizer*, for instance

$$\widehat{S}[[x_1,\ldots,x_r]]_q = \sum_{\pi\in\mathcal{S}_r} [[x_{\pi(1)},\ldots,x_{\pi(r)}]]_q,$$

where $\mathcal{S}_r$ denotes the permutation group of r elements, and similarly for any function of r variables.

PROPOSITION 2.1. *Let $x_1,\ldots,x_r \in \mathbb{R}\setminus\{0\}$, $q>0$, $q\neq 1$. Then*

$$\widehat{S}[x_1,\ldots,x_r] = \frac{1}{x_1\ldots x_r} \quad \textit{and} \quad \widehat{S}[[x_1,\ldots,x_r]] = 1.$$

Proof. If we prove the first relation, the second follows immediately:

$$\widehat{S}[[x_1,\ldots,x_r]] = x_1\ldots x_r\widehat{S}[x_1,\ldots,x_r] = 1.$$

The case $r=1$ of the first relation is clear and the inductive step goes as follows ($\check{}$ denotes an omitted index):

$$\widehat{S}[x_1,\ldots,x_r] = \sum_{\pi\in\mathcal{S}_{r+1}} \frac{1}{x_{\pi(1)}(x_{\pi(1)}+x_{\pi(2)})\ldots(x_{\pi(1)}+\ldots+x_{\pi(r+1)})}$$

$$= \frac{1}{x_1+\ldots+x_{r+1}}\sum_{j=1}^{r+1}\sum_{\pi\in\mathcal{S}_r\{1,\ldots,\check{j},\ldots,r+1\}} \frac{1}{x_{\pi(1)}(x_{\pi(1)}+x_{\pi(2)})\ldots(x_{\pi(1)}+\ldots+x_{\pi(r)})}$$

$$= \frac{1}{x_1+\ldots+x_{r+1}}\sum_{j=1}^{r+1}\frac{1}{x_1\ldots\check{x}_j\ldots x_{r+1}} = \frac{1}{x_1\ldots x_{r+1}}. \ ■$$

Now define another important multivariate q-function:

$$G_N(\beta_1,\ldots,\beta_r\,|\,q) = \sum_{1\le k_1<\ldots<k_r\le N} q^{\beta_1k_1+\ldots+\beta_rk_r}$$

for $q,\beta_1,\ldots,\beta_r\in\mathbb{R}$. Note that for $q=1$ we obtain

$$G_N(\beta_1,\ldots,\beta_r\,|\,1) = \binom{N}{r},$$

so we can think of $G_N(\beta_1,\ldots,\beta_r\,|\,q)$ as q-analogs of the binomial coefficients. However, one has to be careful with the terminology since they are not equal to the q-multinomial coefficients in standard q-analysis.

PROPOSITION 2.2. *Let $q\neq 1,-1$ and $\beta_1,\ldots,\beta_r\neq 0$. The following recurrence relation holds*:

$$G_N(\beta_1,\ldots,\beta_r\,|\,q) = \langle\beta_r\rangle_{q^{-1}}(G_{N-1}(\beta_1,\ldots,\beta_{r-1}+\beta_r\,|\,q) - q^{N\beta_r}G_{N-1}(\beta_1,\ldots,\beta_{r-1}\,|\,q))$$

where we put $G_N(\emptyset\,|\,q)=1$.

Proof. We repeat the proof from [L2] for the reader's convenience. Clearly,

$$G_N(\beta\,|\,q) = \langle\beta\rangle_{q^{-1}}(1-q^{\beta N}).$$

In turn,

$$G_N(\beta_1,\ldots,\beta_r\,|\,q) = \sum_{1\le k_1<\ldots<k_{r-1}\le N-1} q^{\beta_1k_1+\ldots+\beta_{r-1}k_{r-1}} \sum_{k_{r-1}<k_r\le N} q^{k_r\beta_r}$$

$$= \sum_{1\le k_1<\ldots<k_{r-1}\le N-1} q^{\beta_1k_1+\ldots+\beta_{r-1}k_{r-1}+\beta_r(k_{r-1}+1)}\frac{1-q^{\beta_r(N-k_{r-1})}}{1-q^{\beta_r}}$$

$$= \langle\beta_r\rangle_{q^{-1}}(G_{N-1}(\beta_1,\ldots,\beta_{r-1}+\beta_r\,|\,q) - q^{N\beta_r}G_{N-1}(\beta_1,\ldots,\beta_{r-1}\,|\,q)). \ ■$$

LEMMA 2.3. *Under the assumptions of Proposition 2.1 we have the following decomposition*:

$$G_N(\beta_1,\ldots,\beta_r) = \sum_{i=0}^{r} q^{N(\beta_{i+1}+\ldots+\beta_r)}g_i(\beta_1,\ldots,\beta_r\,|\,q)$$

where

$$g_i(\beta_1,\ldots,\beta_r\,|\,q) = q^{\beta_{i+1}+\ldots+\beta_r}\langle\beta_i,\ldots,\beta_1\rangle_{q^{-1}}\langle\beta_{i+1},\ldots,\beta_r\rangle_q.$$

Proof. The general form of the decomposition with some coefficients that depend on $\beta_1,\ldots,\beta_r$ and q follows directly from the recurrence formula in Proposition 2.1. Denote the coefficients by $g_i(\beta_1,\ldots,\beta_r\,|\,q)$. It remains to prove that they take the given form. First, note that Proposition 2.1 used again gives the following recurrence formulas for $g_i(\beta_1,\ldots,\beta_r\,|\,q)$:

$$g_i(\beta_1,\ldots,\beta_r\,|\,q)=\langle\beta_r\rangle_{q^{-1}}(q^{-\beta_{i+1}-\ldots-\beta_r}g_i(\beta_1,\ldots,\beta_{r-1}+\beta_r\,|\,q)$$
$$-\,q^{-\beta_{i+1}-\ldots-\beta_{r-1}}g_i(\beta_1,\ldots,\beta_r\,|\,q))$$

for $i=0,1,\ldots,r-2$ and

$$g_{r-1}(\beta_1,\ldots,\beta_r\,|\,q)=-\langle\beta_r\rangle_{q^{-1}}g_{r-1}(\beta_1,\ldots,\beta_{r-1}\,|\,q),$$
$$g_r(\beta_1,\ldots,\beta_r\,|\,q)=\langle\beta_r\rangle_{q^{-1}}g_{r-1}(\beta_1,\ldots,\beta_{r-1}+\beta_r\,|\,q).$$

From this it is easy to show that $g_i(\beta_1,\ldots,\beta_r\,|\,q)$ given by this lemma satisfy the above relations. The last two of them are immediate. Let us show the first one:

$$\begin{aligned}\mathrm{RHS}&=\langle\beta_r\rangle_{q^{-1}}\langle\beta_i,\ldots,\beta_1\rangle_{q^{-1}}(\langle\beta_{i+1},\ldots,\beta_{r-1}+\beta_r\rangle_q-\langle\beta_{i+1},\ldots,\beta_{r-1}\rangle_q)\\&=\langle\beta_r\rangle_{q^{-1}}\langle\beta_i,\ldots,\beta_1\rangle_{q^{-1}}\langle\beta_{i+1}\rangle_q\ldots\langle\beta_{i+1}+\ldots+\beta_{r-2}\rangle_q\\&\quad\times(\langle\beta_{i+1}+\ldots+\beta_r\rangle_q-\langle\beta_{i+1}+\ldots+\beta_{r-1}\rangle_q)\\&=\langle\beta_r\rangle_{q^{-1}}\langle\beta_i,\ldots,\beta_1\rangle_{q^{-1}}\langle\beta_{i+1}\rangle_q\ldots\langle\beta_{i+1}+\ldots+\beta_{r-2}\rangle_q\\&\quad\times q^{\beta_{i+1}+\ldots+\beta_{r-1}}(1-q^{\beta_r})\langle\beta_{i+1}+\ldots+\beta_{r-1}\rangle_q\langle\beta_{i+1}+\ldots+\beta_r\rangle_q\\&=q^{\beta_{i+1}+\ldots+\beta_r}\langle\beta_i,\ldots,\beta_1\rangle_{q^{-1}}\langle\beta_{i+1},\ldots,\beta_r\rangle_q\\&=g_i(\beta_1,\ldots,\beta_r\,|\,q)=\mathrm{LHS}.\ \blacksquare\end{aligned}$$

In particular, we have

$$g_0(\beta_1,\ldots,\beta_r\,|\,q)=q^{\beta_1+\ldots+\beta_r}\langle\beta_1,\ldots,\beta_r\rangle_q,$$
$$g_r(\beta_1,\ldots,\beta_r\,|\,q)=\langle\beta_r,\ldots,\beta_1\rangle_{q^{-1}}.$$

Lemma 2.3 and the explicit form of $g_0(\beta_1,\ldots,\beta_r\,|\,q)$ and $g_r(\beta_1,\ldots,\beta_r\,|\,q)$ will be used in the proof of Theorem 4.2.

3. Limit theorem for invariant correlations. The two main definitions that lead to assumptions (I)–(P) from the introduction are given below. Let T be an order-preserving injection from $\{1,\ldots,M\}$ into $\{1,\ldots,N\}$, where $M\le N$.

CONDITION I. The correlations are *invariant with respect to order-preserving injections*, or satisfy *Condition* I, iff

$$\widehat{\phi}(j_{T(i_1),N}(v_1)\ldots j_{T(i_p),N}(v_p))=\widehat{\phi}(j_{i_1,M}(v_1)\ldots j_{i_p,M}(v_p))$$

for $1\le i_1,\ldots,i_p\le M$, $1\le T(i_1),\ldots,T(i_p)\le N$ and $M\le N$.

CONDITION P. The correlations satisfy *Condition* P iff the limits

$$\lim_{N\to\infty}N^r\widehat{\phi}(j_{i_1^*,N}(v_1^N)\ldots j_{i_p^*,N}(v_p^N))=M_S(v_1,\ldots,v_p)$$

exist and are finite for the minimal tuple $(i_1^*,\ldots,i_p^*)$ associated with a partition $S=(S_1,\ldots,S_r)$, where $1\le i_1^*,\ldots,i_p^*\le N$.

REMARK 1. In other words, the correlations satisfy Condition P iff they decay polynomially or faster, or iff they decay as N^{-r} or faster, where r is the number of blocks in the partition S associated with the correlation.

REMARK 2. The canonical embeddings in both the infinite tensor product and the reduced free product satisfy Conditions I–P.

THEOREM 3.1. *If the correlations are invariant under order-preserving injections and satisfy Condition* P, *then*

$$\lim_{N\to\infty}\widehat{\phi}(S_N(v_1^N)\ldots S_N(v_p^N))=\sum_{S\in\mathcal{P}^{\mathrm{ord}}\{1,\ldots,p\}}D_SM_S(v_1,\ldots,v_p)$$

where D_S equals $1/r!$, and thus depends only on the number of blocks in the partition S. If, in addition, the correlations are invariant under permutations of sets in the partition S, then

$$\lim_{N\to\infty}\widehat{\phi}(S_N(v_1^N)\ldots S_N(v_p^N))=\sum_{S\in\mathcal{P}\{1,\ldots,p\}}M_S(v_1,\ldots,v_p).$$

In both cases, $M_S(v_1,\ldots,v_p)=0$ for those partitions which decay faster than N^{-r}.

Proof. Tuples of indices can be grouped into equivalence classes represented by minimal tuples associated with ordered partitions. Then each of the tuples from the same equivalence class can be obtained from the same minimal tuple $(i_1^*,\ldots,i_p^*)$ by an order-preserving injection T. We rearrange each tuple $(T(i_1^*),\ldots,T(i_p^*))$ as $\{T(i_1^*),\ldots,T(i_p^*)\}=\{k_1,\ldots,k_r\}$ with $k_1<\ldots<k_r$. Let $S=(S_1,\ldots,S_r)$ be the partition of $\{1,\ldots,p\}$ associated with $(i_1^*,\ldots,i_p^*)$. Namely, S_j consists of all numbers m such that $i_m^*=j$. Note that for a given ordered-partition $S=(S_1,\ldots,S_r)$ and fixed N there are $\binom{N}{r}$ different order-preserving injections. Therefore, using Condition I, we obtain

$$\begin{aligned}&\widehat{\phi}(S_N(v_1^N)\ldots S_N(v_p^N))\\&\quad=\sum_{1\le i_1,\ldots i_p\le N}\widehat{\phi}(j_{i_1,N}(v_1^N)\ldots j_{i_p,N}(v_p^N))\\&\quad=\sum_{S=(S_1,\ldots,S_r)\in\mathcal{P}^{\mathrm{ord}}\{1,\ldots,p\}}\binom{N}{r}\widehat{\phi}(j_{i_1^*,r}(v_1^N)\ldots j_{i_p^*,r}(v_p^N))\end{aligned}$$

and hence, using Condition P, we arrive at

$$\lim_{N\to\infty}\widehat{\phi}(S_N(v_1^N)\ldots S_N(v_p^N))=\sum_{S\in\mathcal{P}^{\mathrm{ord}}\{1,\ldots,p\}}D_SM_S(v_1,\ldots,v_p)$$

where $D_S=\lim_{N\to\infty}N^{-r}\binom{N}{r}=1/r!$. Clearly, the contribution from the correlations which decay faster than N^{-r} vanishes. This finishes the proof of the first part of the theorem. The second part is obvious. $\blacksquare$

The above limit theorem will be generalized in the sequel, but it seems interesting in its own right, since its formulation is relatively simple and it is free from the kind of independence and covers a few types of limit theorems. As its immediate corollaries we obtain, for example, the central limit theorems for commuting, anticommuting, and free independence, as well as the limit theorems for the Poisson law and the free Poisson law (see the discussion below). Moreover, it also covers the q-central limit theorems studied in [Sch, B-S, S-W].

Discussion of special cases. We shall consider the tensor product and the reduced free product. If a correlation vanishes before taking the limit, we say that its rate of decay is equal to $N^{-\infty}$. In the case of central limit theorems, this happens if the factorization produces first moments in the state ϕ. Thus, such correlations will not contribute to the limit. Since the rate of decay of all other correlations is $N^{-p/2}$, it is clear that only pair partitions may survive and only for even p. However, in the case of free independence, the correlation corresponding to any crossing partition can be expressed in terms of products of more than r factors (see [S], Lemma 4). Thus, if it is a pair partition, it must have a first moment in each of the products, and therefore, its rate of decay is $N^{-\infty}$ and it does not appear in the limit Wigner law. In a similar way (skipping the last part of the above argument) we can see that in the limit law for the infinite tensor product of a *-algebra $\mathcal{C}$, *all* pair partitions contribute to the limit, giving the symmetric, antisymmetric, or q-Gaussian laws. The situation is even simpler for theorems of Poisson type. In the classical case, we assume that the rate of decay of all correlations is equal to N^{-r} since $\lim_{N\to\infty} N\phi((v^N)^k) = \lambda$ and therefore for the correlation corresponding to the partition $S = (S_1, \ldots, S_r)$ we have

$$\lim_{N\to\infty} N^r \phi((v^N)^{\gamma_1}) \ldots \phi((v^N)^{\gamma_r}) = \lambda^r$$

and thus all partitions contribute to the Poisson law. However, in the case of free independence, the above calculation holds only for noncrossing partitions, since the crossing ones decay faster than N^{-r} due to the factorization into more than r moments, and thus vanish in the limit. For convenience, we shift the N-dependence from the states to the variables (classically, it is assumed that $Np_N \to \lambda$, where p_N is the probability of heads).

4. Limit theorem for homogeneous correlations. Our aim is to formulate a q-analog of Theorem 3.1 that would also include the q-central limit theorem studied in [L-P, L1, L2] and lead to new limit theorems. In view of the above, we can see that such a q-analog (for positive q, not equal to one, and thus allowing for non-identically distributed variables) could be based on a similar formulation, except that N should be replaced by some q-function of N. Moreover, we know from our previous work that *invariance* with respect to order-preserving injections will not hold. Therefore, Condition I will be replaced by a more general one, namely *homogeneity*.

Let T be an order-preserving injection from $\{1, \ldots, M\}$ into $\{1, \ldots, N\}$, where $M < N$. Further, let α and β be maps from the set of generators of $\mathcal{C}$ into the reals. They will be called *deformation maps*. We set $\alpha_i = \alpha(v_i)$ and $\beta_i = \beta(v_i)$. In turn, for a partition $S = (S_1, \ldots, S_r) \in \mathcal{P}^{\text{ord}}\{1, \ldots, p\}$ and p-tuples of real numbers $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_p)$ associated with the generators $v_1, \ldots, v_p$, we define

$$\alpha_{S_m} = \sum_{k\in S_m} \alpha_k, \qquad \beta_{S_m} = \sum_{k\in S_m} \beta_k$$

where $m = 1, \ldots, r$, which can be viewed as the *degrees of deformation associated with S and $(v_1, \ldots, v_p)$*.

Let us state the conditions which generalize Conditions I–P.

CONDITION H. We say that the correlations are (α, β)-*homogeneous with respect to order-preserving injections*, or that they satisfy *Condition* H, iff

$$\widehat{\phi}\left(\frac{j_{T(i_1),N}(v_1)}{q^{\alpha_1 T(i_1)+\beta_1 N}} \cdots \frac{j_{T(i_p),N}(v_p)}{q^{\alpha_p T(i_p)+\beta_p N}}\right) = \widehat{\phi}\left(\frac{j_{i_1,M}(v_1)}{q^{\alpha_1 i_1+\beta_1 M}} \cdots \frac{j_{i_p,M}(v_p)}{q^{\alpha_p i_p+\beta_p M}}\right)$$

where $q \in \mathbb{R}^+$ and $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p \in \mathbb{R}$ are associated with $v_1, \ldots, v_p \in \mathcal{G}_+ \cup \mathcal{G}_-$.

CONDITION E. We say that the correlations satisfy *Condition* E iff the limits

$$\lim_{N\to\infty} q^{(\beta_{S_1}+\ldots+\beta_{S_r})N} \widehat{\phi}(j_{i_1^*,N}(v_1^N) \ldots j_{i_p^*,N}(v_p^N)) = M_S^q(v_1, \ldots, v_p)$$

for $0 < q < 1$, and

$$\lim_{N\to\infty} q^{(\alpha_{S_1}+\ldots+\alpha_{S_r}+\beta_{S_1}+\ldots+\beta_{S_r})N} \widehat{\phi}(j_{i_1^*,N}(v_1^N) \ldots j_{i_p^*,N}(v_p^N)) = M_S^q(v_1, \ldots, v_p)$$

for $1 < q < \infty$, all exist and are finite, where $(i_1^*, \ldots, i_p^*)$ is the minimal tuple associated with the partition S.

REMARK 1. If $q < 1$ and $\beta < 0$, or $q > 1$ and $\alpha + \beta > 0$, then we say that the correlations *decay β-exponentially*, or $(\alpha+\beta)$-*exponentially*. On the other hand, if $q < 1$ and $\beta > 0$, or if $q > 1$ and $\alpha + \beta < 0$, then we say that they *grow β-exponentially*, or $(\alpha + \beta)$-*exponentially*. It is enough to consider two cases (one for exponential decay and one for exponential growth), but often we will further restrict our attention to exponential decay since exponential growth can be done in a similar fashion.

REMARK 2. Another possibility would be to put σ instead of $\alpha + \beta$ in the second equation, but then Condition H would look more complicated.

REMARK 3. One could say that Condition H means invariance of the correlations with respect to order-preserving injections *for exponentially rescaled variables*. Note also that Condition H is a special case of Condition I for $q = 1$.

Let us now give a few examples of correlations that satisfy the above conditions.

EXAMPLE 1. Let $\mathcal{C} = U_q(su(2))$, the q-deformation of the enveloping algebra $U(su(2))$ generated by v, v^*, t, t^{-1}, with hermitian t, t^{-1}, subject to the usual commutation relations [L-P, L1]:

$$tv = q^{-2}vt, \quad tv^* = q^2 v^* t, \quad tt^{-1} = t^{-1}t = 1, \quad [v^*, v] = \frac{t^2 - t^{-2}}{q^2 - q^{-2}},$$

and the usual coproduct

$$\Delta(w) = t^{-1} \otimes w + w \otimes t, \quad \Delta(t) = t \otimes t, \quad \Delta(t^{-1}) = t^{-1} \otimes t^{-1},$$

where $w \in \{v, v^*\}$. More generally, one can consider the q-bialgebra defined as above, except for one relation, the Lie bracket, which is not assumed. Define injections of $\mathcal{C}$ into $\mathcal{C}^{\otimes\infty}$ as follows:

$$j_{i,N}(w) = (t^{-1})^{\otimes(i-1)} \otimes w \otimes t^{\otimes(N-i)} \otimes 1^{\otimes\infty}.$$

Let $\widehat{\phi} = \phi^{\otimes\infty}$ for the state ϕ on $\mathcal{C}$ such that $\phi(t) = q^{\beta}$ and $\phi(t^{-1}) = q^{-\beta}$, extended homomorphically to $\mathbb{C}[t, t^{-1}]$. Then the correlations satisfy Condition H with $\alpha = -2\beta$. With the appropriate normalization [L-P, L1], namely $v^N = (1/\sqrt{[N]_\alpha})v$, $v^{*N} = (1/\sqrt{[N]_\alpha})v^*$, Condition E can also be satisfied. In particular, when $\beta = 1, \alpha = -2$, we have the vacuum state for the fundamental representation of $U_q(su(2))$, i.e. the cyclic vector is annihilated by v^*. Note that $\Delta_{N-1}(w) = S_N(w)$, where Δ_{N-1} is the $(N-1)$th iteration of the coproduct [L-P, L1]. This noncommutative Jimbo–Drinfeld deformation can be generalized to $U_q(g)$ (see [L2] for details).

EXAMPLE 2. Let $\mathcal{C} = U_{pq}(su(2))$, the pq-deformation of $U(su(2))$, a generalization of $U_q(su(2))$. It is generated by v, v^*, s, t subject to the relations

$$tv = q^{-1}pvt, \quad tv^* = qp^{-1}v^*t, \quad sv = qp^{-1}vs, \quad sv^* = q^{-1}pv^*s$$

and

$$[v^*, v] = \frac{t^2 - s^2}{q^2 - p^2}$$

with hermitian, commuting s, t. The *-injections given by

$$j_{i,N}(w) = s^{\otimes(i-1)} \otimes w \otimes t^{\otimes(N-i)} \otimes 1^{\otimes\infty},$$

where $w \in \{v, v^*\}$, are generalizations of those in Example 1. Let $\phi(t) = q^{\beta}$, $\phi(s) = q^{\alpha+\beta}$, extended homomorphically to $\mathbb{C}[t, s]$. Then the correlations

with $\widehat{\phi} = \phi^{\otimes\infty}$ satisfy Condition H (Condition E can be satisfied with normalization similar to Example 1). In particular, if ϕ is the vacuum state of a 2-dimensional representation of $U_{pq}(su(2))$ indexed by a real number j_0, we have $\phi(t) = q^{j_0+1/2}p^{j_0-1/2}$ and $\phi(s) = q^{j_0-1/2}p^{j_0+1/2}$.

EXAMPLE 3. Introduce q-canonical injections in the infinite tensor product algebra:

$$j_{i,N}(v_k) = b_k^{\otimes(i-1)} \otimes v_k \otimes a_k^{\otimes(N-i)} \otimes 1^{\otimes\infty} = q^{(\alpha_k+\beta_k)(i-1)} j_i(v_k) q^{\beta_k(N-i)}$$

where $b_k = q^{\alpha_k+\beta_k}$ and $a_k = q^{\beta_k}$ and j_i are the canonical embeddings. If we take $\widehat{\phi} = \phi^{\otimes\infty}$, then Conditions H–E are satisfied.

EXAMPLE 4. Replace $(\mathcal{C}^{\otimes\infty}, \phi^{\otimes\infty})$ by $(\mathcal{C}^{*\infty}, \phi^{*\infty})$, the reduced free product of $(\mathcal{C}, \phi)$, and define q-canonical *-injections by

$$j_{i,N}(v_k) = q^{(\alpha_k+\beta_k)(i-1)} j_i(v_k) q^{\beta_k(N-i)}$$

where j_i are the canonical *-homomorphic embeddings. The RHS in Examples 3 and 4 are written in a noncommutative way for the sake of similarity to the previous examples.

Condition H can be expressed in a slightly different form, which will be used in the sequel, given by the following proposition.

PROPOSITION 4.1. *Let $(i_1^*, \ldots, i_p^*)$ be a minimal tuple such that $\{i_1^*, \ldots \ldots, i_p^*\} = \{1, \ldots, r\}$ and let $r \le N$. The correlations are (α, β)-homogeneous with respect to order-preserving injections iff*

$$\widehat{\phi}(j_{T(i_1^*),N}(v_1) \ldots j_{T(i_p^*),N}(v_p)) = q^{TS}\widehat{\phi}(j_{i_1^*,r}(v_1) \ldots j_{i_p^*,r}(v_p))$$

where

$$TS = \sum_{k=1}^{p}((T(i_k^*) - i_k^*)\alpha_k + (N - r)\beta_k)$$

stands for the 'total shift'.

Proof. Obvious.

We are ready to state a quantum limit theorem for the correlations which are (α, β)-homogeneous with respect to order-preserving injections and satisfy Condition E. In this section, we state the theorem with Conditions H–E, thus assuming $q \neq 1$. In Section 5, we will restate the theorem by incorporating the case $q = 1$.

We assume that the deformation map α is positive (the case of negative α is equivalent). The general case of arbitrary α involves certain technical difficulties and will not be treated in this work. There are no restrictions on β, which is the less relevant (easy to handle) deformation map.

THEOREM 4.2. *Assume that the correlations are (α, β)-homogeneous with respect to order-preserving injections and satisfy Condition E for $q > 0$,*

$q \neq 1$. *Then*

$$\lim_{N\to\infty} \widehat{\phi}(S_N(v_1^N)\dots S_N(v_p^N)) = \sum_{S\in\mathcal{P}^{\mathrm{ord}}\{1,\dots,p\}} D_S^q(\alpha,\beta)M_S^q(v_1,\dots,v_p)$$

where

$$D_S^q(\alpha,\beta) = \begin{cases} q^{Y(S,\alpha,\beta)}g_0(\alpha_{S_1},\dots,\alpha_{S_r}\,|\,q) & \text{if } q>1,\\ q^{Y(S,\alpha,\beta)}g_r(\alpha_{S_1},\dots,\alpha_{S_r}\,|\,q) & \text{if } q<1,\end{cases}$$

and $Y(S,\alpha,\beta) = \sum_{m=1}^r(-m\alpha_{S_m} - r\beta_{S_m})$.

Proof. Let first $q>1$. The idea of the proof is similar to that in Theorem 3.1, but we have to keep track of all q-expressions, which complicates the calculations. Note that

$$\begin{aligned} TS &= \sum_{k=1}^p((T(i_k^*) - i_k^*)\alpha_k + (N-r)\beta_k)\\ &= \sum_{m=1}^r((k_m - m)\alpha_{S_m} + (N-r)\beta_{S_m})\\ &= W(S,N,\alpha,\beta) + \sum_{m=1}^r k_m\alpha_{S_m}\end{aligned}$$

where we separated the part depending on the ordered indices from the indices-independent expression

$$W(S,N,\alpha,\beta) = \sum_{m=1}^r(-m\alpha_{S_m} + (N-r)\beta_{S_m}).$$

Thus, using Proposition 4.1, we arrive at

$$\begin{aligned}&\widehat{\phi}(S_N(v_1^N)\dots S_N(v_p^N))\\ &\quad= \sum_{1\le i_1,\dots i_p\le N}\widehat{\phi}(j_{i_1,N}(v_1^N)\dots j_{i_p,N}(v_p^N))\\ &\quad= \sum_{S=(S_1,\dots,S_r)\in\mathcal{P}^{\mathrm{ord}}\{1,\dots,p\}} q^{W(S,N,\alpha,\beta)}G_N(\alpha_{S_1},\dots,\alpha_{S_r}\,|\,q)\\ &\qquad\times \widehat{\phi}(j_{i_1^*,r}(v_1^N)\dots j_{i_p^*,r}(v_p^N))\end{aligned}$$

where

$$G_N(\alpha_{S_1},\dots,\alpha_{S_r}\,|\,q) = \sum_{1\le k_1<\dots<k_r\le N} q^{\alpha_{S_1}k_1+\dots+\alpha_{S_r}k_r}.$$

We now use Proposition 2.2 to calculate

$$D_S^q(\alpha,\beta) = \lim_{N\to\infty} q^{-(\alpha_{S_1}+\dots+\alpha_{S_r}+\beta_{S_1}+\dots+\beta_{S_r})N+W(S,N,\alpha,\beta)}G_N(\alpha_{S_1},\dots,\alpha_{S_r}\,|\,q)$$

for each ordered partition S (recall that the analogous expression in the case of $q=1$ is almost trivial, i.e. we obtain $1/r!$). Here, using Condition E, we arrive at

$$D_S^q(\alpha,\beta) = \lim_{N\to\infty}\sum_{i=0}^r(q^{-N(\alpha_{S_1}+\dots+\alpha_{S_i})})q^{Y(S,\alpha,\beta)}g_i(\alpha_{S_1},\dots,\alpha_{S_r}\,|\,q)$$

where $Y(S,\alpha,\beta) = \sum_{m=1}^r(-m\alpha_{S_m} - r\beta_{S_m})$.

If $q>1$ and $\alpha_i>0$ for every $i\in\{1,\dots,p\}$, then the only term which gives a nonzero contribution is the one with index $i=0$ and thus

$$D_S^q(\alpha,\beta) = q^{Y(S,\alpha,\beta)}g_0(\alpha_{S_1},\dots,\alpha_{S_r}\,|\,q)$$

where $g_0(\alpha_{S_1},\dots,\alpha_{S_r}\,|\,q)$ is defined in Section 2. This finishes the proof for $q>1$.

If $q<1$, we proceed in a similar manner and observe that the only term which survives in the limit is the one with $i=r$. Hence we obtain

$$D_S^q(\alpha,\beta) = q^{Y(S,\alpha,\beta)}g_r(\alpha_{S_1},\dots,\alpha_{S_r}\,|\,q),$$

which finishes the proof. ∎

COROLLARY 4.3. *If, in addition,* $\alpha = -2\beta$, *then we have the inverse-reverse symmetry* $D_S^q(\alpha,\beta) = D_{S^{\mathrm{rev}}}^{q^{-1}}(\alpha,\beta)$.

Proof. If $\alpha=-2\beta$, then

$$D_S^q(\alpha,\beta) = \begin{cases} q^{\sum_{m=1}^r(2m-r-2)\beta_{S_m}}\langle\alpha_{S_1},\dots,\alpha_{S_r}\rangle_q & \text{if } q<1\\ q^{\sum_{m=1}^r(2m-r)\beta_{S_m}}\langle\alpha_{S_r},\dots,\alpha_{S_1}\rangle_{q^{-1}} & \text{if } q>1.\end{cases}$$

Note that

$$\sum_{m=1}^r(2m-r)\beta_{S_m} = \sum_{j=1}^r(r-2j+2)\beta_{S_{r-j+1}},$$

which gives $D_S^q(\alpha,\beta) = D_{S^{\mathrm{rev}}}^{q^{-1}}(\alpha,\beta)$. ∎

5. Another formulation and special cases. We have chosen the above formulation for the sake of simplicity, but we can also rephrase Condition E using the q-deformed N of type

$$[N]_{\delta,\sigma} = \frac{q^{\sigma N} - q^{(\sigma+\delta)N}}{q^\sigma - q^{(\sigma+\delta)}},$$

motivated by q-analysis and quantum groups, used in our previous work. This will be done below. Let us only point out that both formulations have certain advantages and disadvantages. The main advantage of using Condition E is its simplicity. However, if we want to let $q\to 1$, then Condition PE given below becomes more convenient.

Now we introduce a deformed S-dependent N, which will replace N^r from Theorem 3.1 in the assumption of its q-analog (Theorem 5.1). It will be defined multiplicatively, i.e.

$$[N]_{S,\alpha,\beta} = [N]_{S_1} \dots [N]_{S_r}$$

where $[N]_{S_j} = [N]_{\alpha_{S_j},\beta_{S_j}}$. Before we go on, let us mention a few properties of such q-deformations of N. Firstly, we clearly have

$$\lim_{q\to 1} [N]_{S_m} = N.$$

Secondly, note that the deformation of Jimbo–Drinfeld type, as in the q-deformed enveloping algebras $U_q(g)$, is obtained when we put $\alpha = -2\beta$. Thirdly, if we put $\alpha =$ const and $\beta =$ const, then we get a *uniform deformation*, independent of $v_1, \dots, v_p$. In the case of uniform deformation of type $(2,-1)$ the main theorem of this section could be obtained by replacing N^r by

$$[N]_{S,\alpha,\beta} = [N]_{\gamma_1} \dots [N]_{\gamma_r}$$

where $(\gamma_1, \dots \gamma_r)$ is the signature of S. The uniform deformation for *pair partitions* bears even more resemblance to the usual case. Namely, we then obtain $[N]_{S,\alpha,\beta} = [N]_2^r$ for $p = 2r$. Moreover, if $q = 1$, we put $[N]_2 = N$. Therefore, as we shall see later, Theorem 3.1 will be recovered as a special case of the q-limit theorem stated and proved below.

The new version of Condition E takes the following form. In fact, it will not be exactly equivalent to it since its formulation allows for $q = 1$ (with the convention that for $q = 1$, $[N]_{S,\alpha,\beta} = N^r$). One might say that it combines Conditions P and E.

CONDITION PE. Under the assumptions of Condition E the limits

$$\lim_{N\to\infty} [N]_{S,\alpha,\beta} \widehat{\phi}(j_{i_1^*,N}(v_1^N) \dots j_{i_p^*,N}(v_p^N)) = M_S^{q*}(v_1, \dots, v_p)$$

exist and are finite for the minimal tuple $(i_1^*, \dots, i_p^*)$ associated with a partition S, i.e. the correlations associated with (S, α, β) decay as $[N]_{S,\alpha,\beta}^{-1}$ or faster.

We give a version of Theorem 4.2 which lends itself easily to the limit procedure $q \to 1$. As we mentioned before, it will combine Theorems 3.1 and 4.2. Due to different combinatorics, the proofs of both theorems have to be put together to give Theorem 5.1. Its form is similar to the limit theorem presented in [S], except that we can apply it not only to free probability, but also to other kinds of quantum independence. In particular, it includes q-deformed limit theorems, such as the q-analog of the qclt [L1, L2].

THEOREM 5.1. *Assume that the correlations are (α,β)-homogeneous with respect to order-preserving injections and satisfy Condition* PE. *Then*

$$\lim_{N\to\infty} \widehat{\phi}(S_N(v_1^N) \dots S_N(v_p^N)) = \sum_{S\in\mathcal{P}^{\mathrm{ord}}\{1,\dots,p\}} D_S^{q*}(\alpha,\beta) M_S^{q*}(v_1, \dots, v_p)$$

where

$$D_S^{q*}(\alpha,\beta) = \begin{cases} q^{Y^*(S,\alpha,\beta)}[[\alpha_{S_1}, \dots, \alpha_{S_r}]]_q & \text{if } q > 1, \\ 1/r! & \text{if } q = 1, \\ q^{Y^*(S,\alpha,\beta)}[[\alpha_{S_r}, \dots, \alpha_{S_1}]]_{q^{-1}} & \text{if } q < 1, \end{cases}$$

and $Y^*(S,\alpha,\beta) = \sum_{k=1}^r ((1-k)\alpha_{S_k} + (1-r)\beta_{S_k})$.

Proof. For $q > 1$ we have

$$D_S^{q*}(\alpha,\beta) = q^{Y(S,\alpha,\beta)+\sum_{m=1}^r \beta_{S_m}} g_0(\alpha_{S_1}, \dots, \alpha_{S_r} \mid q) \prod_{i=1}^r (q^{\alpha_{S_i}} - 1)$$
$$= q^{Y^*(S,\alpha,\beta)}[[\alpha_{S_1}, \dots, \alpha_{S_r}]]_q.$$

The proof for $q < 1$ is almost identical. If $q = 1$, then we clearly have $D_S^{1*}(\alpha,\beta) = D_S = 1/r!$ by Theorem 3.1. ∎

COROLLARY 5.2. *Suppose that* $\lim_{q\to 1} M_S^q(v_1, \dots, v_p) \equiv M_S(v_1, \dots, v_p)$ *exists for all* $v_1, \dots, v_p$ *and that the assumptions of Theorem* 5.1 *are satisfied. Then*

$$\lim_{q\to 1^+} \lim_{N\to\infty} \widehat{\phi}(S_N(v_1^N) \dots S_N(v_p^N)) = \sum_{S\in\mathcal{P}^{\mathrm{ord}}\{1,\dots,p\}} [[\alpha_{S_1}, \dots, \alpha_{S_r}]] M_S(v_1, \dots, v_p),$$

$$\lim_{q\to 1^-} \lim_{N\to\infty} \widehat{\phi}(S_N(v_1^N) \dots S_N(v_p^N)) = \sum_{S\in\mathcal{P}^{\mathrm{ord}}\{1,\dots,p\}} [[\alpha_{S_r}, \dots, \alpha_{S_1}]] M_S(v_1, \dots, v_p).$$

Proof. Follows directly from Theorem 5.1. ∎

REMARK 1. It is interesting that the form of the limit state is, in general, different from that obtained in Theorem 3.1. However, if we assume invariance under permutations, then they coincide, thus giving the second part of Theorem 3.1. This means that the limits as $N \to \infty$ and $q \to 1$ are, in general, not interchangeable. However, note that $[[\alpha_{S_1}, \dots, \alpha_{S_r}]] = 1/r!$ if $(\gamma_1, \dots, \gamma_r) = (k, \dots, k)$. In particular, this is the case if S is a pair partition. Thus, if we have invariance of correlations with respect to permutations, we obtain the usual result for the central limit theorem (where only pair partitions survive).

REMARK 2. As far as Theorem 5.1 is concerned, the 'strange' q-factor involving $Y(S,\alpha,\beta)$ shows the deviation from commutativity as the argument below will demonstrate. Thus we will argue that it is very natural to redefine $M_S^*(v_1,\ldots,v_p)$.

If $\widehat{\phi}$ is a state, then Theorems 3.1, 4.2 and 5.1 give states on the free *-algebra generated by free generators $\overline{v}$ associated with $v\in\mathcal{G}_+\cup\mathcal{G}_-$. Below, we will refer to the limit state Ψ of Theorem 5.1 and denote its correlations by $\Psi(\overline{v}_1\ldots\overline{v}_p)$.

COROLLARY 5.3. *In the case of commutative independence with q-canonical injections the limit state of Theorem* 5.1 *takes the form*

$$\Psi(\overline{v}_1\ldots\overline{v}_p)=\sum_{S\in\mathcal{P}^{\mathrm{ord}}\{1,\ldots,p\}}[[\alpha_{S_1},\ldots,\alpha_{S_r}]]_q\phi(v_{S_1})\ldots\phi(v_{S_r}).$$

Proof. We take the infinite tensor product algebra with the q-canonical injections

$$j_{i_k,N}(v_k)=b_k^{\otimes(i_k-1)}\otimes v_k\otimes a_k^{\otimes(N-i_k)}\otimes 1^{\otimes\infty}$$

where $a_k=q^{\beta_k}$ and $b_k=q^{\alpha_k+\beta_k}$. For the minimal tuple $(i_1^*,\ldots,i_p^*)$ associated with a partition $S=(S_1,\ldots,S_r)$, we obtain

$$\begin{aligned}\widehat{\phi}(j_{i_1^*,r}(v_1)\ldots j_{i_p^*,r}(v_p))&=q^{\sum_{m=1}^r((\alpha_{S_m}+\beta_{S_m})(m-1)+\beta_{S_m}(r-m))}\phi(v_{S_1})\ldots\phi(v_{S_r})\\&=q^{-Y^*(S,\alpha,\beta)}\phi(v_{S_1})\ldots\phi(v_{S_r})\end{aligned}$$

where $\widehat{\phi}=\phi^{\otimes\infty}$ and $v_{S_i}=\prod_{k\in S_i}v_k$ (in the natural order). ∎

The above proof gives the factorization law for the injections considered. It is worthwhile to compare it with the factorization laws for other types of independent variables. Note that we can write them using unordered partitions and their refinements since there is no need to use the ordered ones.

EXAMPLE 1. The simplest is the *commutative factorization law*:

$$\widehat{\phi}(j_{i_1^*,r}(v_1)\ldots j_{i_p^*,r}(v_p))=\phi(v_{S_1})\ldots\phi(v_{S_r})$$

where S is the partition associated with the minimal tuple $(i_1^*,\ldots,i_p^*)$.

EXAMPLE 2. The case considered in Corollary 5.3 is a generalization of the above to q-canonical injections:

$$\widehat{\phi}(j_{i_1^*,r}(v_1)\ldots j_{i_p^*,r}(v_p))=q^{-Y^*(S,\alpha,\beta)}\phi(v_{S_1})\ldots\phi(v_{S_r}).$$

Let us call it the *commutative factorization law for q-identically distributed variables.*

EXAMPLE 3. The *free factorization law* takes the form

$$\widehat{\phi}(j_{i_1^*,r}(v_1)\ldots j_{i_p^*,r}(v_p))=\sum_{P\preceq S}Q_{SP}^f\phi(v_{P_1})\ldots\phi(v_{P_{r'}})$$

where $P=\{P_1,\ldots,P_{r'}\}\preceq S=\{S_1,\ldots,S_r\}$ means that P is a (not necessarily proper) *refinement* of S. When S is a noncrossing partition, then we have the same law as in the commutative case. Otherwise, only proper refinements appear in the summation.

EXAMPLE 4. If, in Example 3, we take $Q_{SP}^fq^{-Y^*(S,\alpha,\beta)}$ instead of Q_{SP}^f, then we obtain the *free factorization law for q-identically distributed variables.*

EXAMPLE 5. In the case of $U_q(su(2))$ we have the following factorization law [L1, L2]:

$$\widehat{\phi}(j_{i_1^*,r}(v_1)\ldots j_{i_p^*,r}(v_p))=q^{-Y^*(S,\alpha,\beta)+4\#W_{S+}-4\#W_{S-}}\phi(v_{S_1})\ldots\phi(v_{S_r}),$$

which could be called the *q-commutative factorization law for q-identically distributed variables.*

In order to include the known factorizations we could define the following *general factorization law*:

$$\widehat{\phi}(j_{i_1^*,r}(v_1)\ldots j_{i_p^*,r}(v_p))=\sum_{P\preceq S}Q_{SP}(\alpha,\beta\,|\,q)\phi(v_{P_1})\ldots\phi(v_{P_{r'}}).$$

This formula can be inserted in Conditions I–H and the results of all limit theorems in this paper can be rewritten in a new form, dependent only on $Q_{SP}(\alpha,\beta\,|\,q)$. However, we will stick to our notation which is more compact.

One can say that all coefficients except $Q_{SS}(\alpha,\beta\,|\,q)=q^{-Y^*(S,\alpha,\beta)}$ show a deviation from the commutative factorization law. Thus, defining

$$M_S^{q*}(v_1,\ldots,v_p)=q^{-Y^*(S,\alpha,\beta)}\widetilde{M}_S^q(v_1,\ldots,v_p)$$

we can write the limit state obtained in Theorem 5.1 in the form

$$\Psi(\overline{v}_1\ldots\overline{v}_p)=\sum_{S\in\mathcal{P}^{\mathrm{ord}}\{1,\ldots,p\}}[[\alpha_{S_1},\ldots,\alpha_{S_r}]]_q\widetilde{M}_S^q(v_1,\ldots,v_p)$$

for $q>1$ and analogously for $q<1$ (cf. Theorem 3.1).

Let us briefly discuss here certain special cases of Theorem 5.1, including the central limit theorem. For simplicity, we assume that $\beta=0$ and $\alpha=1$. Thus $\alpha_{S_i}=\gamma_{S_i}$ and $\beta_{S_i}=0$, where $(\gamma_{S_1},\ldots,\gamma_{S_r})$ is the signature of the partition S.

COROLLARY 5.4. *Let* $v_i^N=v_i/[N]_s^{1/s}$ *where* $s\in\mathbb{N}$, *and assume that* $\alpha_i=1$, $\beta_i=0$, $i=1,\ldots,p$. *Then*

$$D_S^{q*}(\alpha,\beta)=\begin{cases}(q^s-1)^{p/s}\langle\gamma_{S_1},\ldots,\gamma_{S_r}\rangle_q & \text{if } q>1,\\(q^{-s}-1)^{p/s}\langle\gamma_{S_r},\ldots,\gamma_{S_1}\rangle_{q^{-1}} & \text{if } q<1.\end{cases}$$

Proof. Straightforward calculation gives, for $q > 1$,

$$\frac{[N]_{S,\alpha,\beta}}{([N]_{q^s})^{p/s}} = \left(\frac{q^s - 1}{q^{sN} - 1}\right)^{p/s} \prod_{i=1}^{r} \frac{q^{\gamma_{S_i} N} - 1}{q^{\gamma_{S_i}} - 1} \to \frac{(q^s - 1)^{p/s}}{\prod_{i=1}^{r}(q^{\gamma_{S_i}} - 1)}$$

as $N \to \infty$, which, when multiplied by $[[\gamma_{S_1}, \ldots, \gamma_{S_r}]]_q$, gives the result. The proof for $q < 1$ is similar. ■

COROLLARY 5.5. *Let $s = 2$ and assume that the limit $\lim_{q\to 1} M_S^q(v_1, \ldots, v_p) = M_S(v_1, \ldots, v_p)$ exists. In addition, assume that the correlations which have singletons vanish. Then, for $p = 2k$, we obtain*

$$\lim_{q\to 1} \lim_{N\to\infty} \widehat{\phi}(S_N(v_1^N) \ldots S_N(v_p^N)) = \sum_{S \in \mathcal{P}_{\text{pair}}^{\text{ord}}\{1,\ldots,p\}} D_S M_S(v_1, \ldots, v_p)$$

where $D(S) = 1/r!$. The limit odd correlations vanish.

Proof. Straightforward, in view of Corollary 5.4 and the following simple calculation:

$$\lim_{q\to 1+} \frac{(q^2 - 1)^{j/2}}{q^j - 1} = \begin{cases} 0 & \text{if } j > 2, \\ 1 & \text{if } j = 2, \\ \infty & \text{if } j < 2, \end{cases}$$

and a similar one for the left limit. ■

Thus, we obtain a central limit theorem which is also a special case of Theorem 3.1. Therefore, in this case, the limits as $q \to 1$ and $N \to \infty$ can be interchanged. Note also that the above central limit theorem covers also the free case. Then simply $M_S(v_1, \ldots, v_p) = 0$ for crossing pair partitions since they have a singleton.

6. Limit theorems for convergence of q-Poisson type. The framework of Theorem 4.2 lends itself to limit theorems other than the q-analog of the central limit theorem. Therefore, we present here q-analogs of Poisson convergence leading to certain q-laws. The cases we study are 'one-dimensional' and thus very simple (in the tensor product case—classical), similarly to the free Poisson convergence studied in [S].

In both the classical and free Poisson convergences it is assumed that

$$\lim_{N\to\infty} N\phi((v^N)^k) = \lambda$$

for each natural k. The only difference in the final result (see [S]) comes from different factorization laws as we mentioned in Section 3. Thus, the only question we have to ask ourselves is that concerning the rates of decay of the correlations. That may depend on the factorization law. In fact, we shall discuss two different limit theorems of q-Poisson type, one for the tensor product, and the other for the reduced free product. Thus, we obtain q-analogs of Poisson convergence. However, the limit laws will not be continuous q-deformations of the Poisson law, or free Poisson law, respectively.

We choose a natural way to formulate a special case of Condition E.

CONDITION E-λ. Let $0 < \lambda \le 1$, $\alpha, \beta \in \mathbb{R}$ and let v^N be hermitian elements in a $*$-algebra $\mathcal{C}$. We say that the moments in the state $\phi \in \mathcal{C}^*$ satisfy *Condition* E-λ iff

$$\lim_{N\to\infty} q^{kN(\alpha+\beta)}\phi((v^N)^k) = \lambda$$

for all natural k and $q > 1$, and

$$\lim_{N\to\infty} q^{kN\beta}\phi((v^N)^k) = \lambda$$

for $q < 1$, where $k = 1, 2, \ldots$ In particular, when $\alpha + \beta > 0$ for $q > 1$, or $\beta < 0$ for $q < 1$, we say that the moments *decay exponentially* to λ.

REMARK. It is obvious that for $0 < \lambda \le 1$ such states exist. However, it is not the case for $\lambda > 1$.

In the theorem below, we consider the infinite tensor product state with the q-canonical injections considered in Example 3 of Section 4. We use the notation $S_N = S_N(v^N)$. Also note that due to invariance of λ^r with respect to the symmetrizer $\widehat{S}$ we can use unordered partitions in the expressions for the limit moments.

THEOREM 6.1. *Assume that the moments in the state ϕ satisfy Condition* E-λ. *Then*

$$\lim_{N\to\infty} \widehat{\phi}(S_N^p) = \sum_{S=\{S_1,\ldots,S_r\}\in\mathcal{P}\{1,\ldots,p\}} B_S \lambda^r$$

where

$$B_S = \begin{cases} q^{-\beta p}\widehat{S}\langle \alpha_{S_1}, \ldots, \alpha_{S_r}\rangle_q & \text{if } q > 1, \\ q^{-(\alpha+\beta)p}\widehat{S}\langle \alpha_{S_r}, \ldots, \alpha_{S_1}\rangle_{q^{-1}} & \text{if } q < 1. \end{cases}$$

Proof. It is clear that Condition E-λ for the moments implies Condition E for the correlations. Besides, Condition H is satisfied by the definition of the embeddings and the tensor product state. Therefore, for $q > 1$ we get from Theorem 5.1 the following:

$$\begin{aligned} q^{Y(S,\alpha,\beta)} & g_0(\alpha_{S_1}, \ldots, \alpha_{S_r} \mid q) M_S^q(v^p) \\ &= q^{Y(S,\alpha,\beta) - Y^*(S,\alpha,\beta)} g_0(\alpha_{S_1}, \ldots, \alpha_{S_r} \mid q)\lambda^r \\ &= q^{-\sum_{m=1}^{r}(\alpha s_m + \beta s_m)} g_0(\alpha_{S_1}, \ldots, \alpha_{S_r} \mid q)\lambda^r \\ &= q^{-\beta p}\langle \alpha_{S_1}, \ldots, \alpha_{S_r}\rangle_q \lambda^r. \end{aligned}$$

This finishes the proof for $q > 1$ since the symmetrizer does not affect λ^r. The proof for $q < 1$ is similar. ■

Note that in the limit as $q \to 1$ the correlations become infinite since

$$\lim_{q\to 1}\langle \alpha_{S_1},\ldots,\alpha_{S_r}\rangle_q = \infty.$$

One would like to obtain a law with finite limit for $q \to 1$. An easy modification of Theorem 6.1 is close at hand. Namely, we can replace λ by $(q-1)\lambda$ for $q > 1$ and by $(1-q)\lambda$ for $q < 1$. Both cases are analogous, therefore we restrict our attention to $q > 1$ and assume for simplicity $\alpha = 1$, $\beta = 0$.

COROLLARY 6.2. *Let $q > 1$ and suppose that $0 < \lambda \le 1/(q-1)$. Then, if the moments decay exponentially to $(q-1)\lambda$ with $\alpha = 1, \beta = 0$, we obtain for $\widehat{\phi} = \phi^{\otimes\infty}$,*

$$\lim_{N\to\infty}\widehat{\phi}(S_N^p) = \sum_{S=\{S_1,\ldots,S_r\}\in\mathcal{P}\{1,\ldots,p\}} \widehat{S}[\gamma_{S_1},\ldots,\gamma_{S_r}]_q\lambda^r$$

where $(\gamma_{S_1},\ldots,\gamma_{S_r})$ is the signature of the partition S, which, in turn, gives

$$\lim_{q\to 1^+}\lim_{N\to\infty}\widehat{\phi}(S_N^p) = \sum_{S=\{S_1,\ldots,S_r\}\in\mathcal{P}\{1,\ldots,p\}} \frac{\lambda^r}{\gamma_{S_1}\cdots\gamma_{S_r}}.$$

Proof. The first part follows directly from Theorem 6.1 and the second part is a consequence of Proposition 2.1. ∎

REMARK 1. Note that for any $\lambda > 0$ there exists q close to 1 for which the assumptions of Corollary 6.2 hold, and therefore the second part of the corollary gives a state for arbitrary $\lambda > 0$. Moreover, the moment problem is well-posed since Carleman's condition holds by comparison with the series for the Poisson law.

REMARK 2. If $q < 1$, then an analogous result holds which gives for $q \to 1^-$ the same limit law.

One would like to obtain a q-analog of the Poisson limit theorem that would give a continuous q-deformation of the Poisson law. Another modification of Condition E-λ seems obvious. If we let $q > 1$, $\alpha = 1$, $\beta = 0$ and replace λ by $\lambda(q^k - 1)$, then we would obtain

$$\lim_{N\to\infty}\widehat{\phi}(S_N^p) = \sum_{S=\{S_1,\ldots,S_r\}\in\mathcal{P}\{1,\ldots,p\}} \widehat{S}[[\gamma_{S_1},\ldots,\gamma_{S_r}]]_q\lambda^r,$$

which in the limit $q \to 1^+$ gives the Poisson law by Proposition 2.1. However, one can show that there is no *state* ϕ that would satisfy such a version of Condition E-λ.

PROPOSITION 6.3. *Let $q > 1$ and let v^N be hermitian elements in $\mathcal{C}$. Then for no $\lambda > 0$ does there exist a state $\phi \in \mathcal{C}^*$ satisfying the exponential decay condition*

$$\lim_{N\to\infty} q^{kN}\phi((v^N)^k) = \lambda(q^k - 1).$$

Proof. Assume that the contrary is true. Then, by specific scaling (one can take $w^N = q^N v^N$ as a rescaled hermitian generator), also the limits $m_k = (q^k - 1)\lambda$ must be the moments of a random variable w for some positive normalized functional ψ. This will contradict positivity as the calculation below demonstrates. We have

$$\begin{aligned}\psi\Big(\sum_{j=0}^n \overline{c}_j w^j \sum_{k=0}^n c_k w^k\Big) &= |c_0|^2 + \lambda\sum_{l=1}^{2n}(q^l - 1)\sum_{j+k=l}\overline{c}_j c_k\\ &= |c_0|^2 + \lambda\sum_{l=0}^{2n}\sum_{j+k=l}(q^j\overline{c}_j q^k c_k - \overline{c}_j c_k)\\ &= |c_0|^2 + \lambda(|A|^2 - |a|^2)\end{aligned}$$

where

$$A = \sum_{j=0}^n q^j c_j, \quad a = \sum_{j=0}^n c_j.$$

It is enough to take $c_1 = 1$, $c_2 = x \in \mathbb{R}$ and $c_j = 0$ for $j \notin \{1,2\}$. Then, for $q > 1$, we have $|A| = |q + xq^2|$ and $|a| = |1 + x|$. We must have $|A| \ge |a|$, which gives

$$|q + xq^2| \ge |1 + x|$$

for all x. Squaring both sides of the inequality we obtain

$$x^2(q^4 - 1) + x(2q^3 - 2) + q^2 - 1 \ge 0.$$

The discriminant is $4q^2(q-1)^2 > 0$ for $q > 1$. Hence, there exists x for which the above inequality does not hold, which is a contradiction. ∎

We would like to see what happens when we replace $(\mathcal{C}^{\otimes\infty}, \phi^{\otimes\infty})$ by $(\mathcal{C}^{*\infty}, \phi^{*\infty})$, the reduced free product of $(\mathcal{C}, \phi)$, with the q-canonical *-injections given by

$$j_{i,N}(v^N) = q^{(\alpha+\beta)(i-1)} j_i(v^N) q^{\beta(N-i)}$$

where j_i are the canonical embeddings. We choose the simplest case satisfying Conditions H and E-λ, i.e. $\alpha = 1$, $\beta = 0$. We obtain a free analog of Theorem 6.1.

THEOREM 6.4. *Let $\widehat{\phi} = \phi^{*\infty}$ be the reduced free product state on $\mathcal{C}^{*\infty}$ and let v^N be hermitian elements of $\mathcal{C}$ for which Condition E-λ holds. Then*

$$\lim_{N\to\infty}\widehat{\phi}(S_N^p) = \sum_{P=\{P_1,\ldots,P_m\}\in\mathcal{P}\{1,\ldots,p\}} B_P^f\lambda^m$$

where

$$B_P^f = \sum_{S=\{S_1,\ldots,S_r\}\succeq P} Q_{SP}^f\widehat{S}\langle\gamma_{S_1},\ldots,\gamma_{S_r}\rangle_q$$

and the coefficients Q^f_{SP} come from the factorization law for free independence.

Proof. Essentially, the proof goes along the same lines as in the tensor case. One only has to take into account the free factorization law instead of the commutative one. If S is a noncrossing partition, then the correlation associated with a partition into r blocks factorizes into r moments and gives λ^r. However, if S is a crossing partition, then it can be expressed as a sum of products of at least $r+1$ factors. With each of the summands one can associate a refinement P of a given partition with m blocks and the coefficient Q_{SP}, therefore the contribution to the limit is λ^m. This finishes the proof. ∎

COROLLARY 6.5. *Replacing λ by $(q-1)\lambda$ in Theorem 6.4 and assuming that $0<\lambda\le 1/(q-1)$, we obtain an analogous result, with $[\gamma_{S_1},\ldots,\gamma_{S_r}]_q$ replacing $\langle\gamma_{S_1},\ldots,\gamma_{S_r}\rangle_q$. Moreover, for $\lambda>0$ we obtain*

$$\lim_{q\to 1^+}\lim_{N\to\infty}\widehat{\phi}(S_N^p)=\sum_{S=\{S_1,\ldots,S_r\}\in\mathcal{P}^{\mathrm{nc}}\{1,\ldots,p\}}\frac{\lambda^r}{\gamma_{S_1}\cdots\gamma_{S_r}}$$

where the summation extends over noncrossing partitions only.

Proof. The first part of the theorem follows immediately from Theorem 6.4. In the second part we only need to justify that the contribution from the crossing partitions vanishes as $q\to 1^+$. This follows again from the fact that the factorization law for crossing partitions S with r blocks produces products of at least $r+1$ moments, hence at least $(q-1)^{r+1}\lambda^{r+1}$ as $N\to\infty$. Therefore, there is at least one extra power of $q-1$ left which is not included in $[\gamma_{S_1},\ldots,\gamma_{S_r}]_q$. The latter tends to $1/(\gamma_{S_1}\ldots\gamma_{S_r})$ as $q\to 1^+$, hence this extra power of $q-1$ makes the term vanish in the limit. ∎

EXAMPLE. To illustrate this fact, take the reduced free product with q-canonical injections for the crossing partition associated with the minimal tuple $i_1^*=i_3^*=1$, $i_2^*=i_4^*=2$. We have

$$\widehat{\phi}(j_{1,N}(v^N)j_{2,N}(v^N)j_{1,N}(v^N)j_{2,N}(v^N))=q^2[2\phi((v^N)^2)\phi(v^N)^2-\phi(v^N)^4].$$

Thus, if

$$\lim_{N\to\infty}q^{kN}\phi((v^N)^k)=(q-1)\lambda$$

then the contribution from this partition to the limit law is equal to

$$2[2,2]^q q^2(2(q-1)\lambda^3-(q-1)^2\lambda^4),$$

which tends to zero as $q\to 1$.

7. Connection with partial Bell polynomials. In this section we express the limit law obtained in Corollary 6.2,

$$\Psi(\overline{v}^p)=\sum_{S=\{S_1,\ldots,S_r\}\in\mathcal{P}\{1,\ldots,p\}}\frac{\lambda^r}{\gamma_{S_1}\cdots\gamma_{S_r}},$$

and its characteristic function in terms of *partial Bell polynomials* [T].

Some additional combinatorics is needed. Note that we deal here with unordered partitions. Let $\mathcal{N}=(n_1,n_2,\ldots)$ range over all sequences of nonnegative integers almost all of which are zero. Define its *length* and *weight* by

$$l(\mathcal{N})=\sum_{k\ge1}n_k,\qquad w(\mathcal{N})=\sum_{k\ge1}kn_k,$$

respectively. As noted in [T], such a sequence $\mathcal{N}$ can be regarded as a partition of the *number* $w(\mathcal{N})$ into $l(\mathcal{N})$ parts. A connection with (unordered) partitions of *sets* can be given as follows. We can interpret n_k as the number of blocks of k elements in a partition $S\in\mathcal{P}(I)$, $l(\mathcal{N})=r$ as the total number of blocks in S and $w(\mathcal{N})=p$ as the number of elements in I.

EXAMPLE. Let $S=\{S_1,S_2,S_3,S_4\}$ be a partition of $\{1,\ldots,8\}$ with $\gamma_{S_1}=\gamma_{S_3}=1$, $\gamma_{S_2}=2$ and $\gamma_{S_4}=4$. With this partition we can associate the sequence $\mathcal{N}=(2,1,0,1,0,\ldots)$ of length $l(\mathcal{N})=4$, meaning that we have two 1-element blocks, one 2-element block and one 4-element block. The weight $w(\mathcal{N})$ is 8.

The *partial Bell polynomials* are defined as

$$B_{p,r}(h_1,h_2,h_3,\ldots)=\sum_{w(\mathcal{N})=p,\,l(\mathcal{N})=r}b_{\mathcal{N}}h^{\mathcal{N}}$$

where

$$h^{\mathcal{N}}=h_1^{n_1}h_2^{n_2}\ldots,\qquad b_{\mathcal{N}}=\frac{p!}{\prod_{k\ge1}\{n_k!(k!)^{n_k}\}}.$$

and $h_1,h_2,\ldots$ are indeterminates. As proved in [T], $b_{\mathcal{N}}$ is equal to the number of ways to divide a set of p elements into r disjoint subsets, where the number of subsets containing k elements is n_k for $k\ge1$.

PROPOSITION 7.1. *The moments $\Psi(\overline{v}^p)$ can be written in terms of partial Bell polynomials as follows:*

$$\Psi(\overline{v}^p)=\sum_{r=1}^p\lambda^rB_{p,r}\Big(1,\frac12,\frac13,\ldots,\frac1p\Big).$$

Proof. We have

$$B_{p,r}\left(1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{p}\right)=\sum_{w(\mathcal{N})=p,\,l(\mathcal{N})=r} b_{\mathcal{N}}1^{n_1}\left(\frac{1}{2}\right)^{n_2}\left(\frac{1}{3}\right)^{n_3}\cdots\left(\frac{1}{p}\right)^{n_p}$$
$$=\sum_{S=\{S_1,\ldots,S_r\}\in\mathcal{P}\{1,\ldots,p\}} 1^{n_1}\left(\frac{1}{2}\right)^{n_2}\left(\frac{1}{3}\right)^{n_3}\cdots\left(\frac{1}{p}\right)^{n_p},$$

where $(n_1,n_2,n_3,\ldots)$ in the last formula is associated with S, i.e. it is the sequence of integers which shows the number of blocks consisting of $1,2,3,\ldots$ elements in S. Thus

$$1^{n_1}\left(\frac{1}{2}\right)^{n_2}\left(\frac{1}{3}\right)^{n_3}\cdots\left(\frac{1}{p}\right)^{n_p}=\frac{1}{\gamma_1\cdots\gamma_r},$$

which finishes the proof. ∎

Thus, we can see that the difference between this law and the Poisson law is that we need to evaluate the partial Bell polynomials at $1,1/2,1/3,\ldots$ instead of $1,1,1,\ldots$

We wish to compute the characteristic function of this law. For comparison, let us look at the Poisson law first. Its moments are given by

$$m_p=\sum_{S=\{S_1,\ldots,S_r\}\in\mathcal{P}\{1,\ldots,p\}}\lambda^r=\sum_{r=1}^{p}\lambda^r B_{p,r}(1,\ldots,1).$$

Note that $B_{p,r}(1,\ldots,1)=S_p^r$, where S_p^r are the Stirling numbers of the second kind. Using the composition formula from [T]:

$$f(h(z))=\sum_{p=0}^{\infty}\frac{z^p}{p!}\sum_{r=1}^{p}f_rB_{p,r}(h_1,\ldots,h_p)$$

for formal power series

$$f(z)=\sum_{n\ge 0}f_n\frac{z^n}{n!},\qquad h(z)=\sum_{n\ge1}h_n\frac{z^n}{n!},$$

we see that we can take

$$f(z)=\sum_{n\ge0}\lambda^n\frac{z^n}{n!}=e^{\lambda z},\qquad h(z)=\sum_{n\ge1}\frac{z^n}{n!},$$

to obtain the characteristic function of the Poisson law as

$$\psi_P(t)=f(h(it))=e^{\lambda(e^{it}-1)}.$$

We now write the characteristic function of our limit law in a similar fashion.

THEOREM 7.2. *The logarithm of the characteristic function ψ of the limit law of Proposition 7.1 is given by*

$$\log\psi(t)=\lambda\int_0^1(e^{itx}-1-itx)\frac{1}{x^2}\mu(dx)+\lambda it$$

where $\mu(dx)=\frac{1}{2}d(x^2)$ is the Lebesgue–Stieltjes measure, i.e. the limit process is an infinitely divisible process of mean λ and variance $\mu[0,1]=1/2$.

Proof. Take the same $f(z)$ as in the Poisson case and define

$$h(z)=\sum_{n\ge1}\frac{1}{n}\cdot\frac{z^n}{n!},$$

which gives

$$h(it)=\int_0^t\frac{e^{ix}-1}{x}\,dx.$$

By changing variables, $x\to tx$, we obtain

$$h(it)=\int_0^1\frac{e^{itx}-1}{x}\,dx=it+\int_0^1\frac{e^{itx}-1-itx}{x^2}\,\mu(dx)$$

where $\mu(dx)=\frac{1}{2}d(x^2)$. The formula for $\psi(t)$ is obtained from the composition formula. The second part is clear from the canonical representation for the characteristic functions of infinitely divisible processes. ∎

Acknowledgements. I would like to thank Professor S. L. Woronowicz for an inspiring remark concerning the normalization used in my work on convolution-type limit theorems.

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Hugo Steinhaus Center for Stochastic Methods
Institute of Mathematics
Technical University of Wrocław
50-370 Wrocław, Poland
E-mail: lenczew@im.pwr.wroc.pl

Received February 10, 1997
Revised version November 6, 1997 (3840)

STUDIA MATHEMATICA 129 (3) (1998)

Extremal perturbations of semi-Fredholm operators

by

THORSTEN KRÖNCKE (Berlin)

Abstract. Let T be a bounded operator on an infinite-dimensional Banach space X and Ω a compact subset of the semi-Fredholm domain of T. We construct a finite rank perturbation F such that $\min[\dim N(T+F-\lambda), \operatorname{codim} R(T+F-\lambda)] = 0$ for all $\lambda \in \Omega$, and which is extremal in the sense that $F^2 = 0$ and $\operatorname{rank} F = \max\{\min[\dim N(T-\lambda), \operatorname{codim} R(T-\lambda)] : \lambda \in \Omega\}$.

0. Introduction. Let X and Y be complex Banach spaces and $B(X,Y)$ the space of bounded operators from X to Y. An operator $T \in B(X,Y)$ is called *semi-Fredholm* if its range $R(T)$ is closed and its minimum index is finite. That is,

$$\text{min.ind}(T) = \min[\dim N(T), \operatorname{codim} R(T)] < \infty.$$

(Here $N(T)$ denotes the kernel of T.) In this case the index of T is well defined as

$$\text{ind}(T) = \dim N(T) - \operatorname{codim} R(T).$$

For two operators $S, T \in B(X,Y)$ the *semi-Fredholm domain* is the set

$$\varrho_{\text{S-F}}(T:S) = \{\lambda \in \mathbb{C} : T - \lambda S \text{ is semi-Fredholm}\}.$$

It is well known that $\varrho_{\text{S-F}}(T:S)$ is open and that on its connected components the mapping $\lambda \to \text{ind}(T-\lambda S)$ is constant. The mapping $\lambda \to \text{min.ind}(T-\lambda S)$, however, is constant on each connected component of $\varrho_{\text{S-F}}(T:S)$ except for a discrete subset where its value jumps up (see [Ka66], Chap. IV, §5). Those exceptional points are called *Kato's jumps* or *jumps of* $\lambda \to \text{min.ind}(T-\lambda S)$. They are precisely the points of discontinuity of the mapping $\lambda \to \gamma(T-\lambda S)$ in $\varrho_{\text{S-F}}(T:S)$; here $\gamma(T-\lambda S) = \inf\{\|(T-\lambda S)x\| : \operatorname{dist}[x, N(T-\lambda S)] = 1\}$ denotes the minimum modulus of $T-\lambda S$ (see [Ka58], Thm. 3 in §6 and Thm. 4 in §7).

An analytic, $B(X,Y)$-valued function $\lambda \to A(\lambda)$ is called *uniformly regular* on the open set $D \subseteq \mathbb{C}$ if $\lambda \to \gamma(A(\lambda))$ is strictly positive and continuous

1991 *Mathematics Subject Classification*: 47A53, 47A55.