An ideal characterization of when a subspace of certain Banach spaces has the metric compact approximation property

by

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Abstract. C.-M. Cho and W. B. Johnson showed that if a subspace $E$ of $\ell_p$, $1 < p < \infty$, has the compact approximation property, then $\mathcal{K}(E)$ is an $M$-ideal in $\mathcal{L}(E)$. We prove that for every $r, s \in [0, 1]$ with $r^s + s^r < 1$, the James space can be provided with an equivalent norm such that an arbitrary subspace $E$ has the metric compact approximation property if there is a norm one projection $P$ on $\mathcal{L}(E)^*$ with $\text{Ker } P = \mathcal{K}(E)^*$ satisfying

$$\|f\| \geq r\|Ff\| + s\|\varphi - Pf\| \quad \forall f \in \mathcal{L}(E)^*.$$ 

A similar result is proved for subspaces of upper $p$-spaces (e.g. Lorentz sequence spaces $d(u,p)$ and certain renormings of $L^p$).

1. Introduction. We follow [3] and [7] in assuming that a subspace $X$ of a Banach space $Y$ is said to be an ideal in $Y$ if there exists a norm one projection $P$ on $Y^*$ with $\text{Ker } P = X^\perp$. If, moreover,

$$\|y^*\| \geq r\|Py^*\| + s\|y^* - Py^*\| \quad \forall y^* \in Y^*$$

holds for given $r, s \in [0, 1]$, then we say that $X$ is an ideal satisfying the $M(r, s)$-inequality in $Y$ (for simplicity, we say that $X$ satisfies the $M(r, s)$-inequality if $Y$ is the bidual of $X$, and its associated projection is the canonical projection). If $r = s = 1$, we return to the classical concept of $M$-ideal introduced by Alfsen and Effros [1].

For any Banach spaces $X$ and $Y$, we denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from $X$ to $Y$ and by $\mathcal{K}(X,Y)$ its subspace of compact operators. If $X = Y$, then we simply write $\mathcal{L}(X)$ and $\mathcal{K}(X)$, respectively. Harmand and Lima [9] proved that $X$ with $\mathcal{K}(X)$ being an $M$-ideal in $\mathcal{L}(X)$ must necessarily have the metric compact approximation property (MCAP), and Cho and Johnson [4] showed that for subspaces $E$ of $\ell_p$ (in fact, this holds for subspaces $E$ of $X$ with $\mathcal{K}(X)$ being an $M$-ideal in $\mathcal{L}(X)$ [10, Theorem VI.4.19]) the MCAP already ensures that $\mathcal{K}(E)$ is an

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\( M \)-ideal in \( \mathcal{L}(E) \). It is also known [10, Section VI.5] that for subspaces \( E \) of \( c_0 \), the MCAP moreover entails that \( \mathcal{K}(W, E) \) is an \( M \)-ideal in \( \mathcal{L}(W, E) \) for all Banach spaces \( W \). For subspaces \( E \) of \( l_p \), the MCAP only implies that \( \mathcal{K}(W, E) \) is an \( l_p \)-subspace (a weakening of the notion of \( M \)-ideal) of \( \mathcal{L}(W, E) \) [17].

We write a family of variants of the MCAP that are satisfied by e.g., \( c_0 \), \( l_p \), the Lorentz sequence spaces \( d(w, p), 1 < p < \infty \), and certain renormings of the James space, which are inherited by subspaces having the MCAP.

A net \( (K_\alpha) \) of compact operators on a Banach space \( X \) will be called a compact approximation of the identity (c.a.i.) provided \( \lim_\alpha K_\alpha x = x \) for every \( x \in X \). If, moreover, \( \lim_\alpha K_\alpha^* x = x^* \) for every \( x \in X^* \), we will say that \( (K_\alpha) \) is a shrinking compact approximation of the identity (s.c.a.i.).

Given \( r, s \in [0, 1] \), we say that a Banach space \( X \) satisfies the compact uniform \( r \)-inequality (short, \( r \)-inequality) if \( X \) admits a c.a.i. \( (K_\alpha) \) in \( B_{\mathcal{K}(X)} \) satisfying

\[
\limsup_{\alpha} \sup_{\|x\| = 1} \|xK_\alpha x + s(y - K_\alpha y)\| \leq 1.
\]

Of course, the \( M_p \)-spaces, \( 1 < p \leq \infty \), defined in [10, Section VI.5], satisfy the condition \((*)\) for \( r = s = 1 \) if \( p = \infty \), and for \( r^p + s^p \leq 1 \) if \( 1 < p < \infty \). For more examples the reader can see [3, Sections 4 and 5 below.

For abbreviation, given two Banach spaces \( X \) and \( Y \), we will say that \( \mathcal{K}(X, Y) \) satisfies the \( (r, s) \)-inequality instead of \( \mathcal{K}(X, Y) \) is an ideal satisfying the \( M(r, s) \)-inequality in \( \mathcal{L}(X, Y) \).

We prove the following

**Theorem.** Let \( r, s \in [0, 1] \) be such that \( r + s > 1 \). Assume that \( X \) is a Banach space satisfying the \( M_{cu}(r, s) \)-inequality and \( E \) is a closed subspace of \( X \). Consider the following assertions:

(i) \( E \) has the MCAP.

(ii) \( \mathcal{K}(E) \) is an ideal in \( \mathcal{L}(E) \).

(iii) \( E \) satisfies the \( M_{cu}(r, s) \)-inequality.

(iv) For all Banach spaces \( W \), \( \mathcal{K}(W, E) \) satisfies the \( M(r, s) \)-inequality.

\( \mathcal{K}(E \otimes_{\mathfrak{A}} E) \) satisfies the \( M(r, s) \)-inequality.

Then (i) \( \iff \) (ii) \( \iff \) (iii) \( \iff \) (iv) \( \iff \) (v).

All the above assertions are equivalent if \( r + s/2 > 1 \).

2. Proof of the Theorem. We begin with an expected stability property (cf. [10, Proposition VI.4.2]), whose proof cannot use intersection properties of balls (cf. [3, Lemma 2.3]), but in the classical case \( r = s = 1 \).

**Lemma 2.1.** Let \( X \) and \( Y \) be two Banach spaces and let \( r, s \in [0, 1] \). If \( \mathcal{K}(X, Y) \) satisfies the \( M(r, s) \)-inequality and \( E \subseteq X \) and \( F \subseteq Y \) are 1-complemented subspaces, then \( \mathcal{K}(E, F) \) satisfies the \( M(r, s) \)-inequality.

**Proof.** By hypothesis, there exists a norm one projection \( P \) on \( \mathcal{L}(X, Y)^{\ast} \) with \( \mathrm{Ker} \ P = \mathcal{K}(X, Y)^{\ast} \) satisfying

\[
\|f\| \geq r \|Pf\| + s \|f - Pf\| \quad \forall f \in \mathcal{L}(X, Y)^{\ast}.
\]

Let \( P_1, P_2 \) be two norm one projections on \( X \) and \( Y \) respectively, with \( P_1(X) = E \) and \( P_2(Y) = F \), and denote by \( i_1, i_2 \) the inclusion operators from \( E \) into \( X \) and from \( F \) into \( Y \), respectively. Consider \( \varphi : \mathcal{L}(X, Y) \rightarrow \mathcal{L}(E, F) \) defined by

\[
\varphi(S) = P_2 i_1 \quad \forall S \in \mathcal{L}(X, Y),
\]

and \( \chi : \mathcal{L}(E, F) \rightarrow \mathcal{L}(X, Y) \) defined by

\[
\chi(T) = i_2 TP_1 \quad \forall T \in \mathcal{L}(E, F).
\]

Since \( \varphi \circ \chi = I \), it is straightforward to show that \( Q : \mathcal{L}(E, F)^{\ast} \rightarrow \mathcal{L}(E, F)^{\ast} \) defined by

\[
Q(f)(T) = P(f \circ \varphi)(\chi(T)) \quad \forall f \in \mathcal{L}(E, F)^{\ast}, \quad T \in \mathcal{L}(E, F),
\]

is a norm one projection with \( \mathrm{Ker} \ Q = \mathcal{K}(E, F)^{\ast} \) satisfying

\[
\|f\| \geq r \|Qf\| + s \|f - Qf\| \quad \forall f \in \mathcal{L}(E, F)^{\ast}.
\]

The following lemma, essentially proved in [15], is crucial.

**Lemma 2.2.** Let \( r, s \in [0, 1] \). If \( X \) satisfies the \( M_{cu}(r, s) \)-inequality, then \( \mathcal{K}(X) \) and \( X \) satisfy the \( M(r, s) \)-inequality.

The next result improves [18, Theorem 2], which was proved using intersection properties of balls and Banach algebra techniques. Our proof is based on J. Johnson’s procedure of making projections [12] (cf. [14, Theorem 3.1]), and the unicity of the associated projection [3, Proposition 3.2].

**Proposition 2.3.** Let \( X \) be a Banach space and let \( r, s \in [0, 1] \). Consider the following statements:

(i) \( X \) satisfies the \( M_{cu}(r, s) \)-inequality.

(ii) For all Banach spaces \( W \), \( \mathcal{K}(W, X) \) satisfies the \( M(r, s) \)-inequality.

(iii) \( \mathcal{K}(X \otimes_{\mathfrak{A}} X) \) satisfies the \( \mathcal{K}(r, s) \)-inequality.

Then (i) \( \iff \) (ii) \( \iff \) (iii). All the above statements are equivalent if \( r + s/2 > 1 \).

**Proof.** (i) \( \iff \) (ii). By definition, there is a c.a.i. \( (K_\alpha) \) in \( B_{\mathcal{K}(X)} \) satisfying \((*)\). Let \( W \) be a Banach space and \( T \in B_{\mathcal{L}(W, X)} \). Consider \( L_\alpha = K_\alpha T \). By Johnson’s procedure (see [14, Theorem 3.1]), \( \mathcal{K}(W, X) \) is an ideal in \( \mathcal{L}(W, X) \),
and we can assume that \((L_\alpha)\) converges to \(T\) in the \(\sigma(L(W, X), K(W, X)^*)\)-topology. Hence, by (*),
\[
\lim_\alpha \| rS + s(T) - L_\alpha \| \leq \lim_\alpha \| rK_\alpha S + s(T - K_\alpha T) \| + \lim_\alpha \| rK_\alpha S - S \| \leq 1
\]
holds for every \(S \in B(K(W, X))\). Therefore, by [2, Lemma 2.7], we conclude that (ii) is satisfied.

(ii)\(\Rightarrow\)(iii). This implication follows from the fact that \(Z \oplus_\infty Z\) is an ideal satisfying the \(M(r, s)\)-inequality in \(Y \oplus_\infty Y\) whenever \(Z\) is an ideal satisfying the \(M(r, s)\)-inequality in \(Y\). In fact, we take \(Z = K(X \oplus_\infty X, X)\) and \(Y = L(X \oplus_\infty X, X)\). 

(iii)\(\Rightarrow\)(i). By [3, Theorem 3.1], \(X \oplus_\infty X\) admits a s.c.a.i. \((S_\alpha)\) in \(B(K(X \oplus_\infty X, X))\) satisfying
\[
\lim_\alpha \| rAS_\alpha + sB(I - S_\alpha) \| \leq 1 \quad \forall A, B \in B(L(X \oplus_\infty X, X)).
\]

On the other hand, since \(X\) is a 1-complemented subspace of \(X \oplus_\infty X\), by Lemma 2.1 and [3, Theorem 3.1], \(X\) admits a s.c.a.i. \((L_\beta)\) with \(\| L_\beta \| \leq 1\) for all \(\beta\). It is clear that
\[
\tilde{L}_\beta = \begin{pmatrix} L_\beta & 0 \\ 0 & L_\beta \end{pmatrix}
\]
is another s.c.a.i. in \(B(K(X \oplus_\infty X, X))\).

By Johnson’s procedure, there are two norm one projections \(P_1, P_2\) on \(L(X \oplus_\infty X)^*\) with \(\text{Ker} P_1 = K(X \oplus_\infty X)^*\). Concretely,
\[
P_1(\phi)(T) = \lim_\alpha \phi(IS_\alpha), \quad P_2(\phi)(T) = \lim_\beta \phi(\tilde{L}_\beta T)
\]
for all \(\phi \in L(X \oplus_\infty X)^*\) and \(T \in L(X \oplus_\infty X)\). By [3, Theorem 2.5 and Propositions 2.1 and 3.2], we have \(P_1 = P_2\). We can suppose that both nets are indexed by the same set (after switching to the product index set with the product ordering). In particular, the net \((S_\alpha - \tilde{L}_\alpha)\) is weakly null, and so, by a convex combination argument, we may assume that \(\| S_\alpha - \tilde{L}_\alpha \|\) converges to zero. Then, checking (1) on the operators
\[
A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},
\]
we obtain the condition (*).

Remark. Actually, if \(r + s > 1\), then the c.a.i. \((K_\alpha)\) may be chosen shrinking. In fact, by Lemma 2.2, \(X\) satisfies the \(M(r, s)\)-inequality, so, by [2, Proposition 2.5], \(X^*\) contains no proper norming subspaces. Hence, by [3, Proposition 2.5 and Theorem 2.2], \((K_\alpha)\) is a s.c.a.i.

The next lemma is proved by a standard procedure (cf. [13, Theorem 2.5]). For completeness, we indicate a proof.

Lemma 2.4. Let \(X\) be an Asplund space and let \(E\) be a closed subspace of \(X\). If \(X^*\) and \(E^*\) have the MCAP with adjoint operators, then \(X\) and \(E\) each admit a s.c.a.i. \((K_\alpha)\) and \((H_\alpha)\), respectively, such that
\[
\lim_\alpha \| iK_\alpha - K_\alpha \| = 0,
\]
where \(i : E \to X\) is the inclusion.

Proof. Let \((K_\alpha)\) be a s.c.a.i. in \(B(K(X))\) and let \((H_\alpha)\) be a s.c.a.i. in \(B(K(E))\). We can suppose that both nets are indexed by the same set. It is clear that, for \(x^* \in X^*\) and \(e^{**} \in E^{**}\),
\[
\lim_\alpha e^{**}(i^*K_\alpha x^* - H_\alpha x^*) = 0.
\]
Therefore, it suffices to apply [6, Theorem 1] and a convex combination argument to finish.

Proof of the Theorem. In the first place, note that, by Lemma 2.2 and [2, Proposition 2.1], \(E\) satisfies the \(M(r, s)\)-inequality.

(i)\(\Rightarrow\)(ii). This implication follows from Johnson’s procedure.

(ii)\(\Rightarrow\)(i). This follows from [3, Propositions 2.1 and 3.2].

(iii)\(\Rightarrow\)(iv) and (iv)\(\Rightarrow\)(v) are proved in Proposition 2.3.

(v)\(\Rightarrow\)(ii) is obvious.

Before mentioning applications of our Theorem, we exhibit an interesting example (cf. [2, Example 4.6]).

Example 2.4. Let \(X\) and \(Y\) be two \(M_{\infty}\)-spaces. Given \(0 < \gamma \leq 1\), define
\[
\| (x, y) \| = \max \left\{ \| x \|, \| y \|, \frac{\| x \| + \| y \|}{1 + \gamma} \right\}, \quad x \in X, \ y \in Y.
\]
Then \(Z = (X \times Y, \| \cdot \|)\) satisfies, simultaneously, the \(M_{\infty}(1, \gamma)\)-inequality and the \(M_{\infty}(\gamma, 1)\)-inequality. Moreover, if \(\gamma \neq 1\), then \(Z\) is not an \(M_{\infty}\)-space.

3. The James space. In this section we show a method to provide the James space with a norm which satisfies the \(M_{\infty}(r, s)\)-inequality, and we obtain a Cho–Johnson theorem for the James space.
Proposition 3.1. For $\delta > 0$, let $J_\delta$ be the space of all null sequences $(x_n)$ in $\mathbb{R}$ satisfying
\[
\sup \left\{ \left( \delta x_{k_1} - x_{k_0} \right)^2 + \sum_{i=2}^{n} \left( x_{k_i} - x_{k_{i-1}} \right)^2 + \left( x_{k_n} - \delta x_{k_0} \right)^2 \right\}^{1/2} < \infty,
\]
where the supremum is taken over all $n \in \mathbb{N}$ and all finite increasing sequences $k_1 < \ldots < k_{n+1}$ in $\mathbb{N}$, with norm $\| \cdot \|_\delta$ defined by this supremum. Let $r, s \in [0, 1]$ be such that $r^2 + s^2 < 1$. Then $J_\delta$ satisfies the $M_{en}(r, n)$-inequality for all $\delta > 1$ such that
\[
r^2 + s^2 + \frac{s^2}{2\delta^2} + \frac{2rs}{\delta} \leq 1.
\]

Proof. It follows from [5, Properties I and II, pp. 81–82] that the sequence $(e_n)$, where $e_n = (0, \ldots, 0, 1, 0, \ldots)$, is a monotone shrinking basis. For all $n \in \mathbb{N}$, we define
\[
P_n x = \sum_{i=1}^{n} e_i x_i \quad \forall x = (x_n) \in J_\delta.
\]
It is enough to prove that for every $n \in \mathbb{N}$ and $x, y \in B_{J_\delta}$,
\[
\| \| P_n x + s(y - P_n y) \| \|_{\delta} \leq 1.
\]
Since $\| P_n x \|_{\delta} \leq \| x \|_{\delta}$ for all $n \in \mathbb{N}$, we have
\[
(\delta x_{k_1} - x_{k_0})^2 + \sum_{i=2}^{q} (x_{k_i} - x_{k_{i-1}})^2 + (\delta x_{k_q})^2 \leq \| x \|_{\delta}^2
\]
for every $q \in \mathbb{N}$ and for every finite increasing sequence $k_1 < \ldots < k_{q+1}$ in $\mathbb{N}$. In particular, for every $x = (x_n) \in J_\delta$,
\[
2(\delta x_{k_1})^2 \leq \| x \|_{\delta}^2 \quad \forall n \in \mathbb{N}.
\]
Let $x = (x_n), y = (y_n) \in B_{J_\delta}, p \in \mathbb{N}$, and let $k_1 < \ldots < k_{p+1}$ be a finite sequence in $\mathbb{N}$. Fix $n \in \mathbb{N}$, denote by $\gamma = (\gamma_n)$ the sequence $(\gamma_1, \ldots, \gamma_n, \gamma_{n+1}, \gamma_{n+2}, \ldots)$, and set
\[
S := (\delta \gamma_{k_1} - \gamma_{k_0})^2 + \sum_{i=2}^{p} (\gamma_{k_i} - \gamma_{k_{i-1}})^2 + (\gamma_{k_{p+1}} - \delta \gamma_{k_0})^2.
\]
If $k_1 \geq n+1$, then
\[
S = (\delta y_{k_1} - y_{k_0})^2 + \sum_{i=2}^{p} (y_{k_i} - y_{k_{i-1}})^2 + (y_{k_{p+1}} - \delta y_{k_0})^2
\]
\[
\leq s^2 \| y \|_{\delta}^2 \leq s^2 < 1.
\]
If $k_1 \leq n+1$, then
\[
S = (\delta x_{k_1} - x_{k_0})^2 + \sum_{i=2}^{p} (x_{k_i} - x_{k_{i-1}})^2 + (x_{k_{p+1}} - \delta x_{k_0})^2
\]
\[
\leq r^2 \| x \|_{\delta}^2 \leq r^2 < 1.
\]
Assume that $k_1 \leq n$ and $k_{p+1} \geq n+1$. Set $q = \max\{i \in \{1, \ldots, p\} : k_i \leq n\}$. If $q = 1$, then by (3) and (4),
\[
S = (\delta x_{k_1} - x_{k_0})^2 + \sum_{i=2}^{p} (y_{k_i} - y_{k_{i-1}})^2 + (\gamma_{k_{p+1}} - \delta x_{k_0})^2
\]
\[
\leq 2(\delta x_{k_1})^2 + 2rs|\alpha_{k_1}| |y_{k_1} - y_{k_{p+1}}|
\]
\[
+ \sum_{i=2}^{p} (y_{k_i} - y_{k_{i-1}})^2 + (\gamma_{k_{p+1}})^2
\]
\[
\leq r^2 + \frac{2rs}{\delta} + s^2 + \frac{s^2}{2\delta^2} \leq 1.
\]
If $q > 1$, then again by (3) and (4),
\[
S = (\delta x_{k_1} - x_{k_0})^2 + \sum_{i=2}^{q-1} (x_{k_i} - x_{k_{i-1}})^2 + (x_{k_{p+1}} - y_{k_{q+1}})^2
\]
\[
+ \sum_{i=2}^{p} (y_{k_i} - y_{k_{i-1}})^2 + (\gamma_{k_{p+1}} - \delta x_{k_0})^2
\]
\[
\leq (\delta x_{k_1} - x_{k_0})^2 + \sum_{i=2}^{q-1} (x_{k_i} - x_{k_{i-1}})^2 + (x_{k_{p+1}})^2
\]
\[
+ 2rs|x_{k_1}| |y_{k_{q+1}} - y_{k_{p+1}}|
\]
\[
+ \sum_{i=2}^{q} (y_{k_i} - y_{k_{i-1}})^2 + (\gamma_{k_{p+1}})^2
\]
\[
\leq r^2 + \frac{rs}{\delta} + \frac{r}{\delta} + s^2 + \frac{s^2}{2\delta^2} \leq 1.
\]
Therefore,
\[
\| \| P_n x + s(y - P_n y) \| \|_{\delta} \leq 1,
\]
as required. \(\blacksquare\)

Corollary 3.2. Let $\delta > 1$, and let $E$ be a closed subspace of $J_\delta$. Consider the following statements:

(i) $E$ has the MCAP.
(ii) $K(E)$ is an ideal in $\mathcal{L}(E)$.  

(iii) \( E \) satisfies the \( M_{c_0}(r,s) \)-inequality whenever \( r \) and \( s \) satisfy (2).
(iv) For all Banach spaces \( W, K(W,E) \) satisfies the \( M(r,s) \)-inequality whenever \( r \) and \( s \) satisfy (2).
(v) \( K(E \oplus \infty E) \) satisfies the \( M(r,s) \)-inequality whenever \( r \) and \( s \) satisfy (2).

Then (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii) \( \Leftrightarrow \) (iv) \( \Leftrightarrow \) (v). All the above statements are equivalent if \( \delta > 2 \).

Proof. By Lemma 2.4 and the Theorem, it is enough to observe that if \( \delta > \mu > 0 \), then
\[
\{ (r,s) \text{ satisfying (2)} \} \cap \{ (r,s) : r > s/\mu > 1 \} \neq \emptyset.
\]

In particular, we have the Cho-Johnson theorem:

**Theorem 3.3.** Let \( r, s \in [0,1] \) be such that \( r^2 + s^2 < 1 \). Then there is \( \delta_0 > 1 \) such that for every \( \delta > \delta_0 \), and for every subspace \( E \) of \( J_\delta \), \( E \) has the MCAP iff \( K(E) \) satisfies the \( M(r,s) \)-inequality.

**Remarks.** (i) Observe that, according to Lemma 2.2, \( J_\delta \) satisfies the \( M(r,s) \)-inequality whenever the pair \((r,s)\) satisfies (2) [cf. [2, Example 3.6]].

(ii) As far as we know, it is not clear whether the CAP implies the MCAP for quasi-reflexive Banach spaces (even for subspaces of \( J_\delta \)). Note that if \( Y \) is a quasi-reflexive Banach space having the CAP, then \( Y^* \) has the CAP, so, according to [8, Corollary 1.6], the question could be whether \( Y^* \) has the CAP with adjoint operators.

4. The upper \( p \)-property. We recall the following notion introduced in [10, p. 327] (cf. [17]). We say that a Banach space \( X \) has the upper \( p \)-property (\( 1 < p \leq \infty \)) if \( X \) admits a s.c.a.i. \( (K_n) \) such that
\[
\lim_{n \to \infty} \sup_{\|z\| \leq 1} \|K_nz + (y - K_ny)\| \leq (\|z\|^p + \|y\|^p)^{1/p}.
\]

In fact, they comment [10, p. 327] that an effective way to produce Banach spaces with the upper \( p \)-property (upper \( p \)-spaces) is to look for reflexive sequence spaces whose unit vectors form a Schauder basis and the inequality
\[
\|z + y\| \leq (\|z\|^p + \|y\|^p)^{1/p}
\]
holds for disjointly supported sequences. It is clear that, under this hypothesis, the sequence of coordinate projections is a s.c.a.i. satisfying (5) (and, of course, the inequality \((*)\) for every \((r,s) \in B_{12}\)). Besides the

\( M_p \)-spaces, examples include the Lorentz sequence spaces \( d(u,p) \), and more generally, the \( p \)-convexification of a sequence space whose unit vector basis is \( 1 \)-unconditional. In [10, Proposition 11.6.8] one can see a renorming of \( L^p \) with the upper \( 2 \)-property. On the other hand, if \( p = \infty \), we return to the \( M_{c_0} \)-spaces [10, p. 306].

Given a closed subspace \( X \) of a Banach space \( Y \), according to the Hahn-Banach theorem, each functional on \( X \) admits a norm preserving extension to a functional on \( Y \). Following R. Phelps [19], we shall say that \( X \) has property \( U \) in \( Y \) if for every \( x^* \in X^* \), the norm preserving extension is unique. If, moreover, \( X \) is an \( H \)-space in \( Y \) with associated projection \( P \) such that \( \|I - P\| \leq 1 \), then \( X \) is said to be an \( HB \)-space of \( Y \).

We will say that \( X \) has property \( U^* \) in \( Y \) if there is a norm one projection \( P \) on \( Y^* \) with \( Ker P = X^\perp \) such that for all \( y^* \in Y^* \) with \( Py^* \neq 0 \),
\[
\|y^* - Py^*\| < \|y^*\|.
\]

It is clear that if \( X \) is an \( M \)-ideal in \( Y \), then \( X \) has properties \( U \) and \( U^* \) (in fact, \( X \) is an \( HB \)-subspace) in \( Y \).

In the next lemma, we show that it is not necessary to suppose \( s = 1 \) to have property \( U \), and \( r = 1 \) for property \( U^* \).

**Lemma 4.1.** If \( X \) is an ideal satisfying the \( M(r,s) \)-inequality in \( Y \) with associated projection \( P \), then:

(i) For every \( y^* \in Y^* \), \( Py^* \) is a norm preserving extension of \( y^*|_X \). In particular, \( X^* \) is isometric to \( P(Y^*) \).

(ii) For every \( y^* \in Y^* \),
\[
P_{X^*}(y^*) \subseteq B_{X^*}\left(y^* - Py^*, \frac{1-s}{s} \text{dist}(y^*, X^\perp)\right).
\]

In particular, if \( r = 1 \), then \( X \) has property \( U \) in \( Y \).

(iii) If \( \|I - P\| \leq 1 \), then for every \( y^* \in Y^* \),
\[
P_{X^*}(y^*) \subseteq B_{X^*}\left(Py^*, \frac{1-s}{s} \text{dist}(y^*, X^\perp)\right).
\]

In particular, if \( s = 1 \), then \( X \) has property \( U^* \) in \( Y \).

**Proof.** (i) Let \( y^* \in Y^* \). Since \( y^* - Py^* \in Ker P \), we have \( \text{dist}(y^*, X^\perp) \leq \|Py^*\| \). On the other hand, for every \( x^\perp \in X^\perp \),
\[
\|Py^*\| = \|P(y^* - x^\perp)\| \leq \|y^* - x^\perp\|.
\]

So, \( \|Py^*\| \leq \text{dist}(y^*, X^\perp) \).
(ii) Let \( y^* \in Y^* \) and \( x^* \in P_{X^*}(y^*) \). Then
\[
\| x^* - (y^* - P_y^*) \| = \| x^* - y^* - P(x^* - y^*) \| \\
\leq \frac{1}{s} \left( \| x^* - y^* \| - r \| P_x^* \| \| x^* - y^* \| \right) \\
= \frac{1}{s} \left( \| x^* - y^* \| - r \| P_y^* \| \right) = \frac{1}{s} \left( 1 - \frac{r}{s} \right) \text{dist}(y^*, X^*).
\]

(iii) It is clear that \( \| y^* - P_y^* \| = \text{dist}(y^*, X^*) \) for every \( y^* \in Y^* \), so the proof is similar to the one given in (ii).

In fact, we have proved the following

**Lemma 4.2.** Let \( X \) be an ideal in \( Y \). For all \( \varepsilon > 0 \), define
\[
A_{\varepsilon} = \left\{ (r, s) : 1 - \frac{r}{s} < \varepsilon \right\} \quad \text{and} \quad A^\varepsilon = \left\{ (r, s) : 1 - \frac{s}{r} < \varepsilon \right\}.
\]

Consider the set
\[
B = \left\{ (r, s) : X \text{ satisfies the M}(r, s)\text{-inequality in } Y \right\}.
\]

(i) If \( B \cap A_{\varepsilon} \neq \emptyset \) for all \( \varepsilon > 0 \), then \( X \) has property \( U \) in \( Y \).

(ii) Let \( P \) be the associated projection onto the ideal \( X \). If \( \| P \| \leq 1 \) and \( B \cap A^\varepsilon \neq \emptyset \) for all \( \varepsilon > 0 \), then \( X \) has property \( U^* \) in \( Y \).

The condition \( B \cap A_{\varepsilon} \neq \emptyset \) for all \( \varepsilon > 0 \) cannot be dropped in the above lemma, as shown by the next example.

**Example 4.3** ([3, Example 4.5]). Let \( 0 < \nu < 1 \). Let \( c_0 = K \otimes c_0 \) denote the equivalent renorming of \( c_0 \) with the norm
\[
\| (\alpha, z) \| = \max \{ \| \alpha \| + \nu \| z \|, \| z \| \}, \quad \alpha \in K, \quad z \in c_0,
\]
where \( \| z \| \) is the usual norm in \( c_0 \). Then \( c_0 \) and \( K(c_0) \) satisfy the \( M(1-\nu, 1) \)-inequality without having property \( U \) (in \( c_0^* \) and \( L(c_0) \), respectively).

Again as a consequence of the Theorem, we obtain the Cho–Johnson theorem for upper \( p \)-spaces.

**Theorem 4.4.** Let \( X \) be a Banach space having the upper \( p \)-property, \( 1 < p \leq \infty \). If \( E \) is a closed subspace of \( X \), then the following assertions are equivalent:

(i) \( E \) has the MCAP.

(ii) \( E \) has the upper \( p \)-property.

(iii) \( E \) satisfies the \( M_{\text{UC}}(r, s) \)-inequality for all \( (r, s) \in B_{\text{UC}} \).

(iv) For all Banach spaces \( W \), \( K(W, E) \) satisfies the \( M(r, s) \)-inequality for every \( (r, s) \in B_{\text{UC}} \).

(v) For all Banach spaces \( W \), \( K(W, E) \) is an \( HB \)-subspace of \( L(W, E) \).

**Proof.** By assumption, \( X \) satisfies the \( M_{\text{UC}}(r, s) \)-inequality for all \( (r, s) \in B_{\text{UC}} \). In particular, by Lemma 2.2, \( X \) satisfies the \( M(r, s) \)-inequality. So, by [2, Proposition 2.5], \( X \) is an Asplund space. Now, the implication (i) \( \Rightarrow \) (ii) follows from Lemma 2.4.

The implication (ii) \( \Rightarrow \) (iii) is obvious, and (iii) \( \Rightarrow \) (iv) has been proved in Proposition 2.3.

(iv) \( \Rightarrow \) (v) is proved in Lemma 4.2.

(v) \( \Rightarrow \) (i) is proved in [14, Theorem 3.1].

**Remark.** Another proof of (i) \( \Rightarrow \) (v) can be seen in [17, Proposition 3.1]. The case \( p = \infty \) is essentially known [10, Section VI.5].

**Corollary 4.5.** Let \( X \) be a Banach space having the upper \( p \)-property, \( 1 < p \leq \infty \). If \( E \) is a closed subspace of \( X \) having the \( \text{MCAP} \), then for all Banach spaces \( W \), \( K(W, E) \) has property \( U^* \) in \( L(W, E) \).

**Proof.** Corollary 4.5 follows from the above theorem and Lemma 4.2.

The scope of the above results can be illustrated by supposing that \( E \) is reflexive. A careful reading of the proof of [9, Lemma 5.2] allows us to assert that \( K(E)^{**} = L(E) \), hence, we can apply the results contained in [2].

Finally, let us notice that in [16] one can see how to construct examples satisfying a weakening of the notion of upper \( p \)-property, for which, of course, Theorem 4.4 can be easily adapted.

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**References**


Added in proof (January 1988). After returning the proofs, the authors observed that:

1) All the assertions of the (main) Theorem are equivalent (the condition $r + s/2 > 1$ is not necessary). The proof of the Theorem is the same.

2) All the assertions of Corollary 3.2 are equivalent (the condition $\delta > 2$ is not necessary). The proof is the same.