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# An ideal characterization of when a subspace of certain Banach spaces has the metric compact approximation property

by

J. C. CABELLO and E. NIETO (Granada)

**Abstract.** C.-M. Cho and W. B. Johnson showed that if a subspace $E$ of $\ell_p$, $1<p<\infty$, has the compact approximation property, then $\mathcal{K}(E)$ is an $M$-ideal in $\mathcal{L}(E)$. We prove that for every $r,s\in ]0,1]$ with $r^2+s^2<1$, the James space can be provided with an equivalent norm such that an arbitrary subspace $E$ has the metric compact approximation property iff there is a norm one projection $P$ on $\mathcal{L}(E)^*$ with $\operatorname{Ker} P=\mathcal{K}(E)^\perp$ satisfying

$$\|f\|\ge r\|Pf\|+s\|\varphi-Pf\|\quad \forall f\in\mathcal{L}(E)^*.$$

A similar result is proved for subspaces of upper $p$-spaces (e.g. Lorentz sequence spaces $d(w,p)$ and certain renormings of $L^p$).

**1. Introduction.** We follow [3] and [7] in assuming that a subspace $X$ of a Banach space $Y$ is said to be an *ideal* in $Y$ if there exists a norm one projection $P$ on $Y^*$ with $\operatorname{Ker} P=X^\perp$. If, moreover,

$$\|y^*\|\ge r\|Py^*\|+s\|y^*-Py^*\|\quad \forall y^*\in Y^*$$

holds for given $r,s\in ]0,1]$, then we say that $X$ is an *ideal satisfying the $M(r,s)$-inequality in $Y$* (for simplicity, we say that $X$ *satisfies the $M(r,s)$-inequality* if $Y$ is the bidual of $X$, and its associated projection is the canonical projection). If $r=s=1$, we return to the classical concept of $M$-ideal introduced by Alfsen and Effros [1].

For any Banach spaces $X$ and $Y$, we denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from $X$ to $Y$ and by $\mathcal{K}(X,Y)$ its subspace of compact operators. If $X=Y$, then we simply write $\mathcal{L}(X)$ and $\mathcal{K}(X)$, respectively. Harmand and Lima [9] proved that $X$ with $\mathcal{K}(X)$ being an $M$-ideal in $\mathcal{L}(X)$ must necessarily have the metric compact approximation property (MCAP), and Cho and Johnson [4] showed that for subspaces $E$ of $\ell_p$ (in fact, this holds for subspaces $E$ of $X$ with $\mathcal{K}(X)$ being an $M$-ideal in $\mathcal{L}(X)$ [10, Theorem VI.4.19]) the MCAP already ensures that $\mathcal{K}(E)$ is an

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$M$-ideal in $\mathcal{L}(E)$. It is also known [10, Section VI.5] that for subspaces $E$ of $c_0$, the MCAP moreover entails that $\mathcal{K}(W,E)$ is an $M$-ideal in $\mathcal{L}(W,E)$ for all Banach spaces $W$. For subspaces $E$ of $\ell_p$ the MCAP only implies that $\mathcal{K}(W,E)$ is an $HB$-subspace (a weakening of the notion of $M$-ideal) of $\mathcal{L}(W,E)$ [17].

We investigate a family of variants of the MCAP that are satisfied by e.g., $c_0$, $\ell_p$, the Lorentz sequence spaces $d(w,p)$, $1<p<\infty$, and certain renormings of the James space, which are inherited by subspaces having the MCAP.

A net $(K_\alpha)$ of compact operators on a Banach space $X$ will be called a *compact approximation of the identity* (*c.a.i.*) provided $\lim_\alpha K_\alpha x = x$ for every $x\in X$. If, moreover, $\lim_\alpha K_\alpha^* x^* = x^*$ for every $x^*\in X^*$, we will say that $(K_\alpha)$ is a *shrinking compact approximation of the identity* (*s.c.a.i.*).

Given $r,s\in\,]0,1]$, we say that a Banach space $X$ satisfies the *compact uniform $M(r,s)$-inequality* (for short, $M_{\rm cu}(r,s)$-inequality) if $X$ admits a c.a.i. $(K_\alpha)$ in $B_{\mathcal{K}(X)}$ satisfying

$$(*)\qquad \overline{\lim_\alpha}\sup_{\|x\|,\|y\|\le 1}\|rK_\alpha x + s(y-K_\alpha y)\|\le 1.$$

Of course, the $M_p$-spaces, $1<p\le\infty$, defined in [10, Section VI.5], satisfy the condition $(*)$ for $r=s=1$ if $p=\infty$, and for $r^p+s^p\le 1$ if $1<p<\infty$. For more examples the reader can see [3, Section 4] and Sections 3 and 4 below.

For abbreviation, given two Banach spaces $X$ and $Y$, we will say that $\mathcal{K}(X,Y)$ satisfies the $M(r,s)$-inequality instead of $\mathcal{K}(X,Y)$ is an ideal satisfying the $M(r,s)$-inequality in $\mathcal{L}(X,Y)$.

We prove the following

THEOREM. *Let $r,s\in\,]0,1]$ be such that $r+s>1$. Assume that $X$ is a Banach space satisfying the $M_{\rm cu}(r,s)$-inequality and $E$ is a closed subspace of $X$. Consider the following assertions*:

(i) *$E$ has the MCAP.*
(ii) *$\mathcal{K}(E)$ is an ideal in $\mathcal{L}(E)$.*
(iii) *$E$ satisfies the $M_{\rm cu}(r,s)$-inequality.*
(iv) *For all Banach spaces $W$, $\mathcal{K}(W,E)$ satisfies the $M(r,s)$-inequality.*
(v) *$\mathcal{K}(E\oplus_\infty E)$ satisfies the $M(r,s)$-inequality.*

*Then* (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Rightarrow$(v).

*All the above assertions are equivalent if $r+s/2>1$.*

**2. Proof of the Theorem.** We begin with an expected stability property (cf. [10, Proposition VI.4.2]), whose proof cannot use intersection properties of balls (cf. [3, Lemma 2.3]), as in the classical case ($r=s=1$).



LEMMA 2.1. *Let $X$ and $Y$ be two Banach spaces and let $r,s\in\,]0,1]$. If $\mathcal{K}(X,Y)$ satisfies the $M(r,s)$-inequality and $E\subseteq X$ and $F\subseteq Y$ are 1-complemented subspaces, then $\mathcal{K}(E,F)$ satisfies the $M(r,s)$-inequality.*

Proof. By hypothesis, there exists a norm one projection $P$ on $\mathcal{L}(X,Y)^*$ with $\operatorname{Ker}P=\mathcal{K}(X,Y)^\perp$ satisfying

$$\|f\|\ge r\|Pf\|+s\|f-Pf\|\quad \forall f\in\mathcal{L}(X,Y)^*.$$

Let $P_1,P_2$ be two norm one projections on $X$ and $Y$ respectively, with $P_1(X)=E$ and $P_2(Y)=F$, and denote by $i_1,i_2$ the inclusion operators from $E$ into $X$ and from $F$ into $Y$, respectively. Consider $\varphi:\mathcal{L}(X,Y)\to\mathcal{L}(E,F)$ defined by

$$\varphi(S)=P_2Si_1\quad\forall S\in\mathcal{L}(X,Y),$$

and $\chi:\mathcal{L}(E,F)\to\mathcal{L}(X,Y)$ defined by

$$\chi(T)=i_2TP_1\quad\forall T\in\mathcal{L}(E,F).$$

Since $\phi\circ\chi=I$, it is straightforward to show that $Q:\mathcal{L}(E,F)^*\to\mathcal{L}(E,F)^*$ defined by

$$Q(f)(T)=P(f\circ\varphi)(\chi(T))\quad\forall f\in\mathcal{L}(E,F)^*,\ T\in\mathcal{L}(E,F),$$

is a norm one projection with $\operatorname{Ker}Q=\mathcal{K}(E,F)^\perp$ satisfying

$$\|f\|\ge r\|Qf\|+s\|f-Qf\|\quad\forall f\in\mathcal{L}(E,F)^*.\ \blacksquare$$

The following lemma, essentially proved in [15], is crucial.

LEMMA 2.2. *Let $r,s\in\,]0,1]$. If $X$ satisfies the $M_{\rm cu}(r,s)$-inequality, then $\mathcal{K}(X)$ and $X$ satisfy the $M(r,s)$-inequality.*

The next result improves [18, Theorem 2], which was proved using intersection properties of balls and Banach algebra techniques. Our proof is based on J. Johnson's procedure of making projections [12] (cf. [14, Theorem 3.1], and the unicity of the associated projection [3, Proposition 3.2].

PROPOSITION 2.3. *Let $X$ be a Banach space and let $r,s\in\,]0,1]$. Consider the following statements*:

(i) *$X$ satisfies the $M_{\rm cu}(r,s)$-inequality.*
(ii) *For all Banach spaces $W$, $\mathcal{K}(W,X)$ satisfies the $M(r,s)$-inequality.*
(iii) *$\mathcal{K}(X\oplus_\infty X)$ satisfies the $M(r,s)$-inequality.*

*Then* (i)$\Rightarrow$(ii)$\Rightarrow$(iii). *All the above statements are equivalent if $r+s/2>1$.*

Proof. (i)$\Rightarrow$(ii). By definition, there is a c.a.i. $(K_\alpha)$ in $B_{\mathcal{K}(X)}$ satisfying $(*)$. Let $W$ be a Banach space and $T\in B_{\mathcal{L}(W,X)}$. Consider $L_\alpha=K_\alpha T$. By Johnson's procedure (see [14, Theorem 3.1]), $\mathcal{K}(W,X)$ is an ideal in $\mathcal{L}(W,X)$, and we can assume that $(L_\alpha)$ converges to $T$ in the $\sigma(\mathcal{L}(W,X),\mathcal{K}(W,X)^*)$-topology. Hence, by $(*)$,

$$\overline{\lim_\alpha}\,\|rS+s(T-L_\alpha)\|\le\overline{\lim_\alpha}\,\|rK_\alpha S+s(T-K_\alpha T)\|+\lim_\alpha r\|K_\alpha S-S\|\le 1$$

holds for every $S\in B_{\mathcal{K}(W,X)}$. Therefore, by [2, Lemma 2.7], we conclude that (ii) is satisfied.

(ii)⇒(iii). This implication follows from the fact that $Z\oplus_\infty Z$ is an ideal satisfying the $M(r,s)$-inequality in $Y\oplus_\infty Y$ whenever $Z$ is an ideal satisfying the $M(r,s)$-inequality in $Y$. In fact, we take $Z=\mathcal{K}(X\oplus_\infty X,X)$ and $Y=\mathcal{L}(X\oplus_\infty X,X)$.

(iii)⇒(i). By [3, Theorem 3.1], $X\oplus_\infty X$ admits a s.c.a.i. $(S_\alpha)$ in $B_{\mathcal{K}(X\oplus_\infty X)}$ satisfying

$$\overline{\lim_\alpha}\,\|rAS_\alpha+sB(I-S_\alpha)\|\le 1\quad\forall A,B\in B_{\mathcal{L}(X\oplus_\infty X)}. \tag{1}$$

On the other hand, since $X$ is a 1-complemented subspace of $X\oplus_\infty X$, by Lemma 2.1 and [3, Theorem 3.1], $X$ admits a s.c.a.i. $(L_\beta)$ with $\|L_\beta\|\le 1$ for all $\beta$. It is clear that

$$\widetilde{L}_\beta=\begin{pmatrix}L_\beta & 0\\ 0 & L_\beta\end{pmatrix}$$

is another s.c.a.i. in $B_{\mathcal{K}(X\oplus_\infty X)}$.

By Johnson's procedure, there are two norm one projections $P_1,P_2$ on $\mathcal{L}(X\oplus_\infty X)^*$ with $\operatorname{Ker}P_i=\mathcal{K}(X\oplus_\infty X)^\perp$. Concretely,

$$P_1(\phi)(T)=\lim_\alpha\phi(TS_\alpha),\qquad P_2(\phi)(T)=\lim_\beta\phi(T\widetilde{L}_\beta)$$

for all $\phi\in\mathcal{L}(X\oplus_\infty X)^*$ and $T\in\mathcal{L}(X\oplus_\infty X)$. By [3, Theorem 2.5 and Propositions 2.1 and 3.2], we have $P_1=P_2$. We can suppose that both nets are indexed by the same set (after switching to the product index set with the product ordering). In particular, the net $(S_\alpha-\widetilde{L}_\alpha)$ is weakly null, and so, by a convex combination argument, we may assume that $\|S_\alpha-\widetilde{L}_\alpha\|$ converges to zero. Then, checking (1) on the operators

$$A=\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix},\qquad B=\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix},$$

we obtain the condition $(*)$. ∎

Remark. Actually, if $r+s>1$, then the c.a.i. $(K_\alpha)$ may be chosen shrinking. In fact, by Lemma 2.2, $X$ satisfies the $M(r,s)$-inequality, so, by [2, Proposition 2.5], $X^*$ contains no proper norming subspaces. Hence, by [8, Proposition 2.5 and Theorem 2.2], $(K_\alpha)$ is a s.c.a.i.

The next lemma is proved by a standard procedure (cf. [13, Theorem 2.5]). For completeness, we indicate a proof.



Lemma 2.4. *Let $X$ be an Asplund space and let $E$ be a closed subspace of $X$. If $X^*$ and $E^*$ have the MCAP with adjoint operators, then $X$ and $E$ each admit a s.c.a.i. $(K_\alpha)$ and $(H_\alpha)$, respectively, such that*

$$\lim_\alpha\|iH_\alpha-K_\alpha i\|=0,$$

*where $i:E\to X$ is the inclusion.*

Proof. Let $(K_\alpha)$ be a s.c.a.i. in $B_{\mathcal{K}(X)}$ and let $(H_\beta)$ be a s.c.a.i. in $B_{\mathcal{K}(E)}$. We can suppose that both nets are indexed by the same set. It is clear that, for $x^*\in X^*$ and $e^{**}\in E^{**}$,

$$\lim_\alpha e^{**}(i^*K_\alpha^*x^*-H_\alpha^*i^*x^*)=0.$$

Therefore, it suffices to apply [6, Theorem 1] and a convex combination argument to finish. ∎

*Proof of the Theorem.* In the first place, note that, by Lemma 2.2 and [2, Proposition 2.1], $E$ satisfies the $M(r,s)$-inequality.

(i)⇒(ii). This implication follows from Johnson's procedure.

(ii)⇒(i). This follows from [3, Propositions 2.1 and 3.2].

(i)⇒(iii). On account of the above remark, $E$ admits a s.c.a.i. $(H_\alpha)$ in $B_{\mathcal{K}(E)}$. Let $(K_\alpha)$ be a s.c.a.i. in $B_{\mathcal{K}(X)}$ satisfying $(*)$. By Lemma 2.2 and [2, Proposition 2.5], $X$ is an Asplund space. Therefore, by Lemma 2.4, we can assume that $\|iH_\alpha-K_\alpha i\|$ converges to zero, where $i:E\to X$ is the inclusion. This clearly forces

$$\overline{\lim_\alpha}\sup_{\|x\|,\|y\|\le1}\|rH_\alpha x+s(y-H_\alpha y)\|\le 1,$$

as required.

(iii)⇒(iv) and (iv)⇒(v) are proved in Proposition 2.3.

(v)⇒(ii) is obvious. ∎

Before mentioning applications of our Theorem, we exhibit an interesting example (cf. [2, Example 4.6]).

Example 2.4. *Let $X$ and $Y$ be two $M_\infty$-spaces. Given $0<\gamma\le 1$, define*

$$\|(x,y)\|=\max\left\{\|x\|,\|y\|,\frac{\|x\|+\|y\|}{1+\gamma}\right\},\qquad x\in X,\ y\in Y.$$

*Then $Z=(X\times Y,\|\cdot\|)$ satisfies, simultaneously, the $M_{\mathrm{cu}}(1,\gamma)$-inequality and the $M_{\mathrm{cu}}(\gamma,1)$-inequality. Moreover, if $\gamma\ne1$, then $Z$ is not an $M_\infty$-space.*

**3. The James space.** In this section we show a method to provide the James space with a norm which satisfies the $M_{\mathrm{cu}}(r,s)$-inequality, and we obtain a Cho–Johnson theorem for the James space.

PROPOSITION 3.1. *For $\delta > 0$, let $J_\delta$ be the space of all null sequences $(x_n)$ in $\mathbb{R}$ satisfying*

$$\sup\Big\{(\delta x_{k_1} - x_{k_2})^2 + \sum_{i=2}^{n}(x_{k_i} - x_{k_{i+1}})^2 + (x_{k_{n+1}} - \delta x_{k_1})^2\Big\}^{1/2} < \infty,$$

*where the supremum is taken over all $n \in \mathbb{N}$ and all finite increasing sequences $k_1 < \ldots < k_{n+1}$ in $\mathbb{N}$, with norm $\|\cdot\|_\delta$ defined by this supremum. Let $r, s \in ]0,1]$ be such that $r^2+s^2 < 1$. Then $J_\delta$ satisfies the $M_{\rm cu}(r,s)$-inequality for all $\delta > 1$ such that*

$$r^2 + s^2 + \frac{s^2}{2\delta^2} + \frac{2rs}{\delta} \le 1. \tag{2}$$

Proof. It follows from [5, Properties I and II, pp. 81–82] that the sequence $(e_n)$, where $e_n = (0, \overset{(n-1)}{\ldots}, 0, 1, 0, \ldots)$, is a monotone shrinking basis. For all $n \in \mathbb{N}$, we define

$$P_n x = \sum_{i=1}^{n} e_i x_i \quad \forall x = (x_n) \in J_\delta.$$

It is enough to prove that for every $n \in \mathbb{N}$, and $x, y \in B_{J_\delta}$,

$$\|rP_n x + s(y - P_n y)\|_\delta \le 1.$$

Since $\|P_n x\|_\delta \le \|x\|_\delta$ for all $n \in \mathbb{N}$, we have

$$(\delta x_{k_1} - x_{k_2})^2 + \sum_{i=2}^{q}(x_{k_i} - x_{k_{i+1}})^2 + (x_{k_{q+1}})^2 + (\delta x_{k_1})^2 \le \|x\|_\delta^2 \tag{3}$$

for every $q \in \mathbb{N}$ and for every finite increasing sequence $k_1 < \ldots < k_{q+1}$ in $\mathbb{N}$. In particular, for every $x = (x_n) \in J_\delta$,

$$2(\delta x_n)^2 \le \|x\|_\delta^2 \quad \forall n \in \mathbb{N}. \tag{4}$$

Let $x = (x_n), y = (y_n) \in B_{J_\delta}$, $p \in \mathbb{N}$, and let $k_1 < \ldots < k_{p+1}$ be a finite sequence in $\mathbb{N}$. Fix $n \in \mathbb{N}$, denote by $\gamma = (\gamma_m)$ the sequence $(rx_1, \ldots, rx_n, sy_{n+1}, sy_{n+2}, \ldots)$, and set

$$S := (\delta\gamma_{k_1} - \gamma_{k_2})^2 + \sum_{i=2}^{p}(\gamma_{k_i} - \gamma_{k_{i+1}})^2 + (\gamma_{k_{p+1}} - \delta\gamma_{k_1})^2.$$

If $k_1 \ge n+1$, then

$$\begin{aligned} S &= (\delta s y_{k_1} - s y_{k_2})^2 + \sum_{i=2}^{p}(s y_{k_i} - s y_{k_{i+1}})^2 + (s y_{k_{p+1}} - \delta s y_{k_1})^2 \\ &\le s^2\|y\|_\delta^2 \le s^2 < 1. \end{aligned}$$



If $k_{p+1} \le n$, then

$$\begin{aligned} S &= (\delta r x_{k_1} - r x_{k_2})^2 + \sum_{i=2}^{p}(r x_{k_i} - r x_{k_{i+1}})^2 + (r x_{k_{p+1}} - \delta r x_{k_1})^2 \\ &\le r^2\|x\|_\delta^2 \le r^2 < 1. \end{aligned}$$

Assume that $k_1 \le n$ and $k_{p+1} \ge n+1$. Set $q = \max\{i \in \{1, \ldots, p\} : k_i \le n\}$. If $q = 1$, then by (3) and (4),

$$\begin{aligned} S &= (\delta r x_{k_1} - s y_{k_2})^2 + \sum_{i=2}^{p}(s y_{k_i} - s y_{k_{i+1}})^2 + (s y_{k_{p+1}} - \delta r x_{k_1})^2 \\ &\le 2(\delta r x_{k_1})^2 + 2\delta rs|x_{k_1}|(|y_{k_2}| + |y_{k_{p+1}}|) \\ &\quad + \sum_{i=2}^{p}(s y_{k_i} - s y_{k_{i+1}})^2 + (s y_{k_{p+1}})^2 + (s y_{k_2})^2 \\ &\le r^2 + \frac{2rs}{\delta} + s^2 + \frac{s^2}{2\delta^2} \le 1. \end{aligned}$$

If $q > 1$, then again by (3) and (4),

$$\begin{aligned} S &= (\delta r x_{k_1} - r x_{k_2})^2 + \sum_{i=2}^{q-1}(r x_{k_i} - r x_{k_{i+1}})^2 + (r x_{k_q} - s y_{k_{q+1}})^2 \\ &\quad + \sum_{i=q+1}^{p}(s y_{k_i} - s y_{k_{i+1}})^2 + (s y_{k_{p+1}} - \delta r x_{k_1})^2 \\ &\le (\delta r x_{k_1} - r x_{k_2})^2 + \sum_{i=2}^{q-1}(r x_{k_i} - r x_{k_{i+1}})^2 + (r x_{k_q})^2 + (\delta r x_{k_1})^2 \\ &\quad + 2rs|x_{k_q}| \cdot |y_{k_{q+1}}| + 2\delta rs|x_{k_1}| \cdot |y_{k_{p+1}}| \\ &\quad + \sum_{i=q+1}^{p}(s y_{k_i} - s y_{k_{i+1}})^2 + (s y_{k_{p+1}})^2 + (s y_{k_{q+1}})^2 \\ &\le r^2 + \frac{rs}{\delta^2} + \frac{rs}{\delta} + s^2 + \frac{s^2}{2\delta^2} \le 1. \end{aligned}$$

Therefore,

$$\|rP_n x + s(y - P_n y)\|_\delta \le 1,$$

as required. ■

COROLLARY 3.2. *Let $\delta > 1$, and let $E$ be a closed subspace of $J_\delta$. Consider the following statements:*

(i) *$E$ has the MCAP.*

(ii) *$\mathcal{K}(E)$ is an ideal in $\mathcal{L}(E)$.*

(iii) *$E$ satisfies the $M_{\mathrm{cu}}(r,s)$-inequality whenever $r$ and $s$ satisfy* (2).

(iv) *For all Banach spaces $W$, $\mathcal{K}(W,E)$ satisfies the $M(r,s)$-inequality whenever $r$ and $s$ satisfy* (2).

(v) *$\mathcal{K}(E \oplus_\infty E)$ satisfies the $M(r,s)$-inequality whenever $r$ and $s$ satisfy* (2).

*Then* (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Rightarrow$(v). *All the above statements are equivalent if $\delta > 2$.*

Proof. By Lemma 2.4 and the Theorem, it is enough to observe that if $\delta > \mu > 0$, then

$$\{(r,s) \text{ satisfying } (2)\} \cap \{(r,s) : r + s/\mu > 1\} \neq \emptyset. \ \blacksquare$$

In particular, we have the Cho–Johnson theorem:

THEOREM 3.3. *Let $r,s \in\, ]0,1]$ be such that $r^2+s^2<1$. Then there is $\delta_0 > 1$ such that for every $\delta > \delta_0$, and for every subspace $E$ of $J_\delta$, $E$ has the MCAP iff $\mathcal{K}(E)$ satisfies the $M(r,s)$-inequality.*

REMARKS. (i) Observe that, according to Lemma 2.2, $J_\delta$ satisfies the $M(r,s)$-inequality whenever the pair $(r,s)$ satisfies (2) (cf. [2, Example 3.5]). On the other hand, note that, with this method, $r$ can be chosen as near to 1 as one likes, but the James space cannot be renormed to satisfy the $M(1,s)$-inequality. Actually, the Banach spaces satisfying this inequality contain an isomorphic copy of $c_0$ [2, Corollary 3.4], and this is not true for the James space.

(ii) As far as we know, it is not clear whether the CAP implies the MCAP for quasi-reflexive Banach spaces (even for subspaces of $J_\delta$). Note that if $Y$ is a quasi-reflexive Banach space having the CAP, then $Y^*$ has the CAP, so, according to [8, Corollary 1.6], the question could be whether $Y^*$ has the CAP with adjoint operators.

**4. The upper $p$-property.** We recall the following notion introduced in [10, p. 327] (cf. [17]). We say that a Banach space $X$ has the *upper $p$-property* $(1 < p \le \infty)$ if $X$ admits a s.c.a.i. $(K_\alpha)$ such that

$$\overline{\lim_\alpha} \sup_{\|x\|,\|y\|\le 1} \|K_\alpha x + (y - K_\alpha y)\| \le (\|x\|^p + \|y\|^p)^{1/p}. \tag{5}$$

In fact, they comment [10, p. 327] that an effective way to produce Banach spaces with the upper $p$-property (*upper $p$-spaces*) is to look for reflexive sequence spaces whose unit vectors form a Schauder basis and the inequality

$$\|x+y\| \le (\|x\|^p + \|y\|^p)^{1/p}$$

holds for disjointly supported sequences. It is clear that, under this hypothesis, the sequence of coordinate projections is a s.c.a.i. satisfying (5) (and, of course, the inequality (*) for every $(r,s) \in B_{\ell_p^2}$). Besides the



$M_p$-spaces, examples include the Lorentz sequence spaces $d(w,p)$, and more generally, the $p$-convexification of a sequence space whose unit vector basis is 1-unconditional. In [10, Proposition VI.6.8] one can see a renorming of $L^p$ with the upper 2-property. On the other hand, if $p = \infty$, we return to the $M_\infty$-spaces [10, p. 306].

Given a closed subspace $X$ of a Banach space $Y$, according to the Hahn–Banach theorem, each functional on $X$ admits a norm preserving extension to a functional on $Y$. Following R. Phelps [19], we shall say that $X$ has *property $U$* in $Y$ if for every $x^* \in X^*$, the norm preserving extension is unique. If, moreover, $X$ is an ideal in $Y$ with associated projection $P$ such that $\|I - P\| \le 1$, then $X$ is said to be an *HB-subspace* of $Y$ [11].

We will say that $X$ has *property $U^*$* in $Y$ if there is a norm one projection $P$ on $Y^*$ with $\operatorname{Ker} P = X^\perp$ such that for all $y^* \in Y^*$ with $Py^* \neq 0$,

$$\|y^* - Py^*\| < \|y^*\|.$$

It is clear that if $X$ is an $M$-ideal in $Y$, then $X$ has properties $U$ and $U^*$ (in fact, $X$ is an $HB$-subspace) in $Y$.

In the next lemma, we show that it is not necessary to suppose $s = 1$ to have property $U$, and $r = 1$ for property $U^*$.

LEMMA 4.1. *If $X$ is an ideal satisfying the $M(r,s)$-inequality in $Y$ with associated projection $P$, then*:

(i) *For every $y^* \in Y^*$, $Py^*$ is a norm preserving extension of $y^*|_X$. In particular, $X^*$ is isometric to $P(Y^*)$.*

(ii) *For every $y^* \in Y^*$,*

$$P_{X^\perp}(y^*) \subseteq B_{X^\perp}\Big(y^* - Py^*, \frac{1-r}{s}\operatorname{dist}(y^*, X^\perp)\Big).$$

*In particular, if $r = 1$, then $X$ has property $U$ in $Y$.*

(iii) *If $\|I - P\| \le 1$, then for every $y^* \in Y^*$,*

$$P_{X^*}(y^*) \subseteq B_{X^*}\Big(Py^*, \frac{1-s}{r}\operatorname{dist}(y^*, X^*)\Big).$$

*In particular, if $s = 1$, then $X$ has property $U^*$ in $Y$.*

Proof. (i) Let $y^* \in Y^*$. Since $y^* - Py^* \in \operatorname{Ker} P$, we have $\operatorname{dist}(y^*, X^\perp) \le \|Py^*\|$. On the other hand, for every $x^\perp \in X^\perp$,

$$\|Py^*\| = \|P(y^* - x^\perp)\| \le \|y^* - x^\perp\|.$$

So, $\|Py^*\| \le \operatorname{dist}(y^*, X^\perp)$.

(ii) Let $y^* \in Y^*$ and $x^\perp \in P_{X^\perp}(y^*)$. Then

$$\begin{aligned}\|x^\perp - (y^* - Py^*)\| &= \|(x^\perp - y^*) - P(x^\perp - y^*)\| \\ &\le \frac{1}{s}(\|x^\perp - y^*\| - r\|P(x^\perp - y^*)\|) \\ &= \frac{1}{s}(\|x^\perp - y^*\| - r\|Py^*\|) = \frac{1-r}{s}\operatorname{dist}(y^*, X^\perp).\end{aligned}$$

(iii) It is clear that $\|y^* - Py^*\| = \operatorname{dist}(y^*, X^*)$ for every $y^* \in Y^*$, so the proof is similar to the one given in (ii). ∎

In fact, we have proved the following

LEMMA 4.2. *Let $X$ be an ideal in $Y$. For all $\varepsilon > 0$, define*

$$A_\varepsilon = \left\{(r,s) : \frac{1-r}{s} < \varepsilon\right\} \quad \text{and} \quad A^\varepsilon = \left\{(r,s) : \frac{1-s}{r} < \varepsilon\right\}.$$

*Consider the set*

$$B = \{(r,s) : X \text{ satisfies the } M(r,s)\text{-inequality in } Y\}.$$

(i) *If $B \cap A_\varepsilon \ne \emptyset$ for all $\varepsilon > 0$, then $X$ has property $U$ in $Y$.*

(ii) *Let $P$ be the associated projection onto the ideal $X$. If $\|I - P\| \le 1$ and $B \cap A^\varepsilon \ne \emptyset$ for all $\varepsilon > 0$, then $X$ has property $U^*$ in $Y$.*

The condition $B \cap A_\varepsilon \ne \emptyset$ for all $\varepsilon > 0$ cannot be dropped in the above lemma, as shown by the next example.

EXAMPLE 4.3 ([3, Example 4.5]). *Let $0 < \nu < 1$. Let $\widetilde{c}_0 = \mathbb{K} \otimes c_0$ denote the equivalent renorming of $c_0$ with the norm*

$$\|(\alpha, z)\| = \max\{|\alpha| + \nu\|z\|, \|z\|\}, \qquad \alpha \in \mathbb{K},\ z \in c_0,$$

*where $\|z\|$ is the usual norm in $c_0$. Then $\widetilde{c}_0$ and $\mathcal{K}(\widetilde{c}_0)$ satisfy the $M(1-\nu, 1)$-inequality without having property $U$ (in $\widetilde{c}_0^{**}$ and $\mathcal{L}(\widetilde{c}_0)$, respectively).*

Again as a consequence of the Theorem, we obtain the Cho–Johnson theorem for upper $p$-spaces.

THEOREM 4.4. *Let $X$ be a Banach space having the upper $p$-property, $1 < p \le \infty$. If $E$ is a closed subspace of $X$, then the following assertions are equivalent:*

(i) *$E$ has the MCAP.*

(ii) *$E$ has the upper $p$-property.*

(iii) *$E$ satisfies the $M_{\mathrm{cu}}(r,s)$-inequality for all $(r,s) \in B_{\ell_p^2}$.*

(iv) *For all Banach spaces $W$, $\mathcal{K}(W,E)$ satisfies the $M(r,s)$-inequality for every $(r,s) \in B_{\ell_p^2}$.*

(v) *For all Banach spaces $W$, $\mathcal{K}(W,E)$ is an HB-subspace of $\mathcal{L}(W,E)$.*



Proof. By assumption, $X$ satisfies the $M_{\mathrm{cu}}(r,s)$-inequality for all $(r,s) \in B_{\ell_p^2}$. In particular, by Lemma 2.2, $X$ satisfies the $M(r,s)$-inequality. So, by [2, Proposition 2.5], $X$ is an Asplund space. Now, the implication (i)⇒(ii) follows from Lemma 2.4.

The implication (ii)⇒(iii) is obvious, and (iii)⇒(iv) has been proved in Proposition 2.3.

(iv)⇒(v) is proved in Lemma 4.2.

(v)⇒(i) is proved in [14, Theorem 3.1]. ∎

REMARK. Another proof of (i)⇒(v) can be seen in [17, Proposition 3.1]. The case $p = \infty$ is esentially known [10, Section VI.5].

COROLLARY 4.5. *Let $X$ be a Banach space having the upper $p$-property, $1 < p \le \infty$. If $E$ is a closed subspace of $X$ having the MCAP, then for all Banach spaces $W$, $\mathcal{K}(W,E)$ has property $U^*$ in $\mathcal{L}(W,E)$.*

Proof. Corollary 4.5 follows from the above theorem and Lemma 4.2. ∎

The scope of the above results can be illustrated by supposing that $E$ is reflexive. A careful reading of the proof of [9, Lemma 5.2] allows us to assert that $\mathcal{K}(E)^{**} = \mathcal{L}(E)$, hence, we can apply the results contained in [2].

Finally, let us notice that in [16] one can see how to construct examples satisfying a weakening of the notion of upper $p$-property, for which, of course, Theorem 4.4 can be easily adapted.

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Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, Spain
E-mail: jcabello@goliat.ugr.es

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**Added in proof** (January 1998). After returning the proofs, the authors observed that:

1) All the assertions of the (main) Theorem are equivalent (the condition $r + s/2 > 1$ is not necessary). The proof of the Theorem is the same.

2) All the assertions of Corollary 3.2 are equivalent (the condition $\delta > 2$ is not necessary). The proof is the same.

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