STUDIA MATHEMATICA 129 (2) (1998)

On extremal and perfect σ-algebras for flows

by

B. KAMIŃSKI (Toruń) and Z. S. KOWALSKI (Wrocław)

Abstract. It is shown that there exists a flow on a Lebesgue space with finite entropy and an extremal σ-algebra of it which is not perfect.

1. Introduction. The theory of invariant σ-algebras is an important part of ergodic theory having applications in the spectral theory of dynamical systems ([7], [9]), statistical mechanics ([3], [4]) and probability theory ([10]–[13]).

The extremal and perfect σ-algebras play the key role in the theory of invariant σ-algebras.

The paper [8] contains a positive solution to the Kwiatkowski question whether there exists a $\mathbb{Z}^d$-action on a Lebesgue space and an extremal σ-algebra of it which is not perfect.

The purpose of this paper is to give an affirmative answer to this question in the case of flows. The existence of perfect σ-algebras for flows has been proved by Blanchard [1] and Gurevich [6].

2. Result. Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space and let (T^t) be a measurable flow acting on $(X, \mathcal{B}, \mu)$. Let $\pi((T^t))$ denote the Pinsker σ-algebra of (T^t).

Now we recall the definition of a special flow.

Let $(Y, \mathcal{C}, \nu)$ be a Lebesgue probability space, T be an automorphism of Y and $f : Y \to \mathbb{R}^+$ a measurable function such that $\inf\{f(y) : y \in Y\} > 0$ and $f \in L^1(Y, \nu)$. Let $Y_f = \{(y, u) \in Y \times \mathbb{R}^+ : u < f(y)\}$ and let $\mathcal{C}_f$ be the restriction of the product σ-algebra $\mathcal{C} \otimes \mathcal{L}$ to Y_f, where $\mathcal{L}$ denotes the σ-algebra of Lebesgue sets of $\mathbb{R}^+$. We denote by ν_f the measure on $\mathcal{C}_f$ defined by

1991 *Mathematics Subject Classification*: Primary 28D10.

The first author is supported by KBN grant 2P 30103107 and the second by KBN grant 2 P03A07608.

$$\nu_f = \Big(\int_Y f\,d\nu\Big)^{-1}(\nu\times\lambda),$$

where λ stands for Lebesgue measure. Let (T_f^t) be the measurable flow on $(Y_f, \mathcal{C}_f, \nu_f)$ defined as follows. For $0 \le t < \inf\{f(y) : y \in Y\}$ we put

$$T_f^t(y,u) = \begin{cases} (y, u+t) & \text{if } u+t < f(y),\\ (Ty, u+t-f(y)) & \text{if } u+t \ge f(y).\end{cases}$$

For other values of t the automorphism T_f^t is uniquely determined by the condition that (T_f^t) is a one-parameter group of automorphisms. The flow (T_f^t) is called the *special flow* built under the function f, the automorphism T is called the *base automorphism*, and f is called the *ceiling function* of (T_f^t).

Let $\mathcal{D} \subset \mathcal{C}$ be a σ-algebra such that $T^{-1}\mathcal{D} \subset \mathcal{D}$ and let $\mathcal{D}_f$ be the σ-algebra obtained from $\mathcal{D}$ in the same manner as $\mathcal{C}_f$ from $\mathcal{C}$.

LEMMA 1 ([2]). *$T_f^t\mathcal{D}_f \subset \mathcal{D}_f$ iff f is measurable with respect to $\mathcal{D}$.*

LEMMA 2 ([5]). *If f is measurable with respect to $\mathcal{D}$ then for any $t < \inf\{f(x) : x \in X\}$ we have*

$$H(T_f^t\mathcal{D}_f \,|\, \mathcal{D}_f) = t\frac{H(T\mathcal{D}\,|\,\mathcal{D})}{\int_X f\,d\mu}.$$

LEMMA 3 ([5]). *If T is a K-automorphism with finite entropy, P is a finite generator for T such that the σ-algebra $\bigvee_{n=0}^{\infty} T^{-n}P \cap \bigvee_{n=0}^{\infty} T^nP$ is not non-atomic, f is measurable with respect to the algebra P^* generated by P and the values of f are independent over the field $\mathbb{Q}$ then $\{T_f^t\}$ is a K-flow.*

Let now (T^t) be an arbitrary measurable flow acting on a Lebesgue space $(X, \mathcal{B}, \mu)$. A σ-algebra $\mathcal{A} \subset \mathcal{B}$ is said to be *extremal* for (T^t) if

(i) $T^t\mathcal{A} \subset \mathcal{A}$, $t < 0$,
(ii) $\bigvee_{t\in\mathbb{R}} T^t\mathcal{A} = \mathcal{B}$,
(iii) $\bigcap_{t\in\mathbb{R}} T^t\mathcal{A} = \pi((T^t))$.

If, in addition,

(iv) $h(T^t) = H(T^t\mathcal{A}\,|\,\mathcal{A}), t > 0$

then $\mathcal{A}$ is called *perfect*.

THEOREM. *There exists a measurable flow and an extremal σ-algebra of it which is not perfect.*

Proof. We construct the desired flow as a special flow.

First we describe the base automorphism of the flow. Let the space $Z = \{-1, 1\}^{\mathbb{Z}}$ be equipped with the product σ-algebra $\mathcal{C}$ and let ν be the Bernoulli measure determined by the probability vector $p = (1/2, 1/2)$. We denote by σ the shift transformation of Z.

Let

$$(Y, \mathcal{F}, \lambda) = (Z, \mathcal{C}, \nu) \times (Z, \mathcal{C}, \nu).$$

We consider in Y the skew product transformation T defined as follows:

$$T(x,y) = (\sigma(x), \sigma^{x(0)+1}(y)), \qquad (x,y) \in Y,$$

where σ^0 means the identity transformation on Z.

Let ξ be a measurable partition of Y defined as follows: points (x,y), $(\widetilde{x},\widetilde{y}) \in Y$ are equivalent if $y = \widetilde{y}$ and $x_i = \widetilde{x}_i$ for all $i \ge 0$. We define ξ as the collection of all the equivalence classes of this relation. Let $\mathcal{D}$ be the σ-algebra generated by ξ.

It is shown in [8] that T is a K-automorphism. It is easy to see that $\mathcal{D}$ coincides with the σ-algebra $\mathcal{D}$ defined in the proof of the Theorem in [8] and, therefore, it is extremal and is not perfect, i.e.

$$T^{-1}\mathcal{D} \subset \mathcal{D}, \tag{1}$$

$$\bigvee_{n=0}^{\infty} T^n\mathcal{D} = \mathcal{F}, \tag{2}$$

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{D} = \mathcal{N}, \tag{3}$$

where $\mathcal{N}$ denotes the trivial σ-algebra,

$$H(\mathcal{D}\,|\,T^{-1}\mathcal{D}) < h(T). \tag{4}$$

Let P be the zero-time partition of Z, i.e.

$$P = \{A_{-1}, A_1\}, \quad A_i = \{x(0) = i\}, \quad i = -1, 1,$$

and let $R = P \vee \sigma^{-1}P$.

We consider the partition Q which is the product of P and R, i.e. $Q = P \times R$. First observe that

$$\bigvee_{i=0}^{\infty} T^iQ = \bigvee_{i=0}^{\infty} \sigma^iP \otimes \bigvee_{i=0}^{\infty} \sigma^{2i}P, \quad \bigvee_{i=0}^{\infty} T^{-i}Q = \bigvee_{i=0}^{\infty} \sigma^{-i}P \otimes \bigvee_{i=0}^{\infty} \sigma^{-2i}P. \tag{5}$$

Since $T^{-1}(x,y) = (\sigma^{-1}x, \sigma^{-x(-1)-1}y)$ it is enough to check the first equality. Let n be an arbitrary positive integer. The atoms of $\bigvee_{i=0}^{n} T^iQ$ have the form

$$\bigcap_{k=0}^{n} T^k(P_{i_k} \times R_{j_k}) = \bigcap_{k=0}^{n} \sigma^kP_{i_k} \times \bigcap_{k=0}^{n} \sigma^{\sum_{p=1}^{k}(i_p+1)}R_{j_k}$$

where s_n denotes the number of ones in the sequence $(i_1, \ldots, i_n)$. Therefore, every atom q of $\bigvee_{i=0}^{\infty} T^iQ$ is contained in $\bigcap_{k=0}^{n} T^k(P_{i_k} \times R_{j_k})$.

Since $s_n \to \infty$ modulo a set of measure zero we have $q \subset s \times r$ where s is an atom of $\bigvee_{i=0}^{\infty} \sigma^i P$ and r is an atom of $\bigvee_{i=0}^{\infty} \sigma^{2i} R$.

On the other hand, we have $\bigvee_{i=0}^{n} T^i Q \subset \bigvee_{i=0}^{n} \sigma^i P \otimes \bigvee_{i=0}^{n} \sigma^{2i} P$ and so $s \times r \subset q$. Therefore $s \times r = q$ and (5) is valid.

From (5) it easily follows that Q is a generator and

$$\bigvee_{i=0}^{\infty} T^i Q \cap \bigvee_{i=0}^{\infty} T^{-i} Q = Q. \tag{6}$$

Now we want to check that $h(T) < \infty$. More precisely, we show that

$$h(T) = \log 4. \tag{7}$$

It is easy to see that T is a natural extension of the non-invertible endomorphism

$$\tau(u, v) = (S(u), S^{u(0)+1} v),$$

where $u, v \in \Omega = \{-1, 1\}^{\mathbb{N}}$ and S is the one-sided shift. Let $\pi : Z \to \Omega$ denote the natural projection and let $\Phi(x, y) = (\pi x, \pi y)$. Then $\Phi \circ T = \tau \circ \Phi$.

We denote by Q^+ the partition of $\Omega \times \Omega$ given by the formula

$$Q^+ = P^+ \times (P^+ \vee S^{-1} P^+)$$

where $P^+ = \pi P$. Let A_{-1}^+ and A_1^+ be the atoms of P^+. It is easy to see that Q^+ is a generator for τ. Let J_τ denote the Jacobian of τ and let ν^+ be the Bernoulli measure on Ω determined by the vector $p = (1/2, 1/2)$. We put $\lambda^+ = \nu^+ \times \nu^+$. We have

$$h(T) = h(\tau) = \int_{\Omega \times \Omega} \log J_\tau \, d\lambda^+ = \int_{A_{-1}^+ \times \Omega} \log 2 \, d\lambda^+ + \int_{A_1^+ \times \Omega} \log 8 \, d\lambda^+ = \log 4.$$

Let now f be an arbitrary positive function, constant on the atoms of Q and such that its values are independent over the field of rationals.

Consider the special flow (T^t) with base automorphism T and ceiling function f. Denote by $(X, \mathcal{B}, \mu)$ the corresponding Lebesgue space on which (T^t) acts. It follows from (7) and the well known formula for the entropy of a special flow that the entropy of the flow (T^t) is finite. Let $\mathcal{A} = (\mathcal{D} \otimes \mathcal{L}^1) \cap \mathcal{B}$.

It is easy to see that the atoms of Q belong to $\mathcal{D}$, i.e. f is $\mathcal{D}$-measurable. Therefore, Lemma 1 implies that $T^t \mathcal{A} \subset \mathcal{A}, t < 0$.

The equality $\bigvee_{t \in \mathcal{R}} T^t \mathcal{A} = \mathcal{B}$ follows easily from (2).

Now the equalities (3), (6) and Lemma 3 imply (T^t) is a K-flow, i.e.

$$\bigcap_{t<0} T^t \mathcal{A} = \mathcal{N} = \pi((T^t)).$$

This means that $\mathcal{A}$ is extremal.

Let now $t \in (0, \inf_{y \in Y} f(y))$. Lemma 2 gives

$$H(T^t \mathcal{A} \mid \mathcal{A}) = t \frac{H(T\mathcal{D} \mid \mathcal{D})}{\int_Y f \, d\lambda}.$$

The inequality (4) and the formula for the entropy of a special flow imply

$$H(T^t \mathcal{A} \mid \mathcal{A}) < t \frac{h(T)}{\int_Y f \, d\lambda} = h(T^t),$$

i.e. $\mathcal{A}$ is not perfect. ■

References

[1] F. Blanchard, *Partitions extrêmales des flots d'entropie infinie*, Z. Wahrsch. Verw. Gebiete 36 (1976), 129–136.

[2] —, *K-flots et théorème de renouvellement*, ibid., 345–358.

[3] M. Binkowska and B. Kamiński, *Entropy increase for $\mathbb{Z}^d$-actions on a Lebesgue space*, Israel J. Math. 88 (1994), 307–318.

[4] S. Goldstein and O. Penrose, *A non-equilibrium entropy for dynamical systems*, J. Statist. Phys. 24 (1981), 325–343.

[5] B. M. Gurevich, *Some existence conditions for K-decompositions for special flows*, Trans. Moscow Math. Soc. 17 (1967), 99–126.

[6] —, *Perfect partitions for ergodic flows*, Funktsional. Anal. i Prilozhen. 11 (3) (1977), 20–23 (in Russian).

[7] B. Kamiński, *The theory of invariant partitions for $\mathbb{Z}^d$-actions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 29 (1981), 349–362.

[8] B. Kamiński, Z. S. Kowalski and P. Liardet, *On extremal and perfect σ-algebras for $\mathbb{Z}^d$-actions*, Studia Math. 124 (1997), 173–178.

[9] V. A. Rokhlin, *Lectures on the entropy theory of measure-preserving transformations*, Uspekhi Mat. Nauk 22 (5) (1967), 3–56 (in Russian).

[10] T. Shimano, *An invariant of systems in the ergodic theory*, Tôhoku Math. J. 30 (1978), 337–350.

[11] —, *Multiplicity of helices of a special flow*, ibid. 31 (1979), 49–55.

[12] —, *Multiplicity of helices for a regularly increasing sequence of σ-fields*, ibid. 36 (1984), 141–148.

[13] —, *On helices for Kolmogorov system with two indices*, Math. J. Toyama Univ. 14 (1991), 213–226.

Faculty of Mathematics and Informatics
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: bkam@mat.uni.torun.pl

Institute of Mathematics
Wrocław Technical University
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: kowalski@banach.im.pwr.wroc.pl

Received May 6, 1997
Revised version November 3, 1997
(3880)