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On analytic semigroups and cosine functions in Banach spaces

by

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Abstract. If A generates a bounded cosine function on a Banach space X then the negative square root B of A generates a holomorphic semigroup, and this semigroup is the conjugate potential transform of the cosine function. This connection is studied in detail, and it is used for a characterization of cosine function generators in terms of growth conditions on the semigroup generated by B. The characterization relies on new results on the inversion of the vector-valued conjugate potential transform.

Introduction. In a Banach space X, consider a closed linear operator A which generates a cosine function $C(\cdot)$ (see e.g. Fattorini [6] or Goldstein [7] for more information about cosine operator functions). Then A generates a holomorphic semigroup $T(\cdot)$ of angle $\pi/2$. The semigroup and the cosine function are related by the abstract Weierstrass formula

$$T(t)x = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\tau^2/(4t)} C(\tau)x \, d\tau, \qquad t > 0.$$

On the other hand, assume that A generates a C_0-semigroup $T(\cdot)$. If $T(\cdot)$ is uniformly bounded, then one can define the fractional powers $(-A)^\alpha$ of $-A$ for $0 < \alpha < 1$. We restrict ourselves to the case $\alpha = 1/2$. First define the operator J with domain $D(J) = D(A)$ by

$$Jx = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (\lambda - A)^{-1}(-A)x \, d\lambda, \qquad x \in D(J).$$

Then J is closable and, by definition, $(-A)^{1/2} := \overline{J}$ (see e.g. Yosida [15, p. 260]).

The operator $B := -(-A)^{1/2}$ is the generator of a holomorphic semigroup $T_B(\cdot)$ which has an explicit representation (see [15, p. 268]):

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$$T_B(t)x = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-t^2/(4\tau)} T(\tau)x \frac{d\tau}{\tau^{3/2}}, \quad x \in X,\ t > 0.$$

Combining the above facts, we see that whenever A generates a uniformly bounded cosine function $C(\cdot)$, the negative square root of A generates a bounded holomorphic semigroup of angle $\pi/2$ given by the formula

$$(1) \qquad T_B(t)x = \frac{2}{\pi} \int_0^\infty \frac{t}{t^2+\tau^2} C(\tau)x\, d\tau, \quad x \in X,\ t > 0.$$

It is our intention in this paper to study this connection in more detail. In the first part, we introduce the following general transformation: if $f : (0,\infty) \to X$ is measurable, and if the integral $\int_0^\infty (\|f(\tau)\|/(t^2+\tau^2))\, d\tau$ converges for all $t \in (0,\infty)$, then we define

$$\mathcal{C}f(t) = \frac{2}{\pi} \int_0^\infty \frac{t}{t^2+\tau^2} f(\tau)\, d\tau, \quad t \in (0,\infty),$$

and we call $\mathcal{C}f$ the *conjugate potential transform* of f. We provide a vector-valued inversion theory for the conjugate potential transform in the spirit of [13], using Widder's results on the inversion of convolution transforms [14].

In the second part we consider the relationship (1) and prove that $T_B(\cdot)$ has the semigroup property iff $C(\cdot)$ satisfies the cosine functional equation. A similar relationship was studied by Dettman [4] in connection with the Cauchy problem. Our approach is operator-theoretic.

A remarkable feature is the following: by using the sine function $S(\cdot)$ associated with the cosine function, one can recast formula (1) in the form

$$(2) \qquad T_B(t) = \frac{2}{\pi} \int_0^\infty \frac{t}{t^2+\tau^2}\, dS(\tau), \quad x \in X.$$

Now, if we do not assume that A generates a cosine function but rather that it generates a sine function which is Lipschitz-continuous in the strong operator topology, then we prove that the representation (2) implies that in fact A generates a strongly continuous cosine function. This is to be compared with Arendt [1] where a similar phenomenon occurs in the relationship between resolvents and integrated semigroups. More precisely, Widder's theorem holds for general Banach spaces only in an *integrated form* while it holds in all Banach spaces in the usual form for resolvents of densely defined linear operators.

The results of the first section can then be used to recover $C(\cdot)$ from $T_B(\cdot)$ in the representation (1). We provide an explicit representation to that effect. Another interesting fact is that since the transform of Section 2 was studied for general vector-valued functions, it can be used, along with the inversion formula, to relate the solution of the second order Cauchy problem associated with A to that of the first order Cauchy problem associated with the negative square root of A.

1. Inversion of the conjugate potential transform. If $f : (0,\infty) \to X$ is measurable with $\int_0^\infty (\|f(t)\|/(s^2+t^2))\, dt < \infty$ for all $s \in (0,\infty)$ then we define

$$\mathcal{C}f(s) = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+t^2} f(t)\, dt, \quad s \in (0,\infty).$$

In this section we give an inversion formula which recovers any bounded continuous function f from the transformed function $\mathcal{C}f$, and we characterize those functions $F : (0,\infty) \to X$ which can be represented as

$$F(s) = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+t^2}\, d\phi(t), \quad s \in (0,\infty),$$

where $\phi : (0,\infty) \to X$ is Lipschitz-continuous.

Before we state the inversion formula we introduce some notations. For $\Omega \subseteq \mathbb{R}$ open and $f : \Omega \to X$ differentiable, we set

$$Df(s) = f'(s) \quad \text{and} \quad \Delta f(s) = sf'(s), \quad s \in \Omega.$$

For $n \in \mathbb{N}$, we denote by E_n the polynomial

$$E_n(s) = \prod_{k=0}^{n-1} \left(1 - \frac{s^2}{(2k+1)^2}\right),$$

and put $E_0(s) = 1$. If $f \in C^{2n}$ then we put

$$E_n^D[f] = E_n(D)f \quad \text{and} \quad E_n^\Delta[f] = E_n(\Delta)f.$$

With these notations the inversion formula takes the following form:

THEOREM 1. *If $f : (0,\infty) \to X$ is bounded and continuous then, for all $s \in (0,\infty)$,*

$$\lim_{n\to\infty} E_n^\Delta[\mathcal{C}f](s) = f(s).$$

This theorem will be proven using Widder's results on the inversion of convolution transforms (see [14] and Theorem 2). This is possible because the operator $\mathcal{C}$ can be "translated" into a convolution transform in the following way:

If $f : (0,\infty) \to X$ is any function then, for $u \in \mathbb{R}$, put $\Gamma f(u) = f(e^u)$. If $f \in L_\infty((0,\infty), X)$ then

$$\Gamma(\mathcal{C}f)(s) = \frac{2}{\pi}\int_0^\infty \frac{e^s}{e^{2s}+t^2} f(t)\,dt$$
$$= \frac{2}{\pi}\int_{-\infty}^\infty \frac{e^{s-u}}{e^{2(s-u)}+1}\Gamma f(u)\,du = K * \Gamma f(s),$$

where the convolution kernel $K \in L_1(\mathbb{R})$ is given by

$$K(u) = \frac{2}{\pi}\frac{e^u}{e^{2u}+1}.$$

The convolution transform $g \mapsto K * g$ can be inverted by using the following theorem, which is a special case of [14, Chapter 7, Theorem 7].

THEOREM 2. *Let $K : \mathbb{R} \to \mathbb{R}$ be a measurable function with the following properties*:

(i) *The bilateral Laplace transform of K converges in a strip symmetric about the imaginary axis.*

(ii) *$F(s) = \int_{-\infty}^\infty e^{-su}K(u)\,du$ has no zeros in a strip $|\Re(s)| < \sigma$, and $E(s) = F(s)^{-1}$ can be written as*

$$E(s) = \prod_{k=0}^\infty \left(1 - \frac{s}{a_k}\right),$$

where the numbers $a_k \in \mathbb{R}\setminus\{0\}$ are such that $\lim_{n\to\infty}\sum_{k=0}^n 1/a_k = 0$ and $\sum_{k=0}^\infty 1/a_k^2 < \infty$.

If $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous then $K * g \in C^\infty(\mathbb{R})$, and, for all $s \in \mathbb{R}$,

$$\lim_{n\to\infty}\prod_{k=0}^n \left(1 - \frac{D}{a_k}\right)[K * g](s) = g(s).$$

We next show that the kernel $K(u) = 2\pi^{-1}e^u(e^{2u}+1)^{-1}$ satisfies the assumptions of the foregoing theorem. The bilateral Laplace transform

$$F(s) = \int_{-\infty}^\infty e^{-su}K(u)\,du = \frac{2}{\pi}\int_{-\infty}^\infty e^{-su}\frac{e^u}{e^{2u}+1}\,du$$

of K exists in the strip $|\Re(s)| < 1$, and, by substitution,

$$F(s) = \frac{2}{\pi}\int_0^\infty \frac{t^s}{1+t^2}\,dt = \frac{1}{\cos(s\pi/2)}. \tag{3}$$

Hence F has no zeros in the strip $|\Re(s)| < 1$. Moreover, by [8, p. 484], $E(s) = F(s)^{-1}$ can be written as

$$E(s) = \cos(s\pi/2) = \prod_{k=0}^\infty \left(1 - \frac{s^2}{(2k+1)^2}\right) = \prod_{k=0}^\infty \left(1 - \frac{s}{a_k}\right),$$

where $a_k = k+1$ if k is even, and $a_k = -k$ if k is odd. Moreover,

$$\lim_{n\to\infty}\sum_{k=0}^n \frac{1}{a_k} = 0 \quad\text{and}\quad \sum_{k=0}^\infty \frac{1}{a_k^2} < \infty.$$

Hence K satisfies the assumptions of Theorem 2. Since

$$E(s) = \lim_{n\to\infty} E_n(s)$$

we can use Theorem 2 for the proof of the following proposition.

PROPOSITION 3. *Let $g : \mathbb{R} \to X$ be bounded and continuous. Then $K * g \in C^\infty(\mathbb{R}, X)$ and, for all $s \in \mathbb{R}$,*

$$\lim_{n\to\infty} E_n^D[K * g](s) = g(s).$$

Proof. We first consider a real-valued bounded and continuous function $g : \mathbb{R} \to \mathbb{R}$. Since K satisfies the assumptions of Theorem 2 it follows that, for all $s \in \mathbb{R}$,

$$\lim_{n\to\infty} E_n^D[K * g](s) = g(s). \tag{4}$$

In order to prove the conclusion for X-valued functions we make the following observations:

(a) Let $K_n = E_n^D[K]$ for $n = 0, 1, 2, \ldots$ By induction it can be easily proven that

$$K_n(u) = c_n\frac{e^{(2n+1)u}}{(e^{2u}+1)^{2n+1}},$$

where c_n is a positive constant depending only on n. In particular, K_n is positive for all n.

(b) Let $\widehat{K}_n$ denote the Fourier transform of K_n. Then, by (3),

$$\widehat{K}_n(\omega) = \widehat{E_n^D[K]}(\omega) = E_n(i\omega)\widehat{K}(\omega) = \frac{E_n(i\omega)}{\cos(i\omega\pi/2)}.$$

Consequently, $\int_{-\infty}^\infty K_n(t)\,dt = \widehat{K}(0) = 1$. Since, by (a), K_n is positive we have $\|K_n\|_{L_1} = 1$.

(c) Since K_n belongs to $L_1(\mathbb{R})$ for all $n \in \mathbb{N}$ it follows that

$$E_n^D[K * g] = E_n^D[K] * g = K_n * g.$$

If $g : \mathbb{R} \to X$ is bounded and continuous then $K * g$ belongs to $C^\infty(\mathbb{R}, X)$. For $u, s \in \mathbb{R}$ define $\tau_s(u) = \|g(s) - g(s+u)\|$. Then $\tau_s : \mathbb{R} \to \mathbb{R}$ is bounded and continuous. So we may conclude from (a)–(c) together with (4) that

$$\limsup_{n\to\infty} \|g(s) - E_n^D[K * g](s)\|$$
$$= \limsup_{n\to\infty} \Big\| \int_{-\infty}^{\infty} K_n(u)(g(s) - g(s-u))\,du \Big\|$$
$$\le \lim_{n\to\infty} \int_{-\infty}^{\infty} K_n(u)\tau_s(-u)\,dt = \lim_{n\to\infty} K_n * \tau_s(0) = \tau(0) = 0. \ \blacksquare$$

In order to deduce Theorem 1 from Proposition 3 we note that $\Gamma(\Delta F) = D(\Gamma F)$ if $F \in C^1((0,\infty), X)$, and

$$\Gamma\left(E_n^{\Delta}[F]\right) = E_n^D[\Gamma F] \tag{5}$$

for $f \in C^{2n}((0,\infty), X)$.

Proof of Theorem 1. Let $f : (0,\infty) \to X$ be bounded and continuous. Then $F = \mathcal{C} f$ belongs to $C^\infty((0,\infty), X)$, and by (5),

$$\Gamma(E_n^{\Delta}[F]) = E_n^D[\Gamma F] = E_n^D[K * \Gamma f].$$

Since $\Gamma f : \mathbb{R} \to X$ is bounded and continuous we can apply Proposition 3 to Γf. Hence

$$\lim_{n\to\infty} E_n^{\Delta}[F](s) = \lim_{n\to\infty} \Gamma(E_n^{\Delta}[F])(\log s)$$
$$= \lim_{n\to\infty} E_n^D[K * \Gamma f](\log s) = \Gamma f(\log s) = f(s)$$

for all $s \in (0,\infty)$. ■

In the following section we need the injectivity of $\mathcal{C}$ on $L_\infty([0,\infty), X)$. Therefore, we prove the following corollary to Proposition 3.

COROLLARY 4. *Let $f \in L_\infty([0,\infty), X)$. If $\mathcal{C} f = 0$ then $f = 0$.*

Proof. Since $\Gamma : L_\infty([0,\infty), X) \to L_\infty(\mathbb{R}, X)$ is an isometric isomorphism, and $\Gamma(\mathcal{C} f) = K * \Gamma f$ for $f \in L_\infty([0,\infty), X)$, it is sufficient to prove that $K * g = 0$ implies $g = 0$ for $g \in L_\infty(\mathbb{R}, X)$. If $K * g = 0$ then, for all $h \in L_1(\mathbb{R})$,

$$0 = (K * g) * h = K * (g * h).$$

Since $g * h : \mathbb{R} \to X$ is bounded and continuous, Proposition 3 implies that

$$0 = g * h(0) = \int_{-\infty}^{\infty} g(t)h(-t)\,dt \quad \text{for all } h \in L_1(\mathbb{R}).$$

Consequently, $g = 0$. ■

The inversion formula in Theorem 1 is the key for a characterization of those functions $F : (0,\infty) \to X$ which have a representation

$$F(s) = \frac{2}{\pi} \int_0^{\infty} \frac{s}{s^2 + t^2}\,d\phi(t), \qquad s \in (0,\infty),$$

where $\phi : [0,\infty) \to X$ is Lipschitz-continuous. Our next task is to state and prove such a characterization. To this end, we need some more notations, and we recall some facts about vector-valued Lipschitz-continuous functions, which may be found in [13, Chapter 1, Section 3].

For Lipschitz-continuous functions $\phi : [0,\infty) \to X$ we introduce the Lipschitz norm

$$\|\phi\|_{\mathrm{Lip}} = \sup\left\{ \frac{\|\phi(s) - \phi(t)\|}{s - t} : 0 \le s < t < \infty \right\}. \tag{6}$$

By $\mathrm{Lip}([0,\infty), X)$ we denote the space of all Lipschitz-continuous functions $\phi : [0,\infty) \to X$ with $\phi(0) = 0$. The space $\mathrm{Lip}([0,\infty), X)$ supplied with the norm defined in (6) is a Banach space. Moreover, we have the following proposition (see e.g. [13, Proposition 1.3.5]).

PROPOSITION 5. *The mapping which assigns to $\phi \in \mathrm{Lip}([0,\infty), X)$ the operator $T_\phi : L_1([0,\infty)) \to X$ defined by*

$$T_\phi h = \int_0^{\infty} h(t)\,d\phi(t)$$

is an isometric isomorphism.

If $\psi : \Omega \to X$, $\Omega \subseteq \mathbb{R}$, is any function, and if $x^* \in X^*$, then $x^* \circ \psi$ stands for the scalar-valued function given by $x^* \circ \psi(t) = x^*(\psi(t))$, $t \in \Omega$.

THEOREM 6. *Let $F : (0,\infty) \to X$ be any function, and let M be a positive real number. Then the following two assertions are equivalent*:

(i) *There exists $\phi \in \mathrm{Lip}([0,\infty), X)$, with $\|\phi\|_{\mathrm{Lip}} \le M$, such that, for all $s > 0$,*

$$F(s) = \frac{2}{\pi} \int_0^{\infty} \frac{s}{s^2 + t^2}\,d\phi(t). \tag{7}$$

(ii) *$F \in C^\infty((0,\infty), X)$ and*

$$\sup_{n\in\mathbb{N}\cup\{0\}} \|E_n^{\Delta}[F]\|_\infty \le M. \tag{8}$$

Proof. (i)⇒(ii). Let $\phi \in \mathrm{Lip}([0,\infty), X)$ have Lipschitz norm equal to M. Then F defined by (7) belongs to $C^\infty((0,\infty), X)$. In order to prove (8) it is sufficient to show $\sup_{n\in\mathbb{N}} \|E_n^{\Delta}[x^* \circ F]\|_\infty \le M$ for all $x^* \in X^*$ with $\|x^*\| \le 1$. If $x^* \in X^*$ has norm less than or equal to one then $x^* \circ \phi$ is a scalar-valued Lipschitz-continuous function with $\|x^* \circ \phi\|_{\mathrm{Lip}} \le M$. Hence,

$x^* \circ \phi$ has a Radon–Nikodym derivative f_{x^*} with $\|f_{x^*}\|_\infty \le M$. Moreover, for $s \in (0,\infty)$,

$$x^* \circ F(s) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+t^2}\, d(x^* \circ \phi)(t) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+t^2} f_{x^*}(t)\, dt = \mathcal{C} f_{x^*}(s).$$

Therefore, we have to show $\|E_n^\Delta[\mathcal{C} f_{x^*}]\|_\infty \le M$. But by (5), this estimate is an immediate consequence of

$$\|E_n^D[\Gamma(\mathcal{C} f_{x^*})]\|_\infty = \|K_n * \Gamma f_{x^*}\|_\infty \le \|K_n\|_{L_1}\|\Gamma f_{x^*}\|_\infty \le M.$$

(ii)⇒(i). Let $F \in C^\infty((0,\infty),X)$ satisfy (8). Then for $n \in \mathbb{N}\cup\{0\}$, the operators $T_n : L_1([0,\infty)) \to X$ defined by

$$T_n h = \int_0^\infty h(t) E_n^\Delta[F](t)\, dt$$

each have norm less than or equal to M. We claim that the family (T_n) converges pointwise to an operator $T : L_1([0,\infty)) \to X$ with $\|T\| \le M$. To see this we rewrite $T_n h$ in the following way:

$$T_n h = \int_0^\infty h(t) E_n^\Delta[F](t)\, dt = \int_{-\infty}^\infty e^u h(e^u) E_n^\Delta[F](e^u)\, du = \int_{-\infty}^\infty \Gamma_1 h(u) E_n^D[\Gamma F](u)\, du,$$

where $\Gamma_1 : L_1([0,\infty)) \to L_1(\mathbb{R})$ is given by $\Gamma_1 h(u) = e^u h(e^u)$. Since Γ_1 is an isometric isomorphism it is enough to show that the operators $S_n : L_1(\mathbb{R}) \to X$ given by $S_n g = \int_{-\infty}^\infty g(u) E_n^D[\Gamma F](u)\, du$ converge towards an operator $S : L_1(\mathbb{R}) \to X$. To see this, take $s \in \mathbb{R}$ and consider $K_s(u) = K(s-u)$. Then, by Proposition 3,

$$\lim_{n\to\infty} S_n K_s = \lim_{n\to\infty} \int_{-\infty}^\infty K(s-u) E_n^D[\Gamma F](u)\, du = \lim_{n\to\infty} K * E_n^D[\Gamma F](s) = \lim_{n\to\infty} K_n * \Gamma F(s) = \Gamma F(s).$$

Hence, $S_n g$ converges for all g in the subset $\kappa = \{K_s : s \in \mathbb{R}\} \subseteq L_1(\mathbb{R})$. We know from (3) that the Fourier transform of K has no zeros. Hence, by Wiener's Tauberian theorem [15, Theorem XI.16.3] it follows that κ is total in $L_1(\mathbb{R})$. In addition, the family (S_n) is bounded, since $\|S_n\| = \|T_n\| \le M$. Hence, by the uniform boundedness principle, (S_n) converges pointwise to an operator $S : L_1(\mathbb{R}) \to X$. In particular, $SK_s = \Gamma F(s)$. Consequently, (T_n) converges pointwise to an operator $T : L_1([0,\infty)) \to X$ with $\|T\| \le M$, and S and T are related by $Th = S(\Gamma_1 h)$.

Now, by Proposition 5, there exists $\phi \in \mathrm{Lip}([0,\infty),X)$, with $\|\phi\|_{\mathrm{Lip}} \le \|T\| \le M$, such that T has a representation

$$Th = \int_0^\infty h(t)\, d\phi(t), \quad h \in L_1([0,\infty)).$$

Let

$$k_s(t) = \frac{2}{\pi}\frac{s}{s^2+t^2}.$$

Then $\Gamma_1 k_s(u) = K_{\log s}(u)$. Consequently,

$$F(s) = \Gamma F(\log s) = SK_{\log s} = S(\Gamma_1 k_s) = Tk_s = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+t^2}\, d\phi(t). \ ■$$

2. A characterization of uniformly bounded cosine functions. Let us first recall the following definitions: A mapping $T(\cdot) : (0,\infty) \to \mathbf{L}(X)$ has the *semigroup property* if

$$T(t+u) = T(t)T(u), \quad t,u > 0,$$

and $T(\cdot)$ is a C_0-*semigroup* if, in addition, $T(\cdot)$ is strongly continuous in $[0,\infty)$ and $T(0) = \mathrm{Id}$. A mapping $C(\cdot) : \mathbb{R} \to \mathbf{L}(X)$ satisfies the *cosine functional equation* if

$$(9) \qquad C(t)C(u) = \tfrac{1}{2}[C(t+u) + C(t-u)], \quad t,u \in \mathbb{R},$$

and $S(\cdot) : \mathbb{R} \to \mathbf{L}(X)$ satisfies the *sine functional equation* if S is strongly measurable with

$$(10) \qquad S(t)S(u) = \frac{1}{2}\int_0^u [S(t+\sigma) + S(t-\sigma)]\, d\sigma, \quad t,u \in \mathbb{R}.$$

If, in addition to (9), $C(\cdot)$ is strongly continuous with $C(0) = \mathrm{Id}$ then $C(\cdot)$ is a *cosine function*. $S(\cdot)$ is a *sine function* if, in addition to (10), $S(\cdot)$ is non-degenerate, that is, $S(t)x = 0$ for all $t \in \mathbb{R}$ implies $x = 0$.

If $C(\cdot)$ is a cosine function, then the *generator* A of $C(\cdot)$ is defined by

$$D(A) = \{x \in X : C(\cdot)x \in C^2(\mathbb{R},X)\}, \quad Ax = C''(0)x \quad \text{for } x \in D(A).$$

The *generator* A of a sine function $S(\cdot)$ is given by the condition that x belongs to $D(A)$ if and only if there exists $y \in X$ such that, for all $\tau \in \mathbb{R}$,

$$(11) \qquad S(\tau)x = \tau x + \int_0^\tau (\tau - \sigma) S(\sigma) y\, d\sigma.$$

In this case $Ax = y$. Note that y is uniquely determined by (11) since $S(\cdot)$ is non-degenerate. If we assume that the sine function is exponentially bounded, with densely defined generator, one can provide equivalent definitions using the Laplace transform (see [9] and [12]).

If $C:\mathbb{R}\to \mathbf{L}(X)$ is strongly continuous and even, and if $S:\mathbb{R}\to\mathbf{L}(X)$ is defined by

$$S(t)=\int_0^t C(\tau)\,d\tau,\qquad t\in\mathbb{R},$$

then it follows by straightforward calculations that $C(\cdot)$ satisfies the cosine functional equation if and only if $S(\cdot)$ satisfies the sine functional equation. Consequently, $C(\cdot)$ is a cosine function if and only if $S(\cdot)$ is a sine function. In this case the generators of $C(\cdot)$ and $S(\cdot)$ are the same.

Let A be the generator of a bounded C_0-semigroup $T(\cdot)$. Then, by [15, Chapter IX.11] (see also the Introduction), we can define $B=-(-A)^{1/2}$, and B is the generator of a bounded C_0-semigroup $T_B(\cdot)$. If A generates a cosine function $C(\cdot)$ then we have the fundamental relation (see the Introduction)

$$T_B(t)=\frac{2}{\pi}\int_0^\infty \frac{t}{t^2+\tau^2}C(\tau)\,d\tau, \tag{12}$$

and if $S(\cdot)$ is the sine function generated by A then

$$T_B(t)=\frac{2}{\pi}\int_0^\infty \frac{t}{t^2+\tau^2}\,dS(\tau). \tag{13}$$

Unless otherwise stated, integrals involving operator-valued functions will be understood in the strong operator topology henceforth. Our main goal in this section is to show that the converse of the above assertion holds. More precisely,

THEOREM 7. *Let A be the generator of a bounded C_0-semigroup and let $T_B(\cdot)$ be the C_0-semigroup generated by $B=-(-A)^{1/2}$. Then A generates a bounded cosine function if and only if there exists a strongly Lipschitz-continuous function* $S(\cdot):[0,\infty)\to\mathbf{L}(X)$ such that

$$T_B(t)=\frac{2}{\pi}\int_0^\infty \frac{t}{t^2+\tau^2}\,dS(\tau),\qquad t>0. \tag{14}$$

Before we prove Theorem 7 we need a couple of lemmas and propositions, and we make a few remarks.

REMARK 8. (i) If $F:\mathbb{R}\to\mathbf{L}(X)$ is a strongly Lipschitz-continuous function then, as a consequence of the uniform boundedness principle, F is Lipschitz-continuous with respect to the operator norm. Therefore, it is enough to prove Theorem 7 for Lipschitz-continuous sine functions.

(ii) If the densely defined operator A generates a Lipschitz-continuous sine function $S(\cdot)$ then A generates a bounded strongly continuous analytic semigroup $T(\cdot)$ given by

$$T(t)x=\frac{1}{2\sqrt{\pi}t^{3/2}}\int_0^\infty e^{-\tau^2/(4t)}\tau S(\tau)x\,d\tau \tag{15}$$

(see Arendt–Kellermann [3]). If we proceed as in the Introduction, we find that the semigroup $T_B(\cdot)$ generated by the negative square root B of A has the representation

$$T_B(t)=\frac{4}{\pi}\int_0^\infty \frac{\tau t}{(t^2+\tau^2)^2}S(\tau)\,d\tau. \tag{16}$$

It is well known that there are operators that generate sine functions but do not generate cosine functions (see [3], [9] and [5]). Proposition 10 below states that the semigroup property corresponds to the cosine functional equation via (12) and to the sine functional equation via (13).

(iii) In the case where X has the Radon–Nikodym property (see [13] or [1]), the assumption on $S(\cdot)$ implies the existence of a derivative $S'(\cdot)=C(\cdot)$ which is bounded. The cosine functional equation for $C(\cdot)$ combined with strong measurability implies that $C(\cdot)$ is strongly continuous (see [6, Theorem 1.1, p. 24] or [11]; these results extend the corresponding facts for the semigroup functional equation [10]).

For our further investigations it is useful to introduce the Poisson kernels

$$P_s(\sigma)=\frac{1}{\pi}\frac{s}{s^2+\sigma^2},\qquad s>0,\ \sigma\in\mathbb{R}.$$

We note that the family (P_s) has the following semigroup property:

$$P_s*P_t=P_{s+t},\qquad s,t>0. \tag{17}$$

If f is bounded and measurable on $\mathbb{R}$ then we let

$$\mathcal{P}f(t)=\int_{-\infty}^\infty P_t(\tau)f(\tau)\,d\tau,\qquad t\in\mathbb{R}.$$

We note that $\mathcal{P}f=0$ implies $f=0$ if $f\in L_\infty(\mathbb{R},X)$ is even. This follows from Corollary 4 since, for even functions $f\in L_\infty(\mathbb{R},X)$,

$$\mathcal{P}f(t)=2\int_0^\infty P_t(\tau)f(\tau)\,d\tau=(\mathcal{C}f|_{[0,\infty)})(t).$$

In the sequel we write $Q_t=-P_t'$.

LEMMA 9. *If $f:\mathbb{R}\to X$ is odd and Lipschitz-continuous, and if*

$$\int_{-\infty}^{\infty} Q_t(\tau)f(\tau)\,d\tau = 0 \quad \text{for all } t>0, \tag{18}$$

then $f(\tau)=0$ for all $\tau\in\mathbb{R}$.

Proof. It is enough to prove the lemma for scalar-valued functions. Then the vector-valued case follows by applying the Hahn–Banach theorem. Let f be an odd, scalar-valued Lipschitz-continuous function with the property (18). Then f has an even, bounded Radon–Nikodym derivative f'. By partial integration it follows that

$$0=\int_{-\infty}^{\infty} Q_t(\tau)f(\tau)\,d\tau=\int_{-\infty}^{\infty} P_t(\tau)f'(\tau)\,d\tau.$$

Since the operator $\mathcal{P}$ is injective on even functions we conclude that $f'=0$. Consequently, f is constant. But a constant function which is odd must be 0. ∎

PROPOSITION 10. *Let $T(\cdot):[0,\infty)\to\mathbf{L}(X)$ be bounded and strongly continuous.*

(i) *If $C(\cdot):\mathbb{R}\to\mathbf{L}(X)$ is bounded, strongly continuous and even, and if $C(\cdot)$ and $T(\cdot)$ are related by*

$$T(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{t^2+\tau^2}C(\tau)\,d\tau,\qquad t>0,$$

then $T(\cdot)$ has the semigroup property if and only if $C(\cdot)$ satisfies the cosine functional equation. Moreover, $T(0)=C(0)$.

(ii) *If $S(\cdot):\mathbb{R}\to\mathbf{L}(X)$ is strongly Lipschitz-continuous and odd, and if $S(\cdot)$ and $T(\cdot)$ are related by*

$$T(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{t^2+\tau^2}\,dS(\tau),\qquad t>0,$$

then $T(\cdot)$ has the semigroup property if and only if $S(\cdot)$ satisfies the sine functional equation.

Proof. We first prove (ii). By partial integration it follows that

$$T(t)=\int_{-\infty}^{\infty}P_t(\tau)\,dS(\tau)=\int_{-\infty}^{\infty}Q_t(\tau)S(\tau)\,d\tau.$$

Consequently,

$$T(s)T(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q_s(\sigma)Q_t(\tau)\,S(\sigma)S(\tau)\,d\tau\,d\sigma.$$

The semigroup property of the Poisson kernels gives

$$Q_{s+t}(\tau)=-\frac{d}{d\tau}P_{s+t}(\tau)=-\frac{d}{d\tau}(P_s*P_t)(\tau)=(Q_s*P_t)(\tau).$$

Since S and Q_s are odd it follows that

$$\begin{aligned}T(s+t)&=\int_{-\infty}^{\infty}Q_{s+t}(\varrho)S(\varrho)\,d\varrho\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q_s(\varrho-\tau)P_t(\tau)\,d\tau\,S(\varrho)\,d\varrho\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q_s(\sigma)P_t(\tau)S(\sigma+\tau)\,d\tau\,d\sigma\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q_s(\sigma)P_t(\tau)\frac{1}{2}[S(\sigma+\tau)+S(\sigma-\tau)]\,d\tau\,d\sigma.\end{aligned}$$

Integrating the right hand side of the above equation by parts gives

$$T(s+t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q_s(\sigma)Q_t(\tau)\left(\frac{1}{2}\int_0^{\tau}[S(\sigma+\varrho)+S(\sigma-\varrho)]\,d\varrho\right)d\tau\,d\sigma.$$

If $S(\cdot)$ satisfies the sine functional equation then it follows directly that $T(\cdot)$ has the semigroup property in $(0,\infty)$. That $T(\cdot)$ has the semigroup property in the closed interval $[0,\infty)$ follows from the strong continuity of $T(\cdot)$.

Conversely, if $T(\cdot)$ has the semigroup property then we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q_s(\sigma)Q_t(\tau)\,S(\sigma)S(\tau)\,d\tau\,d\sigma=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q_s(\sigma)Q_t(\tau)\left(\frac{1}{2}\int_0^{\tau}[S(\sigma+\varrho)+S(\sigma-\varrho)]\,d\varrho\right)d\tau\,d\sigma$$

for all $s,t\ge 0$. Since the functions

$$(\sigma,\tau)\mapsto\int_0^{\tau}[S(\sigma+\varrho)+S(\sigma-\varrho)]\,d\varrho\quad\text{and}\quad(\sigma,\tau)\mapsto S(\sigma)S(\tau)$$

are odd in σ for τ fixed, and in τ for σ fixed, it follows from Lemma 9 that

$$S(\sigma)S(\tau)=\frac{1}{2}\int_0^{\tau}[S(\sigma+\varrho)+S(\sigma-\varrho)]\,d\varrho,$$

whence $S(\cdot)$ satisfies the sine functional equation.

(i) Define $S(\cdot):\mathbb{R}\to \mathbf{L}(X)$ by

$$S(t)=\int_0^t C(\tau)\,d\tau.$$

Since $C(\cdot)$ is even it follows that $S(\cdot)$ satisfies the sine functional equation if and only $C(\cdot)$ satisfies the cosine functional equation. Moreover,

$$T(t)=\int_{-\infty}^{\infty} P_t(\tau)C(\tau)\,d\tau=\int_{-\infty}^{\infty} P_t(\tau)\,dS(\tau).$$

Hence it follows from (ii) that $C(\cdot)$ satisfies the cosine functional equation if and only if $T(\cdot)$ has the semigroup property.

Moreover, since the family of Poisson kernels (P_t) is an approximate identity it follows that

$$T(0)=\lim_{t\to 0^+}T(t)=\lim_{t\to 0^+}\int_{-\infty}^{\infty} P_t(\tau)C(\tau)\,d\tau=C(0). \ \blacksquare$$

If A generates an integrated semigroup $U(\cdot)$ then, for all $x\in X$ and $\tau>0$,

$$\int_0^\tau U(\sigma)x\,d\sigma\in D(A)\quad\text{and}\quad U(\tau)x=\tau x+A\int_0^\tau U(\sigma)x\,d\sigma$$

(see Arendt [1, Proposition 3.3]). If we consider sine functions instead of integrated semigroups then, by Arendt [2], we obtain the following result.

LEMMA 11. *Let $S(\cdot)$ be a sine function with generator A. Then, for all $x\in X$ and $\tau\in\mathbb{R}$,*

$$(19)\qquad \int_0^\tau(\tau-\sigma)S(\sigma)x\,d\sigma\in D(A),\qquad S(\tau)x=\tau x+A\int_0^\tau(\tau-\sigma)S(\sigma)x\,d\sigma.$$

Proof. Let $\tau\in\mathbb{R}$, $x\in X$ and set $x_\tau=\int_0^\tau(\tau-\sigma)S(\sigma)x\,d\sigma$. Then

$$\begin{aligned}S(t)x_\tau&=S(t)\int_0^\tau(\tau-\sigma)S(\sigma)x\,d\sigma\\&=\frac12\int_0^\tau(\tau-\sigma)\int_0^\sigma[S(t+\varrho)+S(t-\varrho)]x\,d\varrho d\sigma\\&=\frac12\int_0^\tau(\tau-\sigma)\Big[\int_t^{t+\sigma}S(\varrho)x\,d\varrho-\int_t^{t-\sigma}S(\varrho)x\,d\varrho\Big]\,d\sigma\\&=\frac12\int_0^\tau(\tau-\sigma)\int_{t-\sigma}^{t+\sigma}S(\varrho)x\,d\varrho\,d\sigma.\end{aligned}$$

It follows that

$$(20)\qquad \frac{d}{dt}S(t)x_\tau=\frac12\int_0^\tau(\tau-\sigma)[S(t+\sigma)-S(t-\sigma)]x\,d\sigma.$$

In particular, $S'(0)x_\tau=x_\tau$. From (20) we infer

$$\frac{d}{dt}S(t)x_\tau=\frac12\int_t^{t+\tau}(\tau+t-\sigma)S(\sigma)x\,d\sigma+\frac12\int_t^{t-\tau}(\tau-t+\sigma)S(\sigma)x\,d\sigma,$$

whence

$$\begin{aligned}\frac{d^2}{dt^2}S(t)x_\tau&=\frac12\Big[\int_t^{t+\tau}S(\sigma)x\,d\sigma-\tau S(t)x-\int_t^{t-\tau}S(\sigma)\,d\sigma-\tau S(t)x\Big]\\&=\frac12\int_{t-\tau}^{t+\tau}S(\sigma)x\,d\sigma-\tau S(t)x=[S(\tau)-\tau]S(t)x.\end{aligned}$$

Therefore,

$$\begin{aligned}S(t)x_\tau&=\int_0^t(t-\sigma)S''(\sigma)x_\tau\,d\sigma+tS'(0)x_\tau+S(0)x_\tau\\&=tx_\tau+\int_0^t(t-\sigma)S(\sigma)[S(\tau)-\tau]x\,d\sigma.\end{aligned}$$

Consequently, $x_\tau\in D(A)$ and $Ax_\tau=S(\tau)x-\tau x$. $\blacksquare$

PROPOSITION 12. *Let B generate a C_0-semigroup $T_B(\cdot)$ on X and let A be the generator of a strongly Lipschitz-continuous sine function $S(\cdot)$. If*

$$(21)\qquad T_B(t)=\frac1\pi\int_{-\infty}^{\infty}\frac{t}{\tau^2+t^2}\,dS(\tau),\qquad t>0,$$

then $B^2=-A$.

Proof. T_B is infinitely often differentiable in $t>0$; this follows easily from the representation (21) (actually, $T_B(\cdot)$ is analytic). Hence $T_B(t)x$ belongs to $D(B^n)$ for all $t>0$, $x\in X$, $n\in\mathbb{N}$, and

$$B^nT_B(t)x=\frac{d^n}{dt^n}T_B(t)x.$$

In order to prove that $B^2=-A$, we use integration by parts combined with the estimates $\|S(\tau)x\|\le M\tau\|x\|$ and $\|\int_0^\tau S(\varrho)x\,d\varrho\|\le M\tau^2\|x\|$ for some number $M>0$, and the fundamental formula of Lemma 11, equation (19), for sine function generators:

$$B^2T_B(t)x=\frac{d^2}{dt^2}T_B(t)x=\int_{-\infty}^{\infty}\frac{d^2}{dt^2}P_t(\tau)\,dS(\tau)x$$

$$= \int_{-\infty}^{\infty} -\frac{d^2}{d\tau^2} P_t(\tau)\, d\Big(\tau x + A\int_0^{\tau} (\tau-\sigma) S(\sigma)x\, d\sigma\Big)$$
$$= -A \int_{-\infty}^{\infty} \frac{d^2}{d\tau^2} P_t(\tau) \Big(\int_0^{\tau} S(\sigma)x\, d\sigma\Big)\, d\tau = A \int_{-\infty}^{\infty} \frac{d}{d\tau} P_t(\tau) S(\tau)x\, d\tau$$
$$= -A \int_{-\infty}^{\infty} P_t(\tau)\, dS(\tau)x = -AT_B(t)x.$$

Let $x \in D(B^2)$. Then

$$\lim_{t\to 0^+} -AT_B(t)x = \lim_{t\to 0^+} B^2 T_B(t)x = \lim_{t\to 0^+} T_B(t)B^2 x = B^2 x.$$

Since A is closed and $\lim_{t\to 0^+} T_B(t)x = x$ it follows that $x \in D(A)$ and $-Ax = B^2 x$. Conversely, if $x \in D(A)$ then

$$\lim_{t\to 0^+} B^2 T_B(t)x = -\lim_{t\to 0^+} AT_B(t)x = -\lim_{t\to 0^+} A \int_{-\infty}^{\infty} P_t(\tau)\, dS(\tau)x$$
$$= -\lim_{t\to 0^+} \int_{-\infty}^{\infty} P_t(\tau)\, dS(\tau)Ax = -\lim_{t\to 0^+} T_B(t)Ax = -Ax.$$

Consequently, the closedness of B^2 implies that $x \in D(B^2)$ and $B^2 x = -Ax$. ∎

Now we are in a position to prove the main theorem (Theorem 7).

Proof of Theorem 7. Assume first that A generates a bounded cosine function $C(\cdot)$. Then A is the generator of a sine function $S(\cdot)$ which is given by

$$S(t) = \int_0^t C(\tau)\, d\tau.$$

Hence, since $C(\cdot)$ is bounded, $S(\cdot)$ is Lipschitz-continuous, and (14) follows from (12).

Conversely, assume that there exists a Lipschitz-continuous function $S(\cdot) : [0,\infty) \to \mathbf{L}(X)$ such that $T_B(\cdot)$ and $S(\cdot)$ are related by (14). We may assume without loss of generality that $S(0) = 0$. Then $S(\cdot)$ can be extended to an odd, strongly Lipschitz-continuous function $S : \mathbb{R} \to \mathbf{L}(X)$ by putting $S(t) = -S(-t)$ for $t < 0$. Then

$$T_B(t) = \frac{2}{\pi}\int_0^{\infty} P_t(\tau)\, dS(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty} P_t(\tau)\, dS(\tau).$$

Therefore, Proposition 10 implies that $S(\cdot)$ satisfies the sine functional equation. Moreover, if $S(t)x = 0$ for all $t \in \mathbb{R}$ then it follows from (14) that $T(t)x = 0$ for all $t > 0$, whence $x = 0$. Consequently, $S(\cdot)$ is a sine function, which, by Proposition 12, is generated by $-B^2 = A$.

It remains to show that $S(\cdot)$ has a strong derivative $C(\cdot)$. Let $x \in D(A)$. Then

$$S(t)x = tx + \int_0^t (t-\tau)S(\tau)Ax\, d\tau.$$

Hence $S(t)x$ is continuously differentiable and we can define

$$\Phi(x)(t) = S'(t)x = x + \int_0^t S(\tau)Ax\, d\tau, \quad t \in \mathbb{R}.$$

Since $S(\cdot)$ is Lipschitz-continuous we have

$$(22) \qquad \|\Phi(x)\|_\infty = \|S(\cdot)x\|_{\mathrm{Lip}} \le \|S(\cdot)\|_{\mathrm{Lip}}\|x\|.$$

Hence $\Phi : D(A) \to C_{\mathrm{b}}(\mathbb{R}, X)$ is a bounded linear operator. Consequently, Φ has a unique bounded linear extension to $\overline{D(A)} = X$. Define $C(t)x = \Phi(x)(t)$. Then, for every $t \in \mathbb{R}$,

$$\sup_{\|x\|\le 1} \|C(t)x\| \le \|S(\cdot)\|_{\mathrm{Lip}}.$$

Hence, $C(t) \in \mathbf{L}(X)$ for each $t \in \mathbb{R}$, and $C(\cdot) : \mathbb{R} \to \mathbf{L}(X)$ is bounded and strongly continuous. Moreover, $C(\cdot)$ is a cosine function, since $S(\cdot)$ is a sine function, and $C(\cdot)$ is generated by A since $S(\cdot)$ is. ∎

Combining Theorems 1, 6 and 7 we obtain the following

COROLLARY 13. *Let A be the generator of a bounded C_0-semigroup, and let $B = -(-A)^{1/2}$ generate the semigroup $T_B(\cdot)$. Then A generates a bounded cosine function if and only if there exists $M > 0$ such that*

$$\|E_n^{\Delta}[T_B](t)\| \le M \quad \textit{for all } n = 0,1,2,\ldots \textit{ and } t > 0.$$

In this case, the cosine function $C(\cdot)$ generated by A is given by

$$C(t)x = C(-t)x = \lim_{n\to\infty} E_n^{\Delta}[T_B](t)x, \quad t \ge 0,\ x \in X.$$

We now provide an explicit description of $E_n^{\Delta}[T_B](t)$. We claim first that $E_n^{\Delta}[T_B](t) = p_n(tB)T_B(t)$, where p_n is a polynomial of degree $2n$. This statement is certainly true for $n = 0$, with $p_0(t) = 1$. For any polynomial p let us define $(\Phi p)(t) = t[p(t) + p'(t)]$. If the statement holds for $n > 0$ then

$$\Delta E_n^{\Delta}[T_B](t) = \Delta p_n(tB)T_B(T) = t[Bp_n'(tB)T_B(t) + p_n(tB)BT_B(t)]$$
$$= (\Phi p_n)(tB)T_B(t).$$

Consequently, $E^{\Delta}_{n+1}[T_B](t) = p_{n+1}(tB)T_B(t)$, where

$$p_{n+1} = \left(1 - \frac{\Phi^2}{(2n+1)^2}\right)p_n = \prod_{k=0}^{n}\left(1 - \frac{\Phi^2}{(2k+1)^2}\right)p_0 = E^{\Phi}_n[p_0]$$

is a polynomial of degree $2n+2 = 2(n+1)$.

Secondly, we describe the p_n's explicitly. Let $p_n(t) = a_{2n}t^{2n} + a_{2n-1}t^{2n-1} + \ldots + a_1 t + a_0$. The polynomial p_n is uniquely determined by the equation

$$E^{\Delta}_n[e^t] = p_n(t)e^t = \sum_{j=0}^{2n} a_j t^j \cdot \sum_{l=0}^{\infty}\frac{t^l}{l!} = \sum_{l=0}^{\infty} b_l t^l, \tag{23}$$

where $b_l = \sum_{j=0}^{\min(l,2n)} a_j/(l-j)!$. On the other hand, since $\Delta(t^l) = lt^l$ we have

$$E^{\Delta}_n[e^t] = \sum_{l=0}^{\infty}\frac{E^{\Delta}_n[t^l]}{l!} = \sum_{l=0}^{\infty}\frac{t^l}{l!}\prod_{k=0}^{n-1}\left(1 - \frac{l^2}{(2k+1)^2}\right) = \sum_{l=0}^{\infty} c_l t^l, \tag{24}$$

where $c_l = E_n(l)/l!$. Combining (23) and (24) we have

$$\sum_{j=0}^{l}\frac{a_j}{(l-j)!} = c_l, \qquad l = 0, 1, \ldots, 2n. \tag{25}$$

Let $\alpha = (a_0, \ldots, a_{2n})$ and $\gamma = (c_0, \ldots, c_{2n})$. Then (25) may be written as $\mathcal{A}\alpha = \gamma$, where

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & 0 \\ 1 & 1 & 0 & \cdot & \cdot & \cdots & \cdot & 0 \\ 1/2 & 1 & 1 & 0 & \cdot & \cdots & \cdot & 0 \\ 1/6 & 1/2 & 1 & 1 & 0 & \cdots & \cdot & 0 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & & & & 1 & 1 & 0 & \cdot \\ \cdot & & & & & 1 & 1 & 0 \\ 1/(2n)! & \cdot & \cdot & \cdot & \cdot & \cdots & 1 & 1 \end{pmatrix}.$$

Consequently, $\alpha = \mathcal{A}^{-1}\gamma$, where

$$\mathcal{A}^{-1} = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & 0 \\ -1 & 1 & 0 & \cdot & \cdot & \cdots & \cdot & 0 \\ 1/2 & -1 & 1 & 0 & \cdot & \cdots & \cdot & 0 \\ -1/6 & 1/2 & -1 & 1 & 0 & \cdots & \cdot & 0 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & & & & -1 & 1 & 0 & \cdot \\ \cdot & & & & & -1 & 1 & 0 \\ 1/(2n)! & \cdot & \cdot & \cdot & \cdot & \cdots & -1 & 1 \end{pmatrix}.$$

Since $c_1 = c_3 = \ldots = c_{2n-1} = 0$ we obtain the following representation of $E^{\Delta}_n[T_B](t)$:

PROPOSITION 14. *If $T_B(\cdot)$ is a differentiable semigroup which is generated by B, then*

$$E^{\Delta}_n[T_B](t) = [a_{2n}(tB)^{2n} + \ldots + a_1(tB) + a_0]T_B(t),$$

where

$$a_k = \frac{1}{k!}\sum_{l=0}^{[k/2]}\left[(-1)^k\binom{k}{2l}\prod_{j=0}^{n-1}\left(1 - \frac{(2l)^2}{(2j+1)^2}\right)\right], \qquad k = 0, 1, \ldots, 2n,$$

and $[k/2]$ denotes the greatest non-negative integer not exceeding $k/2$.

Finally, if we consider the Laplace operator on one of the spaces $L_p(\mathbb{R})$, $1 \le p < \infty$, $C_0(\mathbb{R})$ or $BUC(\mathbb{R})$ (with maximal distributional domain for $L_p(\mathbb{R})$, $1 \le p < \infty$), then the semigroup $T_B(\cdot)$ corresponds to the classical Poisson transform, for which an inversion theory has been set out in [13].

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Mapping properties of integral averaging operators

by

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Abstract. Characterizations are obtained for those pairs of weight functions u and v for which the operators $Tf(x) = \int_{a(x)}^{b(x)} f(t)\,dt$ with a and b certain non-negative functions are bounded from $L_u^p(0,\infty)$ to $L_v^q(0,\infty)$, $0 < p, q < \infty$, $p \ge 1$. Sufficient conditions are given for T to be bounded on the cones of monotone functions. The results are applied to give a weighted inequality comparing differences and derivatives as well as a weight characterization for the Steklov operator.

1. Introduction. In this paper we study mapping properties of the operator

$$Tf(x) = \int_{a(x)}^{b(x)} f(t)\,dt, \quad f \ge 0, \tag{1.1}$$

where a and b are increasing, differentiable functions satisfying $a(0) = b(0) = 0$, $a(x) < b(x)$ for $x \in (0,\infty)$ and $a(\infty) = b(\infty) = \infty$. Specifically, conditions on the weight functions u and v are given which are equivalent to

$$\Big(\int_0^\infty \Big(\int_{a(x)}^{b(x)} f\Big)^q v(x)\,dx\Big)^{1/q} \le C\Big(\int_0^\infty f^p u\Big)^{1/p}, \quad 0 < p, q < \infty. \tag{1.2}$$

For example (see Theorem 2.2), if $1 < p \le q < \infty$ then (1.2) holds if and only if

$$\sup \Big(\int_{a(x)}^{b(t)} u^{1-p'}\Big)^{1/p'} \Big(\int_t^x v\Big)^{1/q} = K < \infty, \tag{1.3}$$

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