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On Q-independence, limit theorems and q-Gaussian distribution

by

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Abstract. We formulate the notion of Q-independence which generalizes the classical independence of random variables and free independence introduced by Voiculescu. Here Q stands for a family of polynomials indexed by tiny partitions of finite sets. The analogs of the central limit theorem and Poisson limit theorem are proved. Moreover, it is shown that in some special cases this kind of independence leads to the q-probability theory of Bożejko and Speicher.

1. Introduction. In this paper we are concerned with a certain generalization of the classical notion of independence of random variables. The classical case describes properties of a commutative probability system, i.e. the set of complex measurable functions defined on a measurable space with a normalized positive measure. In [17] D. Voiculescu showed that in order to define a reasonable and essentially different independence one should consider more general concepts of random variables and probability systems.

DEFINITION 1.1. A *probability system* is a pair $(\mathcal{A},\varphi)$, where $\mathcal{A}$ is a unital C^*-algebra and φ is a state on $\mathcal{A}$.

Here $\mathcal{A}$ plays the role of a noncommutative analog of a set of complex random variables and φ is a "noncommutative" probability measure. One can define the distribution of an element of $\mathcal{A}$.

DEFINITION 1.2. Let $(\mathcal{A},\varphi)$ be a probability system and $a\in\mathcal{A}$. A functional $\widetilde{\mu}_a$ on the $*$-algebra $\mathbb{C}[X]$ of complex formal polynomials is called the *distribution* of a if

$$\widetilde{\mu}_a(P)=\varphi(P(a))$$

for every $P\in\mathbb{C}[X]$.

From the well-known Gelfand–Naimark theorem we easily get

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PROPOSITION 1.3. *If $(\mathcal{A},\varphi)$ is a probability system, $a\in\mathcal{A}$ and $a=a^*$, then there exists a probability measure μ_a on $\mathbb{R}$ such that* $\operatorname{supp}\mu_a\subset\sigma(a)$ *and*

$$\int_{\mathbb{R}} x^n\,\mu_a(dx)=\varphi(a^n),\qquad n=1,2,\ldots,$$

where $\sigma(a)$ denotes the spectrum of a.

Therefore, the hermitian elements of $\mathcal{A}$ play the role of real random variables. Observe that Definition 1.1 restricts our considerations to the case of random variables with all moments finite.

In [17, 18] D. Voiculescu introduced the concept of free independent systems of random variables. This allowed him to develop an alternative probability theory. Namely, one can consider the notion of free convolution and the analog of Fourier transform which linearizes that convolution. Moreover, one can prove the central limit theorem which involves the so-called semicircular Wigner distribution instead of the Gaussian measure. A review of the theory can be found in [19]. The combinatorial approach to these questions is presented in [13, 14, 12]. It is based on properties of the lattice of noncrossing partitions. In [6, 4] other notions of independence of noncommutative random variables are defined. They generalize the free independence.

At the same time, in [5] Bożejko and Speicher have constructed a family of probability systems which is indexed by a parameter $q\in[-1,1]$. The main ingredient of these systems is the so-called q-perturbed Fock space. Moreover, it can be considered as a modification of the main model for the free independence. The family interpolates between the classical case ($q=1$) and the free case ($q=0$). Moreover, systems for $q=1$ and $q=-1$ have a direct physical interpretation. They describe bosonic and fermionic quantum systems respectively. In [2, 7, 3, 10, 11] the foundations of q-probability theory based on the above constructions are presented. In particular, q-Gaussian random variables are described. However, a consistent definition of a proper notion of independence with the interpolating properties described above has not been formulated yet. Moreover, the results of [9] indicate strong obstructions to obtaining such a definition.

Other kinds of independence for noncommutative systems were described in [16].

In this paper we propose a combinatorial definition of Q-independence of noncommutative random variables. It depends on a family Q of polynomials which is indexed by all tiny partitions of finite sets. Q-independence reduces to the free or classical independence if Q is properly defined. Similar arguments to those in the free and classical independence theory lead to limit theorems and calculation of moments of Q-Gaussian and Q-Poisson distributions. Moreover, for the special choice of a family Q, this notion is consistent with the q-probability theory.

In Section 2 we set up notation, terminology and technical results about partitions of finite sets. In Section 3 the notion of Q-independence is defined (Definition 3.3). Next, similarly to [13], we use this definition to calculate moments of random variables in terms of elementary moments. As a consequence of our main limit theorem (Theorem 3.12) we obtain the central limit theorem (Theorem 3.13) and Poisson limit theorem (Theorem 3.14). Finally, the special case of q-independence is described. It is shown that in this case the distribution obtained by the Q-central limit theorem is exactly q-Gaussian as described in [4] (Theorem 4.6 and Corollary 4.8). Moreover, this approach allows us to find a new formula for moments of q-Gaussian distribution (Theorem 4.9).

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2. Partitions of finite sets. Let $\mathcal{K}_n=\{1,\ldots,n\}$ for $n=1,2,\ldots$ and $\mathcal{K}_0=\emptyset$. A *partition* of $\mathcal{K}_n$ is a family $\mathcal{V}=\{V_j\}_{j\in J}$ of nonempty subsets of $\mathcal{K}_n$ such that $V_j\cap V_k=\emptyset$ for $j\neq k$, $j,k\in J$ and $\bigcup_{j\in J}V_j=\mathcal{K}_n$. For every partition $\mathcal{V}$ of $\mathcal{K}_n$ we write $d(\mathcal{V})=n$. Let $\mathcal{V}=\{V_j\}_{j\in J}$ be a partition of $\mathcal{K}_n$ for $n\geq 1$. We define the equivalence relation $\sim_{\mathcal{V}}$ on $\mathcal{K}_n$ by the following condition: if $v,w\in\mathcal{K}_n$ and $v\leq w$ then

$$v\sim_{\mathcal{V}} w\Leftrightarrow[\exists j\in J\ \forall u\in\mathcal{K}_n:(v\leq u\leq w\Rightarrow u\in V_j)].$$

We denote by $|A|$ the number of elements of a finite set A.

DEFINITION 2.1. Let $\mathcal{V}=\{V_j\}_{j\in J}$ be a partition.

(a) If $d(\mathcal{V})=0$ or $d(\mathcal{V})\geq 1$ and every equivalence class of $\sim_{\mathcal{V}}$ consists of one element then $\mathcal{V}$ is called a *tiny partition*.

(b) If $|V_j|\geq 2$ for every $j\in J$ then $\mathcal{V}$ is called *nondegenerate*.

(c) If $|V_j|=2$ for every $j\in J$ then $\mathcal{V}$ is called a *2-partition*.

The set of all partitions (respectively tiny partitions, 2-partitions) of $\mathcal{K}_n$ will be denoted by $\mathcal{P}(n)$ (respectively $\mathcal{P}^t(n)$, $\mathcal{P}_2(n)$), and $\mathcal{P}=\bigcup_{n=0}^{\infty}\mathcal{P}(n)$ (respectively $\mathcal{P}^t$, $\mathcal{P}_2$) will denote the set of all partitions (respectively tiny partitions, 2-partitions).

If $V=\{v_1,\ldots,v_k\}$ is a subset of $\mathcal{K}_n$ and $v_1<\ldots<v_k$ then we write $V=(v_1,\ldots,v_k)$.

EXAMPLE 2.2. Let $n=8$, and define $\mathcal{V}_1=\{(1,4),(2,5,8),(3,6),(7)\}$, $\mathcal{V}_2=\{(1,3,7),(2,4,6),(5,8)\}$, $\mathcal{V}_3=\{(1,5),(2,4),(3,7),(6,8)\}$, and $\mathcal{V}_4=\{(1,5,6),(2,3),(4,7),(8)\}$. Then $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ are tiny partitions, while $\mathcal{V}_4$ is not tiny because $2\sim_{\mathcal{V}_4}3$ and $5\sim_{\mathcal{V}_4}6$. Moreover, $\mathcal{V}_2$ is a 2-partition, and $\mathcal{V}_2$, $\mathcal{V}_3$ are nondegenerate partitions.

If $\mathcal{V}=\{V_j\}_{j\in J}$ is any partition of $\mathcal{K}_n$ then let $\mathcal{K}'=\mathcal{K}_n/\sim_{\mathcal{V}}$ and $\mathcal{K}''=\bigcup_{j\in J_0}V_j$ where

$$J_0=\{j\in J:|V_j|\ge 2\}. \tag{2.1}$$

Let $n'=|\mathcal{K}'|$ and $n''=|\mathcal{K}''|$. We can write $\mathcal{K}'=\{w_1,\ldots,w_{n'}\}$ and $\mathcal{K}''=\{v_1,\ldots,v_{n''}\}$ where in both cases the labeling respects the natural orders on $\mathcal{K}'$ and $\mathcal{K}''$ induced from $\mathcal{K}_n$. For $j\in J$, let $V_j^t=\{k\in\mathcal{K}_{n'}: v\in V_j \text{ where } [v]_{\sim_{\mathcal{V}}}=w_k\}$ and for $j\in J_0$, let $V_j^s=\{l\in\mathcal{K}_{n''}:v_l\in V_j\}$. From this construction we immediately get

PROPOSITION 2.3. *Let $\mathcal{V}=\{V_j\}_{j\in J}$ be a partition of $\mathcal{K}_n$. Then $\mathcal{V}^t=\{V_j^t\}_{j\in J}$ is a tiny partition of $\mathcal{K}_{n'}$ and $\mathcal{V}^s=\{V_j^s\}_{j\in J_0}$ is a nondegenerate partition of $\mathcal{K}_{n''}$.*

We will write $\mathcal{V}^{ts}$ instead of $(\mathcal{V}^t)^s$ and inductively $\mathcal{V}^{(ts)^{n+1}}$ instead of $(\mathcal{V}^{(ts)^n})^{ts}$. The definitions of $\mathcal{V}^t$ and $\mathcal{V}^s$ lead to

$$d(\mathcal{V}^t)\le d(\mathcal{V}) \quad\text{and}\quad d(\mathcal{V}^s)\le d(\mathcal{V}). \tag{2.2}$$

Moreover, we have

PROPOSITION 2.4. *If $d(\mathcal{V}^t)=d(\mathcal{V})$ then $\mathcal{V}^t=\mathcal{V}$, and if $d(\mathcal{V}^s)=d(\mathcal{V})$ then $\mathcal{V}^s=\mathcal{V}$.*

Proof. If $d(\mathcal{V}^t)=d(\mathcal{V})$ then every equivalence class of $\sim_{\mathcal{V}}$ has exactly one element and $\mathcal{K}'=\mathcal{K}_n$. Similarly, if $d(\mathcal{V}^s)=d(\mathcal{V})$ then $J_0=J$, and $\mathcal{K}''=\mathcal{K}_n$. ■

Inequalities (2.2) and Proposition 2.4 imply that there is $m_0\in\mathbb{N}$ such that $\mathcal{V}^{(ts)^m}=\mathcal{V}^{(ts)^{m_0}}$ for every $m\ge m_0$. Let $\mathcal{V}^T=\mathcal{V}^{(ts)^{m_0}}$.

The main properties of the operations $\cdot^t$, $\cdot^s$, $\cdot^T$ are described by the following

PROPOSITION 2.5. *Let $\mathcal{V}=\{V_j\}_{j\in J}$ be a partition. Then*

(a) *$\mathcal{V}^t=\mathcal{V}$ if and only if $\mathcal{V}$ is tiny,*
(b) *$\mathcal{V}^{tt}=\mathcal{V}^t$,*
(c) *$\mathcal{V}^s=\mathcal{V}$ if and only if $\mathcal{V}$ is nondegenerate,*
(d) *$\mathcal{V}^{TT}=\mathcal{V}^{tsT}=\mathcal{V}^{tT}=\mathcal{V}^{Tt}=\mathcal{V}^{Tts}=\mathcal{V}^T$,*
(e) *$\mathcal{V}^T$ is tiny and nondegenerate.*

DEFINITION 2.6. A partition $\mathcal{V}$ will be called *admissible* if $\mathcal{V}^T=\emptyset$, where $\emptyset$ denotes the single partition of $\mathcal{K}_0$ (cf. [13]).

REMARK 2.7. We will show in Lemma 2.11 that our notion of admissible partition is equivalent to that given in [13]. It is also called a noncrossing partition (see [12]).

EXAMPLE 2.8. For partitions from Example 2.2: $\mathcal{V}_i^t=\mathcal{V}_i$ for $i=1,2,3$, $\mathcal{V}_4^t\in\mathcal{P}(6)$ and $\mathcal{V}_4^t=\{(1,4),(2),(3,6),(5)\}$. Further, $\mathcal{V}_1^{ts}\in\mathcal{P}(7)$ and $\mathcal{V}_1^{ts}=\{(1,4),(2,5,7),(3,6)\}$, $\mathcal{V}_4^{ts}\in\mathcal{P}(4)$ and $\mathcal{V}_4^{ts}=\{(1,3),(2,4)\}$, $\mathcal{V}_i^{ts}=\mathcal{V}_i$, $i=2,3$. Consequently, $d(\mathcal{V}_1^T)=7$ and $\mathcal{V}_1^T=\{(1,4),(2,5,7),(3,6)\}$, $\mathcal{V}_2^T=\mathcal{V}_2$, $\mathcal{V}_3^T=\mathcal{V}_3$, $d(\mathcal{V}_4^T)=4$ and $\mathcal{V}_4^T=\{(1,3),(2,4)\}$.

LEMMA 2.9. *Let $\mathcal{V}$ be a partition. Then the following conditions are equivalent:*

(a) *$\mathcal{V}$ is admissible,*
(b) *$\mathcal{V}^t$ is admissible,*
(c) *$\mathcal{V}^{ts}$ is admissible.*

Proof. This is an obvious consequence of Proposition 2.5(d). ■

LEMMA 2.10. *If $\mathcal{V}=\{V_j\}_{j\in J}$ is a tiny admissible partition such that $d(\mathcal{V})\ge 1$ then there is $j_0\in J$ such that $|V_{j_0}|=1$.*

Proof. Proposition 2.5(a) implies that $\mathcal{V}^t=\mathcal{V}$. Suppose that $|V_j|\ge 2$ for every $j\in J$. Then $\mathcal{V}^s=\mathcal{V}$ from Proposition 2.5(c), hence $\mathcal{V}^{ts}=\mathcal{V}$ and the definition of $\mathcal{V}^T$ leads to $\mathcal{V}^T=\mathcal{V}\ne\emptyset$, which contradicts the assumption. ■

In the sequel we will need the following characterization of admissible partitions:

LEMMA 2.11. *A partition $\mathcal{V}=\{V_j\}_{j\in J}$ is admissible if and only if*

$$\forall j,k\in J,\ j\ne k\ \forall v_1,v_2\in V_j\ \forall w\in V_k: \quad v_1<w<v_2\Rightarrow(\forall w'\in V_k: v_1<w'<v_2). \tag{2.3}$$

Proof. *Necessity*. Suppose that $\mathcal{V}=\{V_j\}_{j\in J}$ does not satisfy (2.3), i.e. there are different $j,k\in J$ and $v_1,v_2\in V_j$, $w_1,w_2\in V_k$ such that

$$v_1<w_1<v_2<w_2. \tag{2.4}$$

We show that $\mathcal{V}^{ts}$ does not satisfy (2.3) either. Indeed, the inequalities (2.4) imply that v_1, v_2, w_1, w_2 are in different equivalence classes of $\sim_{\mathcal{V}}$ and $[v_1]_{\sim_{\mathcal{V}}}<[w_1]_{\sim_{\mathcal{V}}}<[v_2]_{\sim_{\mathcal{V}}}<[w_2]_{\sim_{\mathcal{V}}}$. Moreover, $[v_1]_{\sim_{\mathcal{V}}},[v_2]_{\sim_{\mathcal{V}}}\in V_j^t$ and $[w_1]_{\sim_{\mathcal{V}}},[w_2]_{\sim_{\mathcal{V}}}\in V_k^t$, so $|V_j^t|\ge 2$ and $|V_k^t|\ge 2$. Finally, the operation $\cdot^s$ does not remove the sets V_j^t, V_k^t from $\mathcal{V}^t$ and preserves the relation (2.4) for corresponding elements. As $\mathcal{V}^T$ also satisfies the negation of (2.3), the set $\mathcal{V}^T$ cannot be empty.

Sufficiency. Firstly, observe that if $\mathcal{V}$ satisfies (2.3), then so do $\mathcal{V}^t$ and $\mathcal{V}^{ts}$. This results from the fact that both $\cdot^t$ and $\cdot^s$ preserve the inequalities from (2.3). Secondly, let us show that if a tiny partition $\mathcal{V} = \{V_j\}_{j\in J}$ satisfies (2.3) then there is $j \in J$ such that $|V_j| = 1$. This will be done by induction with respect to the cardinality of J. If $J = \{j\}$, then from Definition 2.1 we have $d(\mathcal{V}) = 1$ and $|V_j| = 1$. Now, assume that our statement is true for every partition with $|J| < p$ for some $p \in \mathbb{N}$, and let $\mathcal{V} = \{V_j\}_{j\in J}$ be such that $|J| = p$. Choose some $j_0 \in J$. Then either $|V_{j_0}| = 1$ or $|V_{j_0}| \geq 2$. The first case ends our proof, while in the second one we fix $v_1, v_2 \in V_{j_0}$ such that $v_1 < v_2$. Let $J' = \{j \in J : v_1 < w < v_2 \text{ for every } w \in V_j\}$. Definition 2.1(a) and the condition (2.3) imply that $J' \neq \emptyset$. We can define a partition $\mathcal{V}' = \{V'_j\}_{j\in J'}$ in the same way as in the construction of $\mathcal{V}^s$, where J' replaces J_0. Obviously, $\mathcal{V}'$ is also tiny and $|J'| \leq |J \setminus \{j_0\}| < p$. Thus, from the inductive assumption there is $j \in J' \subset J$ such that $|V_j| = |V'_j| = 1$.

Further, observe that the above first step implies that for every $m \in \mathbb{N}$, if $\mathcal{V}^{(ts)^m} \neq \emptyset$ then $\mathcal{V}^{(ts)^m}$ and $(\mathcal{V}^{(ts)^m})^t$ satisfy (2.3). Moreover, $(\mathcal{V}^{(ts)^m})^t$ is tiny. So, using the second step, one can prove that it contains a singleton. Thus

$$d(\mathcal{V}^{(ts)^{m+1}}) = d((\mathcal{V}^{(ts)^m})^{ts}) < d((\mathcal{V}^{(ts)^m})^t) \leq d(\mathcal{V}^{(ts)^m})$$

provided that $d(\mathcal{V}^{(ts)^m}) \neq 0$. Hence $d(\mathcal{V}^T) = 0$ and $\mathcal{V}^T = \emptyset$. ∎

Let $\mathcal{V} = \{V_j\}_{j\in J}$ and $\mathcal{W} = \{W_k\}_{k\in K}$ be two partitions such that $d(\mathcal{V}) = d(\mathcal{W})$. We say that $\mathcal{W}$ is *finer* than $\mathcal{V}$ if for every $k \in K$ there is $j \in J$ such that $W_k \subset V_j$. In this case we write $\mathcal{W} \leq \mathcal{V}$. Let us remark that if $\mathcal{W} \leq \mathcal{V}$ and $\mathcal{V}$ is a tiny partition, then so is $\mathcal{W}$.

Let us end this section with some remarks and technical lemmas about 2-partitions.

LEMMA 2.12. *Let $\mathcal{V} = \{V_j\}_{j\in J}$ be a 2-partition and $V_j = (v_j, w_j)$ for every $j \in J$. Then the following conditions are equivalent*:

(a) $\mathcal{V}^T = \mathcal{V}$,
(b) $\mathcal{V}$ *is tiny*,
(c) $w_j > v_j + 1$ *for every* $j \in J$.

Proof. (a)⇒(b). This follows from Proposition 2.5(e).
(b)⇒(a). $|V_j| = 2$ for $j \in J$, so $\mathcal{V}^{ts} = \mathcal{V}^s = \mathcal{V}$.
(b)⇔(c). This is a simple consequence of Definition 2.1. ∎

LEMMA 2.13. *If $\mathcal{V}$ is a 2-partition such that $\mathcal{V}^T = \mathcal{V}$ and $d(\mathcal{V}) \geq 4$, then there exists a pair $(j,k) \in J \times J$ such that $v_j < v_k < w_j < w_k$ and $v_k + 1 = w_j$.*

Proof. We use induction with respect to n, where $d(\mathcal{V}) = 2n$. If $n = 2$, then $\{(1,3),(2,4)\}$ is the only 2-partition such that $\mathcal{V}^T = \mathcal{V}$.

Now, let $\mathcal{V} = \{(v_j, w_j)\}_{j=1,\ldots,n+1}$ be a 2-partition such that $\mathcal{V}^T = \mathcal{V}$. Let j_0 be such that $v_{j_0} = 1$. Then either $v_j + 1 = w_{j_0}$ for some $j = 1,\ldots,n+1$, $j \neq j_0$, and this ends our proof, or a partition $\mathcal{V}'$ of $\mathcal{K}_{2n}$ obtained from $\mathcal{V}$ by excluding the pair (v_{j_0}, w_{j_0}) satisfies the condition (c) of Lemma 2.12. In the second case the statement results from the inductive assumption. ∎

For a 2-partition $\mathcal{V} = \{V_j\}_{j\in J}$ where $V_j = (v_j, w_j)$ for $j \in J$, we define

$$i(\mathcal{V}) = |\{(j,k) \in J \times J : v_j < v_k < w_j < w_k\}|. \tag{2.5}$$

Then we have

LEMMA 2.14. (a) *If $\mathcal{V} \in \mathcal{P}_2$ then $\mathcal{V}^{ts} \in \mathcal{P}_2$ and $\mathcal{V}^T \in \mathcal{P}_2$.*
(b) *If $\mathcal{V} \in \mathcal{P}_2$, then $i(\mathcal{V}^{ts}) = i(\mathcal{V})$ and $i(\mathcal{V}^T) = i(\mathcal{V})$.*
(c) *If $\mathcal{V} \in \mathcal{P}_2$ and $\mathcal{V}^t \in \mathcal{P}_2$, then $i(\mathcal{V}^t) = i(\mathcal{V})$.*

Proof. (a) If $\mathcal{V}^t = \{V_j^t\}_{j\in J}$ then $|V_j^t| \leq 2$ for every $j \in J$. The definition of the operation $\cdot^s$ implies that $\mathcal{V}^{ts} = \{V_j^{ts}\}_{j\in J_0}$ where $J_0 = \{j \in J : |V_j^t| \geq 2\}$ is defined as in (2.1) and $|V_j^{ts}| = |V_j^t| = 2$ for every $j \in J_0$.

(b) This is a simple consequence of the observation that if $v_j < v_k < w_j < w_k$, then $j,k \in J_0$ and $v_j^{ts} < v_k^{ts} < w_j^{ts} < w_k^{ts}$ where $V_j^{ts} = (v_j^{ts}, w_j^{ts})$ for $j \in J_0$. The second equality results from the definition of the operation $\cdot^T$.

(c) Use a similar argument to that in (b). ∎

The above lemma justifies the following

DEFINITION 2.15. The number $i(\mathcal{V}^T)$ is called the *index* of the partition $\mathcal{V} \in \mathcal{P}$ such that $\mathcal{V}^T \in \mathcal{P}_2$. For simplicity, the index of $\mathcal{V}$ will be denoted by $i(\mathcal{V})$.

We denote by S_n the set of all permutations of the set $\mathcal{K}_n$. Let $\mathcal{V} = \{V_j\}_{j\in J} \in \mathcal{P}(n)$ and $\sigma \in S_n$. Then $\mathcal{V}^\sigma \in \mathcal{P}(n)$ is a partition defined as follows:

$$\mathcal{V}^\sigma = \{V_j^\sigma\}_{j\in J}, \quad \text{where} \quad V_j^\sigma = \{\sigma(v) : v \in V_j\}. \tag{2.6}$$

Recall that a permutation of the form

$$\pi_v = \begin{pmatrix} 1 & \ldots & v-1 & v & v+1 & v+2 & \ldots & n \\ 1 & \ldots & v-1 & v+1 & v & v+2 & \ldots & n \end{pmatrix}$$

for $v = 1,\ldots,n-1$ is called an *inversion*. Now let

$$m(\mathcal{V}) = \min\{k \in \mathbb{N} \cup \{0\} : \text{there are } k \text{ inversions } \sigma_1,\ldots,\sigma_k \in S_n \text{ such that } \mathcal{V}^{T\sigma_1 T\sigma_2 \ldots T\sigma_k T} = \emptyset\}. \tag{2.7}$$

Here a similar convention as in the paragraph following Proposition 2.3 is used.

LEMMA 2.16. *If $\mathcal{V} \in \mathcal{P}_2$ then $i(\mathcal{V}) = m(\mathcal{V})$.*

Proof. Observe that if $\mathcal{V}$ is a tiny 2-partition then

$$(2.8)\quad i(\mathcal{V}^{\pi_u}) = \begin{cases} i(\mathcal{V})-1 & \text{if } \begin{pmatrix} u=v_j,\ u+1=w_k \\ \text{or } u=w_j,\ u+1=w_k,\ v_j<v_k \\ \text{or } u=v_j,\ u+1=v_k,\ w_j<w_k \end{pmatrix} \\ & \qquad \text{for some } j,k\in J, \\ i(\mathcal{V})+1 & \text{if } \begin{pmatrix} u=w_j,\ u+1=v_k \\ \text{or } u=w_j,\ u+1=w_k,\ v_j>v_k \\ \text{or } u=v_j,\ u+1=v_k,\ w_j>w_k \end{pmatrix} \\ & \qquad \text{for some } j,k\in J, \end{cases}$$

for every $u=1,\ldots,d(\mathcal{V})$. Thus, $i(\mathcal{V}^{T\sigma})\ge i(\mathcal{V})-1$ for every 2-partition $\mathcal{V}$ and inversion σ. Hence, for every $k<i(\mathcal{V})$ and every sequence of inversions $\sigma_1,\ldots,\sigma_k$ we have $i(\mathcal{V}^{T\sigma_1T\sigma_2\ldots T\sigma_k})\ge i(\mathcal{V})-k>0$. Lemma 2.11 implies that $\mathcal{V}^{T\sigma_1T\sigma_2\ldots T\sigma_kT}\ne\emptyset$. So, $i(\mathcal{V})\le m(\mathcal{V})$.

In order to prove that $i(\mathcal{V})\ge m(\mathcal{V})$ we construct by induction a sequence $\sigma_1,\ldots,\sigma_{i(\mathcal{V})}$ of inversions and a sequence $\mathcal{V}_0,\mathcal{V}_1,\ldots,\mathcal{V}_{i(\mathcal{V})}$ of partitions such that $\mathcal{V}_l=\mathcal{V}_{l-1}^{T\sigma_l}$ and $i(\mathcal{V}_l)=i(\mathcal{V}_{l-1})-1$ for $l=1,\ldots,i(\mathcal{V})$. Suppose that for some $k<i(\mathcal{V})$ we have partitions $\mathcal{V}_0=\mathcal{V},\mathcal{V}_1,\ldots,\mathcal{V}_k$ and inversions $\sigma_1,\ldots,\sigma_k$ such that $\mathcal{V}_l=\mathcal{V}_{l-1}^{T\sigma_l}$ and $i(\mathcal{V}_l)=i(\mathcal{V}_{l-1})-1$ for $l=1,\ldots,k$. The conditions imply that $i(\mathcal{V}_k)=i(\mathcal{V})-k>0$, so using Lemma 2.11 we get $\mathcal{V}_k^T\ne\emptyset$. Proposition 2.5(d) and Lemma 2.14(a) imply that $\mathcal{V}_l^T$ satisfies the assumption of Lemma 2.13, hence if $\mathcal{V}_l^T=\{(v_j,w_j)\}_{j\in J}$ then there are $j,k\in J$ such that $j\ne k$ and $v_j+1=w_k$. Let $\sigma_{k+1}=\pi_{v_j}$ and $\mathcal{V}_{k+1}=\mathcal{V}_k^{T\sigma_{k+1}}$. Then, using (2.8), $i(\mathcal{V}_{k+1})=i(\mathcal{V}_k)-1$. From the above construction $i(\mathcal{V}_{i(\mathcal{V})})=0$, so Lemma 2.11 implies that $\mathcal{V}_{i(\mathcal{V})}^T=\emptyset$. ■

3. *Q*-independence.

A family Q of polynomials indexed by all tiny partitions will play the crucial role in this section.

DEFINITION 3.1. For $\mathcal{V}=\{V_j\}_{j\in J}\in\mathcal{P}^t$, let $Q_\mathcal{V}$ be a formal polynomial in commuting variables from the set

$$X^\mathcal{V}=\{X_A : A\subset V_j \text{ for some } j\in J\},$$

of the form

$$Q_\mathcal{V}(X^\mathcal{V})=\sum_{\mathcal{W}\le\mathcal{V}} q_{\mathcal{W},\mathcal{V}}\prod_{k\in K}X_{W_k},$$

where $\mathcal{W}=\{W_k\}_{k\in K}$, and $q_{\mathcal{W},\mathcal{V}}$ are real coefficients. Then $Q=\{Q_\mathcal{V}\}_{\mathcal{V}\in\mathcal{P}^t}$ is called a *consistent family of polynomials*.

REMARK 3.2. A consistent family Q of polynomials is determined by the system $\{q_{\mathcal{W},\mathcal{V}} : \mathcal{V}\in\mathcal{P}^t, \mathcal{W}\le\mathcal{V}\}$ of coefficients, so for convenience, the latter will also be denoted by Q.

Now, let $(\mathcal{A},\varphi)$ be a probability system, $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ be a family of *-subalgebras of $\mathcal{A}$, and Q be a consistent family of polynomials.

Further, let $(a_1,\ldots,a_n)$ be a system of elements of $\mathcal{A}$ such that for every $v=1,\ldots,n$ there is $i_v\in\mathcal{I}$ such that $a_v\in\mathcal{A}_{i_v}$. We define the partition $\mathcal{V}=\{V_j\}_{j\in J}\in\mathcal{P}(n)$ *associated with* $(a_1,\ldots,a_n)$ by the following condition:

$$(3.1)\qquad \forall v,w\in\mathcal{K}_n:\quad [(\exists j\in J: v\in V_j\wedge w\in V_j)\Leftrightarrow i_v=i_w].$$

Observe that this partition is tiny if and only if $i_v\ne i_{v+1}$ for every $v=1,\ldots,n-1$.

For simplicity, $\varphi(A)$ denotes $\varphi(a_{v_1}\ldots a_{v_l})$ where $A=(v_1,\ldots,v_l)\subset\mathcal{K}_n$ is an ordered subset. If $\mathcal{V}=\{V_j\}_{j\in J}$ is a partition of $\mathcal{K}_n$ then $\varphi(\mathcal{V})=\prod_{j\in J}\varphi(V_j)$ and $\varphi^\mathcal{V}=\{\varphi(A) : A\subset V_j \text{ for some } j\in J\}$. We denote by $\mathrm{Alg}(a)$ the *-subalgebra of $\mathcal{A}$ generated by an element a.

Now we can formulate the following

DEFINITION 3.3. (a) Suppose that $Q=\{Q_\mathcal{V}\}_{\mathcal{V}\in\mathcal{P}^t}$ is a consistent family of polynomials. We say that a family $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ of subalgebras of $\mathcal{A}$ is *independent with respect to* Q, or briefly *Q-independent*, if

$$(3.2)\qquad \varphi(a_1\ldots a_n)=Q_\mathcal{V}(\varphi^\mathcal{V})$$

for every $n\in\mathbb{N}$ and for every system $(a_1,\ldots,a_n)$ of elements of $\mathcal{A}$ such that $a_v\in\mathcal{A}_{i_v}$, $i_1\ne i_2\ne\ldots\ne i_n$, where $\mathcal{V}$ is the partition associated with $(a_1,\ldots,a_n)$.

(b) A family $\{a_i\}_{i\in\mathcal{I}}$ of elements of $\mathcal{A}$ is called *Q-independent* if the family $\{\mathrm{Alg}(a_i)\}_{i\in\mathcal{I}}$ is Q-independent.

EXAMPLE 3.4. Let

$$q_{\mathcal{W},\mathcal{V}}=\begin{cases}1 & \text{if } \mathcal{W}=\mathcal{V},\\ 0 & \text{if } \mathcal{W}\ne\mathcal{V}.\end{cases}$$

Then we obtain the classical independence of random variables.

EXAMPLE 3.5. The results of [18] and [13] clearly show that the free independence introduced by Voiculescu (see [17]) is a special case of our notion of Q-independence. In this case

$$q_{\mathcal{V},\mathcal{V}}=\begin{cases}1 & \text{if } \mathcal{V} \text{ is admissible},\\ 0 & \text{if } \mathcal{V} \text{ is not admissible}.\end{cases}$$

The values of $q_{\mathcal{W},\mathcal{V}}$ for $\mathcal{W}\ne\mathcal{V}$ are determined by the above condition (cf. [15]).

EXAMPLE 3.6. "Boolean" independence (see [16]):

$$q_{\mathcal{W},\mathcal{V}}=\begin{cases}1 & \text{if } |W_k|=1 \text{ for every } k\in K,\\ 0 & \text{otherwise}.\end{cases}$$

This kind of independence is important for nonunital subalgebras ([1]).

([1]) Precise calculations of $q_{\mathcal{W},\mathcal{V}}$ for Examples 3.4–3.6 can be found in [15].

EXAMPLE 3.7. Let $q \in [-1,1]$. Following [10], if a consistent family Q satisfies

$$q_{\mathcal{V},\mathcal{V}} = q^{i(\mathcal{V})}, \quad \mathcal{V} \in \mathcal{P}_2 \cap \mathcal{P}^t, \tag{3.3}$$

then we obtain the "q-independence" (cf. [5, 2]).

Let us fix some Q-independent family $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ of subalgebras of $\mathcal{A}$. To simplify our terminology, the value $\varphi(a_1\ldots a_n)$ will be called a *moment* of the system $(a_1,\ldots,a_n)$. The value $\varphi(a_{v_1}\ldots a_{v_l})$ where $v_1 < \ldots < v_l$ and $i_{v_1} = \ldots = i_{v_l}$ will be called an *elementary moment* of $(a_1,\ldots,a_n)$. The next three lemmas show how (3.2) can be used to express the moment of $(a_1,\ldots,a_n)$ in terms of elementary moments.

LEMMA 3.8. *Suppose that $a_1,\ldots,a_n \in \mathcal{A}$ satisfy the following conditions:*

(i) *$a_v \in \mathcal{A}_{i_v}$, where $i_1 \neq i_2 \neq \ldots \neq i_n$,*
(ii) *there is $v_0 \in \mathcal{K}_n$ such that $i_{v_0} \neq i_v$ for $v \neq v_0$.*

Then $\varphi(a_1\ldots a_n) = \varphi(a_{v_0})\varphi(a_1\ldots a_{v_0-1}a_{v_0+1}\ldots a_n)$.

Proof. Firstly, suppose that $\varphi(a_{v_0}) = 0$. Let $\mathcal{V} = \{V_j\}_{j\in J}$ be the associated partition of $\mathcal{K}_n$. By (i), $\mathcal{V}$ is tiny, while (ii) implies that $\mathcal{V}$ is degenerate, i.e. there is $j_0 \in J$ such that $V_{j_0} = \{v_0\}$. Observe that for every partition $\mathcal{W} = \{W_k\}_{k\in K}$ finer than $\mathcal{V}$ there is $k_0 \in K$ such that $W_{k_0} = \{v_0\}$. By (3.2) we have

$$\varphi(a_1\ldots a_n) = \sum_{\mathcal{W}\le\mathcal{V}} q_{\mathcal{W},\mathcal{V}}\varphi(\mathcal{W}) = \sum_{\mathcal{W}\le\mathcal{V}} q_{\mathcal{W},\mathcal{V}}\varphi(a_{v_0}) \prod_{k\in K\setminus\{k_0\}} \varphi(W_k) = 0.$$

Secondly, for arbitrary $\varphi(a_{v_0})$ we can write $a_{v_0} = a^0_{v_0} + \varphi(a_{v_0})\mathbf{1}$ where $\varphi(a^0_{v_0}) = 0$. Now

$$\begin{aligned}\varphi(a_1\ldots a_n) &= \varphi(a_1\ldots a_{v_0-1}a^0_{v_0}a_{v_0+1}\ldots a_n)\\ &\quad + \varphi(a_{v_0})\varphi(a_1\ldots a_{v_0-1}a_{v_0+1}\ldots a_n)\\ &= \varphi(a_{v_0})\varphi(a_1\ldots a_{v_0-1}a_{v_0+1}\ldots a_n). \ \blacksquare\end{aligned}$$

REMARK 3.9. If $(a_1,\ldots,a_n)$ is such that $a_v \in \mathcal{A}_{i_v}$ for $v = 1,\ldots,n$ and $\mathcal{V}$ is the associated partition, then we can write $a_1\ldots a_n = b_1\ldots b_{n'}$ where $b_w \in \mathcal{A}_{i_w}$ and the partition associated with $(b_1,\ldots,b_{n'})$ is $\mathcal{V}^t$. Moreover, $\varphi(\mathcal{V}) = \varphi(\mathcal{V}^t)$.

COROLLARY 3.10. *If $(a_1,\ldots,a_n)$ is such that $a_v \in \mathcal{A}_{i_v}$ for $v = 1,\ldots,n$ and the associated partition $\mathcal{V}$ is admissible, then*

$$\varphi(a_1\ldots a_n) = \varphi(\mathcal{V}). \tag{3.4}$$

Proof. Let $\mathcal{V} = \{V_j\}_{j\in J}$. If $|J| = 1$ then (3.4) is obvious. Now, assume that we have $(a_1,\ldots,a_n)$ with associated partition $\mathcal{V}$ with $|J| = p+1$. By Remark 3.7 we can write $a_1\ldots a_n = b_1\ldots b_{n'}$ where the partition associated with $(b_1,\ldots,b_{n'})$ is $\mathcal{V}^t$, i.e. it is a tiny partition. We note $(\mathcal{V}^t)^T = \mathcal{V}^T = \emptyset$. Then, by Lemma 2.10, there is $j_0 \in J$ such that $V^t_{j_0} = \{w_0\}$ for some $w_0 = 1,\ldots,n'$. Hence, by Lemma 3.8,

$$\begin{aligned}\varphi(b_1\ldots b_{n'}) &= \varphi(b_{w_0})\varphi(b_1\ldots b_{w_0-1}b_{w_0+1}\ldots b_{n'})\\ &= \varphi(V^t_{j_0})\varphi(b_1\ldots b_{w_0-1}b_{w_0+1}\ldots b_{n'})\\ &= \varphi(V_{j_0})\varphi(b_1\ldots b_{w_0-1}b_{w_0+1}\ldots b_{n'}).\end{aligned}$$

Next, consider the partition associated with $(b_1,\ldots,b_{w_0-1},b_{w_0+1},\ldots,b_{n'})$. If it is degenerate then we can repeat the above procedure. Hence

$$\varphi(b_1\ldots b_{n'}) = \Big(\prod_{j\in J\setminus J_0}\varphi(V_j)\Big)\varphi(c_1\ldots c_{n''})$$

where J_0 is defined for $\mathcal{V}^t$ as in (2.1) and the partition associated with $(c_1,\ldots,c_{n''})$ is $\mathcal{V}^{ts}$. This partition has fewer than $p+1$ elements, and $(\mathcal{V}^{ts})^T = \mathcal{V}^T = \emptyset$. Thus by the inductive assumption we have

$$\varphi(c_1\ldots c_{n''}) = \varphi(\mathcal{V}^{ts}) = \prod_{j\in J_0}\varphi(V^t_j) = \prod_{j\in J_0}\varphi(V_j). \ \blacksquare$$

LEMMA 3.11. *Let $a_v \in \mathcal{A}_{i_v}$ for $v = 1,\ldots,n$ and let $\mathcal{V} = \{V_j\}_{j\in J}$ be the associated partition. Then*

$$\varphi(a_1\ldots a_n) = q_{\mathcal{V}^t,\mathcal{V}^t}\varphi(\mathcal{V}) + U \tag{3.5}$$

where U is a sum of products of elementary moments and each product contains at least $|J|+1$ factors.

Proof. Let $a_1\ldots a_n = b_1\ldots b_{n'}$ where the partition associated with $(b_1,\ldots b_{n'})$ is $\mathcal{V}^t$. Then (3.2) implies

$$\varphi(b_1\ldots b_{n'}) = \sum_{\mathcal{W}\le\mathcal{V}^t} q_{\mathcal{W},\mathcal{V}^t}\varphi(\mathcal{W}) = q_{\mathcal{V}^t,\mathcal{V}^t}\varphi(\mathcal{V}^t) + \sum_{\mathcal{W}\le\mathcal{V},\,\mathcal{W}\ne\mathcal{V}^t} q_{\mathcal{W},\mathcal{V}^t}\varphi(\mathcal{W}).$$

If $\mathcal{W} = \{W_k\}_{k\in K}$ and $\mathcal{W}\le\mathcal{V}$, $\mathcal{W}\ne\mathcal{V}$ then $|K| > |J|$. Therefore, $\varphi(\mathcal{W}) = \prod_{k\in K}\varphi(W_k)$ is a product of at least $|J|+1$ elementary moments. $\blacksquare$

Now, we can formulate the following

THEOREM 3.12. *Let $a_{1,N}, a_{2,N},\ldots,a_{N,N} \in \mathcal{A}$ be Q-independent and identically distributed for every $N \in \mathbb{N}$. Moreover, suppose that for every $s \in \mathbb{N}$ the limit*

$$R(s) = \lim_{N\to\infty} N\varphi(a^s_{1,N})$$

exists. Let $S_N = a_{1,N} + a_{2,N} + \ldots + a_{N,N}$. Then, for every $m \in \mathbb{N}$,

$$\lim_{N\to\infty}\varphi(S^m_N) = \sum_{\mathcal{V}\in\mathcal{P}(m)} q_{\mathcal{V}^t,\mathcal{V}^t}R(\mathcal{V})$$

where $R(\mathcal{V}) = \prod_{j\in J} R(|V_j|)$ for $\mathcal{V} = \{V_j : j \in J\}$.

Proof. Let $m \in \mathbb{N}$ and $M_N = \varphi(S_N^m)$. Then

$$M_N = \varphi((a_{1,N} + \ldots + a_{N,N})^m) = \sum_{i_1,\ldots,i_m=1}^{N} \varphi(a_{i_1,N} \ldots a_{i_m,N}).$$

Observe that if the partitions associated with $(a_{i_1,N}, \ldots, a_{i_m,N})$ and with $(a_{i'_1,N}, \ldots, a_{i'_m,N})$ are equal then $\varphi(a_{i_1,N} \ldots a_{i_m,N}) = \varphi(a_{i'_1,N} \ldots a_{i'_m,N})$. This follows from Definition 3.3(a) and from the fact that all $a_{1,N}, \ldots, a_{N,N}$ have the same distribution. For a given partition $\mathcal{V} = \{V_j : j \in J\} \in \mathcal{P}(m)$ such that $|J| = p$ there are $\lambda_{p,N} = N(N-1)\ldots(N-p+1)$ systems $(a_{i_1,N}, \ldots, a_{i_m,N})$ for which $\mathcal{V}$ is the associated partition. Hence

$$M_N = \sum_{p=1}^{m} \lambda_{p,N} \sum_{\mathcal{V}\in\mathcal{P}(m),\,|J|=p} \varphi(\mathcal{V})$$

where $\varphi(\mathcal{V}) = \varphi(a_{i_1,N} \ldots a_{i_m,N})$ for some $(a_{i_1,N}, \ldots, a_{i_m,N})$ with associated partition $\mathcal{V}$. So, from Lemma 3.11,

$$\varphi(\mathcal{V}) = q_{\mathcal{V}^t,\mathcal{V}^t} \prod_{j\in J} \varphi(V_j) + U$$

where U is a sum of products of moments of $a_{1,N}, \ldots, a_{N,N}$ with at least $p+1$ factors. Then

$$\lim_{N\to\infty} \lambda_{p,N}\varphi(\mathcal{V}) = q_{\mathcal{V}^t,\mathcal{V}^t} \prod_{j\in J} R(|V_j|). \ \blacksquare$$

As a corollary we get the following version of the central limit theorem.

THEOREM 3.13. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of Q-independent and identically distributed elements of $\mathcal{A}$. Assume that $\varphi(a_n) = 0$ and $\varphi(a_n^2) = \sigma^2$. If*

$$S_N = \frac{1}{\sqrt{N}}(a_1 + \ldots + a_N),$$

then, for every $m \in \mathbb{N}$,

$$\lim_{N\to\infty} \varphi(S_N^m) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \sum_{\mathcal{V}\in\mathcal{P}_2(m)} q_{\mathcal{V}^t,\mathcal{V}^t}\sigma^m & \text{if } m \text{ is even.} \end{cases}$$

Proof. Let $a_{j,N} = N^{-1/2}a_N$ for $j = 1, \ldots, N$. Then the $a_{j,N}$ are as in the preceding theorem. Let us calculate:

$$R(1) = \lim_{N\to\infty} N\varphi(a_{1,N}) = 0,$$
$$R(2) = \lim_{N\to\infty} N\varphi(N^{-1}a_N^2) = \varphi(a_N^2) = \sigma^2,$$
$$R(s) = \lim_{N\to\infty} N^{1-s/2}\varphi(a_N^s) = 0, \qquad s \geq 3.$$

Now, we can easily complete the proof by using Theorem 3.12. $\blacksquare$

The next theorem gives a Q-version of the Poisson limit theorem.

THEOREM 3.14. *Suppose that $a_{1,N}, \ldots, a_{N,N} \in \mathcal{A}$ are Q-independent and identically distributed for every $N \in \mathbb{N}$. Moreover, assume that*

$$\lim_{N\to\infty} N\varphi(a_{1,N}^s) = \alpha$$

for every $s \in \mathbb{N}$ and $S_N = a_{1,N} + a_{2,N} + \ldots + a_{N,N}$. Then

$$\lim_{N\to\infty} \varphi(S_N^m) = \sum_{p=1}^{m} \sum_{\mathcal{V}\in\mathcal{P}(m),\,|J|=p} q_{\mathcal{V}^t,\mathcal{V}^t}\alpha^p$$

for every $m \in \mathbb{N}$.

Proof. $R(s) = \alpha$ for every $s \in \mathbb{N}$, so $R(\mathcal{V}) = \alpha^{|J|}$. $\blacksquare$

4. q-Gaussian distribution. In this section we consider a consistent family Q of polynomials which satisfies the following supplementary condition (cf. Definition 3.3):

$$q_{\mathcal{V},\mathcal{V}} = q^{i(\mathcal{V})} \quad \text{for every } \mathcal{V} \in \mathcal{P}_2 \cap \mathcal{P}^t, \tag{4.1}$$

where q is a fixed number from $[-1,1]$ (see Example 3.7).

The central limit theorem (Theorem 3.13) for this kind of Q-independence defines the so-called *q-Gaussian distribution* $\mu_{q,\sigma}$ with moments

$$\mu_{q,\sigma}(X^m) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \sum_{\mathcal{V}\in\mathcal{P}_2(m)} q^{i(\mathcal{V})}\sigma^m & \text{if } m \text{ is even.} \end{cases} \tag{4.2}$$

In the sequel we will describe some q-Gaussian random variables and find another useful formula for moments of $\mu_{q,\sigma}$.

Let us recall the definition of a probability system based on the q-deformed Fock space (for details see [5]). If $\mathcal{H}$ is a Hilbert space, then $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$ will be called the *full Fock space* over $\mathcal{H}$. Here $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ is the one-dimensional Hilbert space spanned by the so-called *vacuum vector* Ω with $\|\Omega\| = 1$. Let $q \in [-1,1]$. Then, for $n \geq 0$, we define an operator $P_q^{(n)}$ on $\mathcal{H}^{\otimes n}$ by

$$P_q^{(0)}\Omega = \Omega, \tag{4.3}$$

and

$$P_q^{(n)}(f_1 \otimes \ldots \otimes f_n) = \sum_{\sigma\in S_n} q^{l(\sigma)} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)} \tag{4.4}$$

where $n > 0$, $f_1, \ldots, f_n \in \mathcal{H}$ and $l(\sigma)$ is the number of inversions of $\sigma \in S_n$, i.e. $l(\sigma) = |\{(v,w) \in \mathcal{K}_n \times \mathcal{K}_n : v < w \text{ and } \sigma(v) > \sigma(w)\}|$. The operator $P_q = \bigoplus_{n=0}^{\infty} P_q^{(n)}$ is strictly positive for $q \in (-1,1)$ ([7]). We can define the following sesquilinear form $\langle\cdot,\cdot\rangle_q$ on the set of simple tensors in $\mathcal{F}(\mathcal{H})$:

$$\langle f^{(n)}, g^{(m)}\rangle_q = \langle f^{(n)}, P_q g^{(m)}\rangle \tag{4.5}$$

for every $n, m \geq 0$ and $f^{(n)} \in \mathcal{H}^{\otimes n}$, $g^{(m)} \in \mathcal{H}^{\otimes m}$. The completion of the linear span of simple tensors with respect to the scalar product $\langle\cdot,\cdot\rangle_q$ will be denoted by $\mathcal{F}_q(\mathcal{H})$. Now, we define a state ω on the C^*-algebra $\mathcal{B}(\mathcal{F}_q(\mathcal{H}))$ of all bounded operators on $\mathcal{F}_q(\mathcal{H})$ by

$$\omega(A) = \langle \Omega, A\Omega\rangle_q, \quad A \in \mathcal{B}(\mathcal{F}_q(\mathcal{H})). \tag{4.6}$$

Thus we have defined the probability system $(\mathcal{B}(\mathcal{F}_q(\mathcal{H})), \omega)$.

Let $C^*(f)\Omega = f$ and

$$C^*(f)(f_1 \otimes \ldots \otimes f_n) = f \otimes f_1 \otimes \ldots \otimes f_n \tag{4.7}$$

for every $f \in \mathcal{H}$, $n > 0$ and $f_1 \otimes \ldots \otimes f_n \in \mathcal{H}^{\otimes n}$. (4.7) defines the so-called *creation operator* $C^*(f)$ on $\mathcal{F}_q(\mathcal{H})$. Its adjoint operator $C(f)$, called the *annihilation operator*, satisfies $C(f)\Omega = 0$ and

$$C(f)(f_1 \otimes \ldots \otimes f_n) = \sum_{i=1}^{n} q^{i-1}\langle f, f_i\rangle f_1 \otimes \ldots \otimes f_{i-1} \otimes f_{i+1} \otimes \ldots \otimes f_n \tag{4.8}$$

for $n > 0$, $f_1 \otimes \ldots \otimes f_n \in \mathcal{H}^{\otimes n}$. The creation and annihilation operators obey the following "q-commutation" relations:

$$C(g)C^*(f) - qC^*(f)C(g) = \langle g, f\rangle \mathbf{1}. \tag{4.9}$$

The aim of the next lemmas is to show that the distribution of the "position operator" $C(f) + C^*(f)$ is q-Gaussian within the probability system $(\mathcal{B}(\mathcal{F}_q(\mathcal{H})), \omega)$ for every $f \in \mathcal{H}$.

Define $(n)_q = 1 + q + \ldots + q^{n-1}$ for $n = 1, 2, \ldots$ We start with

LEMMA 4.1. *Let $C = C(f)$ for some $f \in \mathcal{H}$. Then*

$$\omega((C + C^*)^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \sum_{(i_1,\ldots,i_n)\in A_{n/2}} \omega(C^{i_n}\ldots C^{i_1}) & \text{if } n \text{ is even,} \end{cases}$$

where A_k is the set of sequences $(i_1, \ldots, i_{2k})$ with values in $\{1, *\}$ such that:

(i) $|\{j : i_j = 1\}| = |\{j : i_j = *\}| = k$,

(ii) $|\{j = 1, \ldots, r : i_j = 1\}| \leq |\{j = 1, \ldots, r : i_j = *\}|$ *for all* $r \leq 2k$.

Proof. We have

$$\omega((C + C^*)^n) = \sum_{i_1,\ldots,i_n \in \{1,*\}} \langle \Omega, C^{i_n}\ldots C^{i_1}\Omega\rangle_q. \tag{4.10}$$

For every sequence $(i_1, \ldots, i_n)$ the vector $C^{i_n}\ldots C^{i_1}\Omega$ is either zero or a nonzero multiple of $f^{\otimes k}$ for some $k \in \mathbb{N} \cup \{0\}$ (we set $f^{\otimes 0} = \Omega$). Note that $f^{\otimes k}$ is orthogonal to Ω for every $k > 0$. Therefore,

$$\omega((C + C^*)^n) = \sum_{(i_1,\ldots,i_n)\in B_n} \langle \Omega, C^{i_n}\ldots C^{i_1}\Omega\rangle_q,$$

where B_n is the set of all sequences $(i_1, \ldots, i_n)$ such that $C^{i_n}\ldots C^{i_1}\Omega = \lambda\Omega$ where $\lambda \neq 0$.

For a fixed sequence $(i_1, \ldots, i_n)$ define $\alpha(r) = |\{j = 1, \ldots, r : i_j = *\}|$ and $\beta(r) = |\{j = 1, \ldots, r : i_j = 1\}|$ for every $r = 1, \ldots, n$. In order to prove our lemma we should show that $B_n = \emptyset$ for n odd and $B_n = A_{n/2}$ for n even. To this end we prove the following three simple statements:

1. $C^{i_n}\ldots C^{i_1}\Omega = \lambda\Omega$, $\lambda \neq 0 \Rightarrow \alpha(n) = \beta(n)$.

Let $\xi_r = C^{i_r}\ldots C^{i_1}\Omega$ for $r = 1, \ldots, n$ and $\xi_0 = \Omega$. If $\xi_n \neq 0$ then $\xi_r \neq 0$ for $r = 1, \ldots, n$ or more precisely $\xi_r = \lambda_r f^{\otimes k_r}$ where $k_r \in \mathbb{N} \cup \{0\}$ and $\lambda_r \neq 0$. From (4.7) and (4.8) we have

$$C^* f^{\otimes k} = f^{\otimes(k+1)}, \quad k = 0, 1, 2, \ldots, \tag{4.11}$$

$$Cf^{\otimes k} = \begin{cases} 0 & \text{if } k = 0, \\ (k)_q\|f\|^2 f^{\otimes(k-1)} & \text{if } k > 0. \end{cases} \tag{4.12}$$

Hence, $i_r = *$ if and only if $k_r = k_{r-1} + 1$, and $i_r = 1$ if and only if $k_r = k_{r-1} - 1$, for every $r = 1, \ldots, n$. Thus, $\alpha(n) = \beta(n)$ because $k_0 = k_n = 0$.

2. $\alpha(n) = \beta(n) \Rightarrow C^{i_n}\ldots C^{i_1}\Omega = \lambda\Omega$.

If $C^{i_n}\ldots C^{i_1}\Omega \neq 0$ then from $\alpha(n) = \beta(n)$ we derive that $k_n = k_0 = 0$ as in the proof of statement 1.

3. $C^{i_n}\ldots C^{i_1}\Omega = 0 \Leftrightarrow \exists r = 1, \ldots, n : \alpha(r) < \beta(r)$.

Necessity. Let $r_0 = \min\{r = 1, \ldots, n : \xi_r = 0\} - 1$. Then $\xi_{r_0+1} = C^{i_{r_0+1}}\xi_{r_0} = 0$ and $\xi_{r_0} \neq 0$. Because $\ker C = \mathbb{C}\Omega$ and $\ker C^* = \{0\}$, we conclude that $i_{r_0+1} = 1$ and $\xi_{r_0} = \lambda\Omega$ where $\lambda \neq 0$. Statement 1 implies $\alpha(r_0) = \beta(r_0)$, hence $\alpha(r_0 + 1) < \beta(r_0 + 1)$.

Sufficiency. If $r_0 = \min\{r = 1, \ldots, n : \alpha(r) < \beta(r)\} - 1$, then $\alpha(r_0) \geq \beta(r_0)$ and $\alpha(r_0+1) < \beta(r_0+1)$. Obviously $\alpha(r_0+1) \geq \alpha(r_0)$, thus $\beta(r_0+1) > \beta(r_0)$. This implies that $i_{r_0+1} = 1$, $\beta(r_0 + 1) = \beta(r_0) + 1$ and $\alpha(r_0) = \beta(r_0)$. From statement 2 and (4.12) we conclude that $\xi_{r_0} = \lambda\Omega$, $\xi_{r_0+1} = C(\lambda\Omega) = 0$, and consequently $\xi_n = 0$. ∎

In the next lemma we will need the following disjoint union decomposition:

$$A_k = \bigcup_{s=1}^{k} A_k^{(s)}, \tag{4.13}$$

where

$$A_k^{(s)} = \{(i_1, \ldots, i_{2k}) \in A_k : i_1 = \ldots = i_s = *, i_{s+1} = 1\} \tag{4.14}$$

for $s = 1, \ldots, k$. We also set $A_k^{(0)} = \emptyset$ for every $k \in \mathbb{N}$.

REMARK 4.2. Let $k \in \mathbb{N}$ and $s = 1, \ldots, k$. Then there is a one-to-one correspondence between elements of $A_{k+1}^{(s)}$ and $\bigcup_{r=s-1}^{k} A_k^{(r)}$: for $(i_1, \ldots, i_{2k+2}) \in A_{k+1}^{(s)}$ we define

$$i'_l = \begin{cases} * & \text{for } l = 1, \ldots, s-1, \\ i_{l+2} & \text{for } l = s, \ldots, 2k. \end{cases}$$

Clearly, $(i'_1, \ldots, i'_{2k}) \in \bigcup_{r=s-1}^{k} A_k^{(r)}$.

Conversely, for $(i'_1, \ldots, i'_{2k}) \in \bigcup_{r=s-1}^{k} A_k^{(r)}$ we set

$$i_l = \begin{cases} * & \text{for } l = 1, \ldots, s, \\ 1 & \text{for } l = s+1, \\ i'_{l-2} & \text{for } l = s+2, \ldots, 2k+2. \end{cases}$$

LEMMA 4.3. *Suppose that* k, s, $(i_1, \ldots, i_{2k+2}) \in A_{k+1}^{(s)}$ *and* $(i'_1, \ldots, i'_{2k}) \in \bigcup_{r=s-1}^{k} A_k^{(r)}$ *are as in Remark 4.2. Then*

$$C^{i_{2k+2}} \ldots C^{i_1} \Omega = (s)_q \|f\|^2 C^{i'_{2k}} \ldots C^{i'_1} \Omega.$$

Proof. From (4.11) and (4.12) we have

$$\begin{aligned} C^{i_{2k+2}} \ldots C^{i_1} \Omega &= C^{i_{2k+2}} \ldots C^{i_s+2} C \overbrace{C^* \ldots C^*}^{s \text{ times}} \Omega = C^{i_{2k+2}} \ldots C^{i_{s+2}} C f^{\otimes s} \\ &= (s)_q \|f\|^2 C^{i_{2k+2}} \ldots C^{i_{s+2}} f^{\otimes(s-1)} \\ &= (s)_q \|f\|^2 C^{i_{2k+2}} \ldots C^{i_{s+2}} \overbrace{C^* \ldots C^*}^{s-1 \text{ times}} \Omega \\ &= (s)_q \|f\|^2 C^{i'_{2k}} \ldots C^{i'_1} \Omega. \ \blacksquare \end{aligned}$$

LEMMA 4.4. *Let* $(i_1, \ldots, i_{2k}) \in A_k$ *and let* $\mathcal{P}(i_1, \ldots, i_{2k})$ *be the set of 2-partitions* $\{(v_j, w_j)\}_{j=1,\ldots,k}$ *of* $\mathcal{K}_{2k}$ *such that* $i_{v_j} = *$, $i_{w_j} = 1$ *for* $j = 1, \ldots, k$. *For every* $\mathcal{V} \in \mathcal{P}(i_1, \ldots, i_{2k})$ *define an operator* $B_{\mathcal{V}} = B_{2k} \ldots B_2 B_1$, *where*

$$B_u = \begin{cases} C_j^* & \text{if } u = v_j, \\ C_j & \text{if } u = w_j, \end{cases}$$

and $C_j = C(f_j)$ *for some orthogonal system* $f_1, \ldots, f_k$ *of vectors from* $\mathcal{H}$ *such that* $\|f_j\| = \|f\|$ *for* $j = 1, \ldots, k$. *Then*

$$C^{i_{2k}} \ldots C^{i_1} \Omega = \sum_{\mathcal{V} \in \mathcal{P}(i_1, \ldots, i_{2k})} B_{\mathcal{V}} \Omega. \tag{4.15}$$

Proof. We use induction with respect to k.

We have $A_1 = \{(*, 1)\}$, $\mathcal{P}(*, 1) = \{\mathcal{V}\}$, where $\mathcal{V} = \{(1,2)\}$ and $B_{\mathcal{V}} = C_1 C_1^*$. Obviously $CC^* \Omega = \|f\|^2 \Omega = C_1 C_1^* \Omega$, which proves (4.15) for $k = 1$.

Suppose now that (4.15) holds for every sequence of length $2k$ for some $k \in \mathbb{N}$. Let $(i_1, \ldots, i_{2k+2}) \in A_{k+1}$. (4.13) implies that there is $s = 1, \ldots, k+1$ such that $(i_1, \ldots, i_{2k+2}) \in A_{k+1}^{(s)}$. For $r = 1, \ldots, s$, denote by $\mathcal{P}^{(r)}$ the set of all partitions $\mathcal{V}$ from $\mathcal{P}(i_1, \ldots, i_{2k+2})$ such that $(r, s+1) \in \mathcal{V}$. Clearly,

$$\mathcal{P}(i_1, \ldots, i_{2k+2}) = \bigcup_{r=1}^{s} \mathcal{P}^{(r)} \tag{4.16}$$

and $\mathcal{P}^{(r)} \cap \mathcal{P}^{(r')} = \emptyset$ for $r \neq r'$. Let $\mathcal{V} \in \mathcal{P}(i_1, \ldots, i_{2k+2})$. Then $\mathcal{V} \in \mathcal{P}^{(r)}$ for some $r = 1, \ldots, s$. Assume that $\mathcal{V} = \{V_1, \ldots, V_{k+1}\}$ where $V_j = (v_j, w_j)$ for $j = 1, \ldots, k+1$, and $v_1 < \ldots < v_{k+1}$. We define the partition $\mathcal{T}_r \mathcal{V}$ of $\mathcal{K}_{2k}$ by $\mathcal{T}_r \mathcal{V} = \{V'_1, \ldots, V'_k\}$ where $V'_j = (v'_j, w'_j)$ and

$$v'_j = \begin{cases} v_j & \text{if } j < r, \\ v_{j+1} - 1 & \text{if } r \le j < s, \\ v_{j+1} - 2 & \text{if } j \ge s, \end{cases} \qquad w'_j = \begin{cases} w_j - 2 & \text{if } j < r, \\ w_{j+1} - 2 & \text{if } j \ge r, \end{cases} \tag{4.17}$$

for $j = 1, \ldots, k$. Observe that $\mathcal{T}_r \mathcal{V} \in \mathcal{P}(i'_1, \ldots, i'_{2k})$ where $(i'_1, \ldots, i'_{2k})$ corresponds to $(i_1, \ldots, i_{2k+2})$ as in Remark 4.2. To see this it is enough to notice that (4.17) implies

$$v'_j = \begin{cases} j & \text{if } j < s, \\ v_{j+1} - 2 & \text{if } j \ge s, \end{cases}$$

and the definition in Remark 4.2 gives

$$i'_{v'_j} = \left\{ \begin{matrix} i_{v'_j} & \text{if } v'_j < s \\ i_{v'_j+2} & \text{if } v'_j \ge s \end{matrix} \right\} = i_{v_j} = * \quad \text{for every } j = 1, \ldots, k.$$

Similarly one can show that $i'_{w'_j} = 1$ for every $j = 1, \ldots, k$. It is easy to see that $\mathcal{T}_r$ is a bijection between $\mathcal{P}^{(r)}$ and $\mathcal{P}(i'_1, \ldots, i'_{2k})$. Now we can calculate:

$$\begin{aligned} B_{\mathcal{V}} \Omega &= B_{2k+2} \ldots B_{s+2} C_r C_s^* \ldots C_r^* \ldots C_1^* \Omega \\ &= B_{2k+2} \ldots B_{s+2} C_r (f_s \otimes \ldots \otimes f_r \otimes \ldots \otimes f_1) \\ &= q^{s-r} \|f\|^2 B_{2k+2} \ldots B_{s+2} (f_s \otimes \ldots \otimes f_{r+1} \otimes f_{r-1} \otimes \ldots \otimes f_1) \\ &= q^{s-r} \|f\|^2 B_{2k+2} \ldots B_{s+2} B_s \ldots B_{r+1} B_{r-1} \ldots B_1 \Omega = q^{s-r} \|f\|^2 B_{\mathcal{T}_r \mathcal{V}} \Omega. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{\mathcal{V} \in \mathcal{P}(i_1, \ldots, i_{2k+2})} B_{\mathcal{V}} \Omega &= \sum_{r=1}^{s} \sum_{\mathcal{V} \in \mathcal{P}^{(r)}} B_{\mathcal{V}} \Omega = \sum_{r=1}^{s} q^{s-r} \|f\|^2 \sum_{\mathcal{W} \in \mathcal{P}(i'_1, \ldots, i'_{2k})} B_{\mathcal{W}} \Omega \\ &= (s)_q \|f\|^2 \sum_{\mathcal{W} \in \mathcal{P}(i'_1, \ldots, i'_{2k})} B_{\mathcal{W}} \Omega = (s)_q \|f\|^2 C^{i'_{2k}} \ldots C^{i'_1} \Omega \end{aligned}$$

where the last equality follows from the inductive assumption. Hence taking into account Lemma 4.3 we arrive at

$$\sum_{\mathcal{V} \in \mathcal{P}(i_1, \ldots, i_{2k+2})} B_{\mathcal{V}} \Omega = C^{i_{2k+2}} \ldots C^{i_1} \Omega. \ \blacksquare$$

LEMMA 4.5. *Given $\mathcal{V} \in \mathcal{P}_2(2k)$ let $B_{\mathcal{V}}$ be the operator determined in Lemma* 4.4. *Then*

$$B_{\mathcal{V}}\Omega = q^{i(\mathcal{V})}\|f\|^{2k}\Omega. \tag{4.18}$$

Proof. We apply arguments similar to those in the proof of Lemma 2.16. Let us start with the case where $\mathcal{V}^T \neq \mathcal{V}$. Then there is $v \in \mathcal{K}_{2k}$ such that $(v, v+1) \in \mathcal{V}$, and $B_v = C_j^*$ and $B_{v+1} = C_j$ for some $j = 1, \ldots, k$. Hence

$$\begin{aligned} B_{\mathcal{V}}\Omega &= B_{2k}\ldots B_{v+2}C_jC_j^*B_{v-1}\ldots B_1\Omega \\ &= qB_{2k}\ldots B_{v+2}C_j^*C_jB_{v-1}\ldots B_1\Omega + \|f\|^2B_{2k}\ldots B_{v+2}B_{v-1}\ldots B_1\Omega \\ &= \|f\|^2B_{2k}\ldots B_{v+2}B_{v-1}\ldots B_1\Omega. \end{aligned}$$

The second equality comes from (4.9) while the third follows from the fact that $f_1, \ldots, f_k$ are orthogonal. Repetition of the above argument leads to

$$B_{\mathcal{V}}\Omega = \|f\|^{2k-d(\mathcal{V}^T)}B_{\mathcal{V}^T}\Omega.$$

The proof is thus finished when $\mathcal{V}^T = \emptyset$. Otherwise, Lemma 2.13 shows that we can find $j, k \in J$ such that $v_j < v_k < w_j < w_k$ and $v_k + 1 = w_j$. Thus, $B_s = C_k^*$ and $B_{s+1} = C_j$ for some $s \in \mathcal{K}_{d(\mathcal{V}^T)}$. The relations (4.9) lead to

$$\begin{aligned} B_{\mathcal{V}^T}\Omega &= B_{d(\mathcal{V}^T)}\ldots B_{s+2}C_jC_k^*B_{s-1}\ldots B_1\Omega \\ &= qB_{d(\mathcal{V}^T)}\ldots B_{s+2}C_k^*C_jB_{s-1}\ldots B_1\Omega = qB_{\mathcal{V}^T\sigma_1}\Omega \end{aligned}$$

where σ_1 is as in the proof of Lemma 2.16. The above procedure repeated $i(\mathcal{V})$ times yields (4.18). ∎

As a corollary we show that moments of the position operator are the same as in (4.2).

THEOREM 4.6. *Let $f \in \mathcal{H}$. Then*

$$\omega((C(f) + C^*(f))^{2k}) = \|f\|^{2k}\sum_{\mathcal{V}\in\mathcal{P}_2(2k)} q^{i(\mathcal{V})}. \tag{4.19}$$

Proof. It is easy to see that

$$\mathcal{P}_2(2k) = \bigcup_{(i_1,\ldots,i_{2k})\in A_k} \mathcal{P}(i_1, \ldots, i_{2k}).$$

Thus, (4.19) follows from Lemmas 4.1, 4.4 and 4.5. ∎

REMARK 4.7. Observe that for $q = 1$ one has $\omega((C(f) + C^*(f))^{2k}) = R_k\|f\|^{2k}$ where R_k is the number of all 2-partitions of $\mathcal{K}_{2k}$. By induction we can prove that

$$R_k = (2k-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2k-1)$$

for $k = 1, 2, \ldots$ In this way, the moments of the normal distribution can be obtained. For $q = 0$ we have $\omega((C(f) + C^*(f))^{2k}) = R_k^a\|f\|^{2k}$ where R_k^a is the number of all admissible 2-partitions of $\mathcal{K}_{2k}$. The R_k^a's are the so-called *Catalan numbers* (see [8]):

$$R_k^a = \frac{1}{k+1}\binom{2k}{k} = \frac{2^k}{(k+1)!}1 \cdot 3 \cdot \ldots \cdot (2k-1)$$

for $k = 1, 2, \ldots$, equal to the moments of the Wigner distribution with density

$$f(x) = \frac{2}{\pi}\sqrt{1-x^2}\chi_{[-1,1]}(x)$$

(see also [17, 13]).

COROLLARY 4.8. *Let $q \in (-1, 1)$. Then*

$$\operatorname{supp}\mu_{q,1} = \left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right]$$

and the density f_q of the measure $\mu_{q,1}$ is

$$f_q(x) = \frac{\sqrt{1-q}}{\pi}\sin\theta\prod_{n=1}^{\infty}(1-q^n)|1-q^ne^{-2i\theta}|^2 \tag{4.20}$$

where $x = \frac{2}{\sqrt{1-q}}\cos\theta$, $0 \le \theta \le \pi$.

Proof. Suppose that $T_n(x)$, $n = 0, 1, 2, \ldots$, are polynomials which satisfy the following recurrence relations:

$$T_0(x) = 1, \quad T_1(x) = x,$$
$$xT_n(x) = T_{n+1}(x) + (n)_qT_{n-1}(x), \quad n = 1, 2, \ldots$$

Let ν_q be the measure with density f_q. In [1] it is proved that the set $\{T_n\}_{n=0}^{\infty}$ is complete and orthogonal with respect to ν_q. More precisely,

$$\int_{\mathbb{R}} T_n(x)T_m(x)\,d\nu_q(x) = (n)_q!\delta_{n,m} \tag{4.21}$$

for $n, m = 0, 1, 2, \ldots$ and $(n)_q! = (1)_q(2)_q\ldots(n)_q$. Let

$$\psi_n(x) = \frac{1}{\sqrt{(n)_q!}}T_n(x)\sqrt{f_q(x)}, \quad x \in \left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right].$$

Relations (4.21) imply that $\{\psi_n(x)\}_{n=0}^{\infty}$ is an orthonormal basis in

$$\mathcal{H}_0 = L^2\left(\left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right], dx\right)$$

where dx denotes the Lebesgue measure. Define

$$l^*\psi_n = \sqrt{(n+1)_q}\psi_{n+1}, \quad n = 0, 1, 2, \ldots, \tag{4.22}$$

$$l\psi_n = \begin{cases} 0 & \text{for } n = 0, \\ \sqrt{(n)_q}\psi_{n-1} & \text{for } n = 1, 2, \ldots \end{cases} \tag{4.23}$$

Then l, l^* are bounded operators on $\mathcal{H}_0$ and l^* is adjoint to l. Moreover, they satisfy the q-commutation relation

$$ll^* - ql^*l = \mathbf{1}.$$

The correspondence

$$\psi_n \mapsto \frac{1}{\sqrt{(n)_q!}} 1^{\otimes n}, \qquad n = 0, 1, 2, \ldots,$$

defines an isometric isomorphism between $\mathcal{H}_0$ and $\mathcal{F}_q(\mathbb{C})$. The operators l, l^* correspond to $C(1)$, $C^*(1)$ under this isomorphism. Thus, the moments of $C(1)+C^*(1)$ are the same as the moments of $l+l^*$ in the probability system $(\mathcal{B}(\mathcal{H}_0), \omega_0)$, where $\omega_0(T) = \langle\psi_0, T\psi_0\rangle_{\mathcal{H}_0}$ for $T \in \mathcal{B}(\mathcal{H}_0)$. By Theorem 4.6 the operator $C(1) + C^*(1)$ has q-Gaussian distribution, as does $l + l^*$. But

$$\begin{aligned}(l+l^*)\psi_n(x) &= \sqrt{(n)_q}\psi_{n-1}(x) + \sqrt{(n+1)_q}\psi_{n+1}(x)\\ &= \sqrt{\frac{(n)_q}{(n-1)_q!}}T_{n-1}(x)\sqrt{f_q(x)} + \sqrt{\frac{(n+1)_q}{(n+1)_q!}}T_{n+1}(x)\sqrt{f_q(x)}\\ &= \frac{1}{\sqrt{(n)_q!}}((n)_qT_{n-1} + T_{n+1})\sqrt{f_q(x)} = x\psi_n(x)\end{aligned}$$

so

$$\begin{aligned}&\omega_0((l+l^*)^n)\\ &\quad= \langle\psi_0, (l+l^*)^n\psi_0\rangle_{\mathcal{H}_0} = \int x^n\psi_0(x)^2\,dx = \int x^n f_q(x)\,dx = \int x^n\,d\nu_q(x).\end{aligned}$$

Hence ν_q is the q-Gaussian distribution. ■

The following theorem gives another expression for moments of the q-Gaussian distribution.

THEOREM 4.9. *Suppose $f \in \mathcal{H}$ is such that $\|f\| = 1$. Then*

$$\omega((C(f)+C^*(f))^{2k}) = \sum_{r_1=1}^{1}\sum_{r_2=1}^{r_1+1}\cdots\sum_{r_k=1}^{r_{k-1}+1}(r_1)_q(r_2)_q\ldots(r_k)_q \tag{4.24}$$

for every $k = 1, 2, \ldots$

Proof. Let $k \in \mathbb{N}$, $s = 1, \ldots, k$ and

$$m_k^{(s)} = \sum_{(i_1,\ldots,i_{2k})\in A_k^{(s)}} \langle\Omega, C^{i_{2k}}\ldots C^{i_1}\Omega\rangle_q$$

where $A_k^{(s)}$ is as in (4.14). Moreover, set $m_k^{(s)} = 0$ for every $k \in \mathbb{N}$ and $s \le 0$. Lemma 4.3 and Remark 4.2 imply that for every k and $s = 1, \ldots, k+1$ we have

$$m_{k+1}^{(s)} = (s)_q\sum_{r=s-1}^{k} m_k^{(r)}.$$

Lemma 4.1, (4.13) and the above recurrence formula lead to

$$\begin{aligned}&\omega((C(f)+C^*(f))^{2k})\\ &\quad= \sum_{r_1=1}^{k} m_k^{(r_1)} = \sum_{r_1=1}^{k}(r_1)_q\sum_{r_2=r_1-1}^{k-1} m_{k-1}^{(r_2)} = \ldots\\ &\quad= \sum_{r_1=1}^{k}\sum_{r_2=r_1-1}^{k-1}\cdots\sum_{r_k=r_{k-1}-1}^{1}(r_1)_q(r_2)_q\ldots(r_{k-1})_q m_1^{(r_k)}\end{aligned}$$

where $(r)_q = 0$ for $r < 0$. By changing the order of summation we get

$$\begin{aligned}&\omega((C(f)+C^*(f))^{2k})\\ &\quad= \sum_{r_k=-k+2}^{1}\sum_{r_{k-1}=-k+3}^{r_k+1}\cdots\sum_{r_2=0}^{r_3+1}\sum_{r_1=1}^{r_2+1}(r_1)_q(r_2)_q\ldots(r_{k-1})_q m_1^{(r_k)}\\ &\quad= \sum_{r_k=1}^{1}\sum_{r_{k-1}=1}^{r_k+1}\cdots\sum_{r_2=1}^{r_3+1}\sum_{r_1=1}^{r_2+1}(r_k)_q(r_{k-1})_q\ldots(r_2)_q(r_1)_q.\end{aligned}$$

Here, we have written $(r_k)_q$ instead of $m_1^{(r_k)}$ because both expressions are equal to 1 for $r_k = 1$. If we write r_j instead of r_{k+1-j} for $j = 1, \ldots, k$ in the above formula, then we obtain (4.24). ■

5. Concluding remarks. To conclude our paper from a probabilistic point of view, let us discuss the convolution induced by the notion of Q-independence.

Let $\mathcal{M} = \{\mu : \mathbb{C}[X] \to \mathbb{C} : \mu \text{ is linear}\}$. Then $\mathcal{M}$ contains the set of probability measures on $\mathbb{R}$, but in general we do not assume that elements of $\mathcal{M}$ are positive. Given $\mu_1, \mu_2 \in \mathcal{M}$ one can define a "Q-product distribution" $\mu_1 \circ_Q \mu_2$ on $\mathbb{C}[X_1, X_2] = \mathbb{C}[X_1] * \mathbb{C}[X_2]$, where $\mathcal{A}*\mathcal{B}$ denotes the free product of *-algebras $\mathcal{A}$ and $\mathcal{B}$. To this end let us remind that $\mathbb{C}[X_1, X_2]$ is spanned by words of the form $X_{i_1}^{p_1}\ldots X_{i_n}^{p_n}$, where $n \in \mathbb{N}$, $i_1, \ldots, i_n \in \{1, 2\}$, $i_j \ne i_{j+1}$ and $p_j \ge 1$. The partition associated with such a word is $\mathcal{V} = \{V_1, V_2\}$, where $V_l = \{j = 1, \ldots, n : i_j = l\}$ for $l = 1, 2$. Observe that each V_l consists only of odd numbers or only of even numbers. Now, let

$$\mu_1 \circ_Q \mu_2(\mathbf{1}) = 1$$

and

$$\mu_1 \circ_Q \mu_2(X_{i_1}^{p_1} \dots X_{i_n}^{p_n}) = \sum_{\mathcal{W} \le \mathcal{V}} q_{\mathcal{W},\mathcal{V}} \prod_{k \in K} \mu_{s_k}(X_{s_k}^{r_k}),$$

where $\mathcal{W} = \{W_k\}_{k\in K}$, $s_k = 1,2$ for $k \in K$, $s_k = l$ if $W_k \subset V_l$, and $r_k = \sum_{j\in W_k} p_j$.

Definition 3.3 implies that elements a_1, a_2 with distributions μ_1, μ_2 are Q-independent if and only if $\mu_1 \circ_Q \mu_2$ is the joint distribution of a_1, a_2 (cf. Definition 2.3.4 in [19]).

Now, we can define

$$\mu_1 *_Q \mu_2(X^p) = \mu_1 \circ_Q \mu_2((X_1 + X_2)^p)$$

for every $p \in \mathbb{N}$. The element $\mu_1 *_Q \mu_2 \in \mathcal{M}$ is called the *Q-convolution* of μ_1, μ_2. It is an open problem whether this operation is positive, i.e. whether given any pair of probability measures μ_1, μ_2 the distribution $\mu_1 *_Q \mu_2$ is also a probability measure. Equivalently, the problem of positivity can be stated in the following form: for every pair of probability measures μ_1, μ_2 find a probability system with two Q-independent elements a_1, a_2 such that μ_i is the distribution of a_i for $i = 1, 2$. So far, we can answer this question only for the cases listed in Examples 3.4–3.6. The classical case leads to a construction of the tensor product of algebras, the free case to the reduced free product, and the case described by 3.6 leads to the so-called Boolean product of algebras (2).

Another open problem is to find some analytic tools for a proper description of Q-convolution. In [18] Voiculescu defined the R-transform which linearizes the free convolution of measures and plays the role of the logarithm of the Fourier transform for the classical case. In [10] Nica generalized this construction to the case of q-probability, while Speicher and Woroudi ([16]) did that for the Boolean convolution. It would be interesting to generalize this construction to an arbitrary family Q.

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