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The Grothendieck–Pietsch domination principle for nonlinear summing integral operators

by

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Abstract. We transform the concept of p-summing operators, $1 \le p < \infty$, to the more general setting of nonlinear Banach space operators. For 1-summing operators on $B(\Sigma, X)$-spaces having weak integral representations we generalize the Grothendieck–Pietsch domination principle. This is applied for the characterization of 1-summing Hammerstein operators on $C(S,X)$-spaces. For p-summing Hammerstein operators we derive the existence of control measures and p-summing extensions to $B(\Sigma, X)$-spaces.

J. Batt [1] gave a generalization of the Riesz representation theorem for a certain class of not necessarily linear Banach space valued operators on $C(S,X)$-spaces, where S is a compact Hausdorff space and X a Banach space. This class is defined by operators $T : C(S,X) \to Y$ which are uniformly continuous on bounded sets, fix the origin and have the Hammerstein property

$$\forall f, f_1, f_2 \in C(S,X), \operatorname{supp}(f_1) \cap \operatorname{supp}(f_2) = \emptyset :$$
$$T(f + f_1 + f_2) = T(f + f_1) + T(f + f_2) - T(f).$$

Furthermore, Batt proved that every Hammerstein operator has weak integral representations with kernels satisfying the Carathéodory conditions. In our earlier work [10] we proved that many of the classical characterizations of linear weakly compact operators on $C(S)$-spaces can be extended to Hammerstein operators on $C(S)$-spaces. This paper deals with p-summing operators ($1 \le p < \infty$). We extend the notion of p-summing operators to the setting of not necessarily linear Banach space operators.

According to the Grothendieck–Pietsch domination principle [11] every p-summing, linear and continuous operator $T : X \to Y$ can be dominated with the help of a probability measure $\mu \in C(B_{X^*})^*$:

$$\|Tx\| \le \pi_p(T)\Big(\int\limits_{B_{X^*}} |\langle x, x^*\rangle|^p \, d\mu(x^*)\Big)^{1/p}.$$

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Even more, this property characterizes p-summing operators. We generalize this theorem to Banach space valued integral operators on $B(\Sigma, X)$, the space of totally measurable functions on S with values in X: if a 1-summing operator has weak integral representations with kernels satisfying the Carathéodory conditions, then it can be dominated with the help of a probability measure in a generalized sense.

If these integral operators start on $B(\Sigma)$, the space of scalar valued totally measurable functions, then they are even characterized as those which can be dominated by probability measures in the above sense.

Applying our result to operators with the Hammerstein property leads to a generalized version of the domination principle for 1-summing Hammerstein operators on $C(S, X)$-spaces. In complete analogy to the linear theory, we can show that p-summing Hammerstein operators on $C(S, X)$ admit control measures and p-summing extensions to the space $B(\Sigma, X)$.

1. Preliminaries. We use standard Banach space terminology and we denote Banach spaces by $X, Y, Z, \ldots \in \mathbf{Ban}$. If $X \in \mathbf{Ban}$, then X^* will be its dual and X_α, $\alpha > 0$, its closed α-ball. S will denote a *compact Hausdorff space* and Σ its *Borel σ-algebra*. Let $C(S, X)$ and $B(\Sigma, X)$ be the Banach spaces of all continuous and all totally measurable X-valued functions on S, respectively (see [7]). We write $C(S)$ and $B(\Sigma)$ if X is the scalar field $\mathbb{R}$ or $\mathbb{C}$.

As usual χ_A, $A \in \Sigma$, denotes a characteristic function. A Σ-simple function $g \in B(\Sigma, X)$ has the form $g = \sum_{i=1}^n \chi_{A_i} x_i$, where $A_1, \ldots, A_n \in \Sigma$ are pairwise disjoint and $x_1, \ldots, x_n \in X$. For a family of pairwise disjoint sets $A_1, \ldots, A_n \in \Sigma$ with $\bigcup_{i=1}^n A_i = A$ we briefly write $\{A_1, \ldots, A_n\}$ Σ-P of A. $\mathrm{rca}(\Sigma)_+$ denotes the space of all bounded, regular, countably additive and positive measures on Σ.

We use the term operator for any map $T : X \to Y$. $\mathcal{M}(X, Y)$ denotes the linear space of all operators $T : X \to Y$ which are uniformly continuous on bounded subsets of X and which fix the origin. Given $T : X \to Y$, the modulus of boundedness is denoted by

$$M_\alpha(T) = \sup\{\|Tx\| : x \in X_\alpha\} \quad (\alpha > 0).$$

If T is linear and belongs to a subclass $\mathcal{A}(X, Y)$ of $\mathcal{M}(X, Y)$ we write $T \in \mathcal{A}^\ell(X, Y)$.

Linear summing operators were first introduced by A. Grothendieck [8]; the general theory, however, was developed by A. Pietsch [11]. In the following we use standard terminology which can be found in [11, 9, 5, 6].

Given $X \in \mathbf{Ban}$ and $1 \le p < \infty$, we define the Banach space $(l_p(X), \|\cdot\|_p)$ of so-called *strong l_p-sequences in X* by

$$l_p(X) := \Big\{(x_n)_n \in X^{\mathbb{N}} : \|(x_n)_n\|_p := \Big(\sum_n \|x_n\|^p\Big)^{1/p} < \infty\Big\}$$

and the Banach space $(W_p(X), w_p(\cdot))$ of *weak l_p-sequences in X* by

$$W_p(X) := \Big\{(x_n)_n \in X^{\mathbb{N}} : w_p((x_n)_n) := \sup_{x^* \in B_{X^*}} \Big(\sum_n |\langle x^*, x_n\rangle|^p\Big)^{1/p} < \infty\Big\}.$$

Note that

$$\|(x_n)_n\|_p = \lim_{N\to\infty} \|(x_n)_{n\le N}\|_p \quad \text{and} \quad w_p((x_n)_n) = \lim_{N\to\infty} w_p((x_n)_{n\le N}).$$

PROPOSITION 1.1. *For any operator $T : X \to Y$ the following conditions are equivalent*:

(i) $\forall \beta > 0 : \sup\{\|(Tx_n)_n\|_p \cdot \beta^{-1} : (x_n)_n \in W_p(X)_\beta\} < \infty$.

(ii) $\forall \beta > 0\ \exists c_\beta \ge 0\ \forall (x_n)_n \in W_p(X)_\beta : \|(Tx_n)_n\|_p \le c_\beta w_p((x_n)_n)$.

Write $\pi_{p,\beta}(T)$ for the supremum in (i) *and $s_{p,\beta}(T)$ for the infimum of all c_β in* (ii). *Then*

(a) $\pi_{p,\beta}(T) \le s_{p,\beta}(T) \le s_{p,\beta_0}(T)$, $\pi_{p,\beta}(T) \le \pi_{p,\beta_0}(T)\beta_0/\beta$, $\beta \le \beta_0$,

(b) $s_{p,\beta}(T) \le \gamma\pi_{p,\gamma}(T)$ *for* $\gamma = (1 + \beta^p)^{1/p}$,

(c) $s_{p,\beta}(T) \le 2^{1/p}\pi_{p,\gamma}(T)$ *for* $\gamma = 2^{1/p}\beta$.

Proof. For (i)⇒(ii) it is sufficient to prove inequality (b). Let $\xi = (x_n)_n \in W_p(X)_\beta$, $N \in \mathbb{N}$ and $\varepsilon > 0$ be such that

$$\Big(\frac{\|(Tx_n)_{n\le N}\|_p}{w_p(\xi)}\Big)^p \ge \Big(\frac{\|(Tx_n)_n\|_p}{w_p(\xi)}\Big)^p - \varepsilon. \tag{1}$$

Let $M := [w_p(\xi)^{-p}] + 1$ ([ ] denotes the integer part) and define a sequence $(y_n)_n \in l_p(X)$ by

$$y_n := \begin{cases} x_l & \text{with } 1 \le l \le N \text{ and } l = n \bmod N \text{ whenever } n \le NM, \\ 0 & \text{if } n > NM. \end{cases} \tag{2}$$

We obtain the inequalities

$$w_p((y_n)_n)^p \le M w_p(\xi)^p \le 1 + w_p(\xi)^p \le 1 + \beta^p$$

and

$$\Big(\frac{\|(Tx_n)_n\|_p}{w_p(\xi)}\Big)^p - \varepsilon \le \Big(\frac{\|(Tx_n)_{n\le N}\|_p}{w_p(\xi)}\Big)^p \le M\|(Tx_n)_{n\le N}\|_p^p$$
$$= \|(Ty_n)_n\|_p^p \le \pi_{p,(1+\beta^p)^{1/p}}(T)^p(1+\beta^p).$$

The implication (ii)⇒(i) and inequality (a) are trivial. It remains to prove assertion (c). For $\xi = (x_n)_n$ with $0 < w_p(\xi) \le \beta$ and $\varepsilon > 0$ we fix $N \in \mathbb{N}$ as in (1). Now, we choose $M \in \mathbb{N}$ with $\beta^p \le M w_p((x_n)_{n\le N})^p \le 2\beta^p$. For $(y_n)_n \in l_p(X)$ defined as in (2), we conclude

$$\beta^p \le w_p((y_n)_n)^p = M w_p((x_n)_{n\le N})^p \le 2\beta^p$$

and

$$\left(\frac{\|(Tx_n)_n\|_p}{w_p(\xi)}\right)^p - \varepsilon \le \frac{M\|(Tx_n)_{n\le N}\|_p^p}{Mw_p((x_n)_{n\le N})^p} = \left(\frac{\|(Ty_n)_n\|_p}{w_p((y_n)_n)}\right)^p$$
$$\le (\beta^{-1}\|(Ty_n)_n\|_p)^p \le 2\pi_{p,2^{1/p}\beta}(T)^p. \blacksquare$$

We call an operator $T : X \to Y$ *p-(absolutely) summing* if it satisfies one (and then all) of the conditions in Proposition 1.1. The class of these operators will be denoted by $\mathcal{P}_p(X,Y)$.

The conditions (a), (b), (c) of 1.1 illustrate an essential difference of linear and nonlinear summing operators. If $T \in \mathcal{P}_p^\ell(X,Y)$ then

$$\pi_{p,\beta}(T) = s_{p,\beta}(T) = \pi_{p,\beta_0}(T), \quad \beta \le \beta_0,$$

and so we may write $\pi_p(T) = \pi_{p,\beta}(T)$.

For $T : X \to Y$, $\beta > 0$ and $N \in \mathbb{N}$ let

$$\pi_{p,\beta}^N(T) := \beta^{-1}\sup\{\|(Tx_n)_{n\le N}\|_p : (x_n)_n \in X^{\mathbb{N}},\ w_p((x_n)_n) \le \beta\}.$$

Each $\pi_{p,\beta}^N(\cdot)$ is a *seminorm* on the space of the operators $T : X \to Y$ which are bounded on X_β and we have $\pi_{p,\beta}^N(\cdot) \le \pi_{p,\beta}^{N+1}(\cdot)$. Moreover, if we compare $\pi_{p,\beta}^N(\cdot)$ with $M_\beta(\cdot)$ we get

$$M_\beta(\cdot) = \pi_{p,\beta}^1(\cdot)\beta \le \pi_{p,\beta}^N(\cdot)\beta \le N^{1/p}M_\beta(\cdot).$$

Note that an operator $T : X \to Y$ is in $\mathcal{P}_p(X,Y)$ iff $\lim_{N\to\infty}\pi_{p,\beta}^N(T) < \infty$ for all $\beta > 0$. In this case

$$\pi_{p,\beta}(T) = \lim_{N\to\infty}\pi_{p,\beta}^N(T). \tag{3}$$

The domination principle (see [11], 17.3.2) is one of the fundamentals of the theory of linear summing operators. We are going to examine those nonlinear operators for which such a principle is valid. For this we introduce the class $\mathcal{AD}_p(X,Y)$ of *p-dominated* ($1 \le p < \infty$) *operators*, defined to be those $T : X \to Y$ such that

$$\forall\beta > 0\ \exists Z \in \mathbf{Ban}\ \exists L \in \mathcal{P}_p^\ell(X,Z)\ \forall x \in X_\beta : \quad \|Tx\| \le \|Lx\|.$$

For $T \in \mathcal{AD}_p(X,Y)$ and $\beta > 0$, we set

$$\mathrm{ad}_{p,\beta}(T) := \inf\{\pi_p(L) : L \in \mathcal{P}_p^\ell(X,\cdot),\ \|Tx\| \le \|Lx\|, x \in X_\beta\}.$$

$(\mathcal{AD}_p(X,Y), \mathrm{ad}_{p,\beta}(\cdot))$ is a seminormed space for each $\beta > 0$ and if $T \in \mathcal{AD}_p(X,Y)$ then

$$\mathrm{ad}_{p,\beta}(T) \ge s_{p,\beta}(T) \ge \beta^{-1}M_\beta(T). \tag{4}$$

THEOREM 1.2. *For $T : X \to Y$ the following conditions are equivalent:*

(i) $T \in \mathcal{AD}_p(X,Y)$.

(ii) $\exists Z \in \mathbf{Ban}\ \exists L \in \mathcal{P}_p^\ell(X,Z)\ \forall\beta > 0\ \exists c_\beta \ge 0\ \forall x \in X_\beta$:
$$\|Tx\| \le c_\beta\|Lx\|.$$

(iii) $\exists$ *probability measure* $\mu \in C(B_{X^*})^*\ \forall\beta > 0\ \exists d_\beta \ge 0\ \forall x \in X_\beta$:
$$\|Tx\| \le d_\beta\Big(\int_{B_{X^*}} |\langle x, x^*\rangle|^p\, d\mu(x^*)\Big)^{1/p}.$$

In that case
$$\mathrm{ad}_{p,\beta}(T) = \inf\{c_\beta\pi_p(L) : c_\beta, L \textit{ as in } \text{(ii)}\}$$
$$= \inf\{d_\beta : d_\beta, \mu \textit{ as in } \text{(iii)}\}, \quad \beta > 0.$$

Proof. (i)$\Rightarrow$(ii). We fix $\varepsilon > 0$. We can find a family $(Z_n)_n$ of Banach spaces and a sequence $(L_n)_n$ of operators $L_n \in \mathcal{P}_p^\ell(X, Z_n)$ with

$$\mathrm{ad}_{p,n}(T) \ge \pi_p(L_n) - \varepsilon, \quad \|Tx\| \le \|L_nx\|, \quad x \in X_n,\ n \in \mathbb{N}.$$

Defining the l_1-sum of the Z_n by

$$Z := (\bigoplus_n Z_n)_1 := \{(z_n)_n : z_n \in Z_n,\ \|(z_n)_n\| := \|(z_n)_n\|_1 < \infty\}$$

and fixing $N, M \in \mathbb{N}$ with $M > N$ and $\sum_{n\ge M} 2^{-n} < \varepsilon$, we define an operator $L_0 : X \to Z$ by

$$L_0x := L_Nx + \sum_{n\ge M}\frac{2^{-n}}{\pi_p(L_n)}L_nx, \quad x \in X.$$

Hence, $\pi_p(L_0) \le \pi_p(L_N) + \varepsilon$ and $L_0 \in \mathcal{P}_p^\ell(X,Z)$. For $n \ge M$ and $x \in X_n$ we have

$$\|Tx\| \le 2^n\pi_p(L_n)\|L_0x\|,$$

which proves (ii). The inequality $\|Tx\| \le \|L_0x\|$ for $x \in X_N$ leads to

$$\inf\{c_N\pi_p(L) : c_N, L \text{ as in (ii)}\} \le \pi_p(L_0) \le \pi_p(L_N) + \varepsilon$$
$$\le \mathrm{ad}_{p,N}(T) + 2\varepsilon.$$

(ii)$\Rightarrow$(iii). We choose $L \in \mathcal{P}_p^\ell(X,Z)$ according to (ii). The domination principle ([11], 17.3.2) provides us with a probability measure $\mu \in C(B_{X^*})^*$ such that

$$\|Lx\| \le \pi_p(L)\Big(\int_{B_{X^*}} |\langle x^*, x\rangle|^p\, d\mu(x^*)\Big)^{1/p}.$$

For $x \in X_\beta$, $\beta > 0$, we get

$$\|Tx\| \le c_\beta\|Lx\| \le c_\beta\pi_p(L)\Big(\int_{B_{X^*}} |\langle x^*, x\rangle|^p\, d\mu\Big)^{1/p}.$$

Moreover,

$$\inf\{d_\beta : d_\beta, \mu \text{ as in (iii)}\} \le \inf\{c_\beta \pi_p(L) : c_\beta, L \text{ as in (ii)}\}.$$

(iii)⇒(i) is obvious. ■

With the help of a simple weak* argument it is possible to majorize an operator $T \in \mathcal{AD}_p(X,Y)$ in an optimal sense:

$\forall \beta > 0\ \exists$ probability measure $\lambda_\beta \in C(B_{X^*})^*\ \forall x \in X_\beta$:

$$\|Tx\| \le \mathrm{ad}_{p,\beta}(T)\Big(\int\limits_{B_{X^*}} |\langle x, x^*\rangle|^p \, d\lambda_\beta(x^*)\Big)^{1/p}.$$

The Grothendieck–Pietsch domination principle is nothing but the statement $\mathcal{AD}_p^\ell = \mathcal{P}_p^\ell$, and we have

$$\mathrm{ad}_{p,\beta}(T) = \pi_{p,\beta}(T) = \pi_p(T), \qquad \beta > 0,$$

for every operator $T \in \mathcal{P}_p^\ell(X,Y)$.

Various inclusions known from the linear case carry over directly to the classes $\mathcal{AD}_p$, $1 \le p < \infty$. For example, using Lebesgue's theorem (see [9], 19.5.4, 19.6.2), we get $\mathcal{AD}_p \subset \mathcal{AD}_q$, $\mathcal{AD}_p \subset \mathcal{P}_q$, $1 \le p \le q < \infty$, and every operator of $\mathcal{AD}_p$ maps weakly null sequences to norm null sequences.

2. Nonlinear summing integral operators. In this main section we investigate summing integral operators on $B(\Sigma, X)$-spaces which admit an integral representation with kernels satisfying the Carathéodory conditions. For those operators we prove a generalized version of the domination principle.

We first state a useful dependence of the strong and weak l_p-norms on totally measurable functions:

LEMMA 2.1. *Let $(g_j)_j$ be a sequence in $W_p(B(\Sigma, X))$. Then*

$$w_p((g_j)_j) = \sup\{w_p((g_j(t))_j) : t \in S\}.$$

Proof. Define p^* by $1/p + 1/p^* = 1$. Then

$$\begin{aligned} w_p((g_j)_j) &= \sup\{\|(\langle \xi, g_j\rangle)_j\|_p : \xi \in B_{B(\Sigma,X)^*}\} \\ &= \sup\Big\{\Big\|\sum_j a_j g_j\Big\| : (a_j)_j \in l_{p^*}\Big\} \\ &= \sup\Big\{\sum_j |\langle x^*, a_j g_j(t)\rangle| : t \in S,\ (a_j)_j \in l_{p^*},\ x^* \in B_{X^*}\Big\} \\ &= \sup\{\|(\langle x^*, g_j(t)\rangle)_j\|_p : t \in S,\ x^* \in B_{X^*}\}. \ ■ \end{aligned}$$

Recall that in order to compute the seminorms $\pi_{1,\beta}(T)$ and $s_{1,\beta}(T)$ of an operator $T : X \to Y$, it is only necessary to know T on the β-ball.

LEMMA 2.2. *Let S be a compact Hausdorff space, $\beta > 0$ and let $T : B(\Sigma, X)_\beta \to \mathbb{R}$, $Tg = \int_S \varphi(t, g(t))\, d\lambda(t)$, $g \in B(\Sigma, X)_\beta$, with $\lambda \in \mathrm{rca}(\Sigma)_+$ and a nonnegative kernel φ with the properties*

(i) *$\varphi : S \times X_\beta \to \mathbb{C}$,*
(ii) *$\varphi(t, g(t)) \in L_1(\lambda)$ for every $g \in B(\Sigma, X)_\beta$,*
(iii) *$\varphi(t, \cdot)$ is continuous λ-a.e.*

If $\pi_{1,\beta}(T) < \infty$, then there exists a probability measure $\mu \in C(S)^*$ such that

$$\|(Tg_j)_j\|_1 \le 2\pi_{1,\beta}(T) \int\limits_S w_1((g_j(t))_j)\, d\mu(t), \qquad w_1((g_j)_j) \le \beta/2.$$

Proof. Note first that by using the constant sequence $(0)_{j\in\mathbb{N}}$, $\pi_{1,\beta}(T) < \infty$ implies $T0 = 0$. Hence $\varphi(\cdot, 0) = 0$ λ-a.e.

We define $\varphi_n := \varphi \wedge n$, $n \in \mathbb{N}$. Obviously $\varphi_n(\cdot, x) \in \mathcal{L}_\infty(\lambda)$ for $x \in X_\beta$. According to [7], 11.4.1, we can assume the existence of a lifting ϱ of $\mathcal{L}_\infty(\lambda)$. Define

$$\widehat{\varphi}_n(\cdot, x) := \varrho(\varphi_n(\cdot, x)), \qquad x \in X_\beta.$$

The lifting properties lead to $\widehat{\varphi}_n(\cdot, x) = \varphi_n(\cdot, x)$ λ-a.e., $x \in X_\beta$, and

$$\sup_{t\in S} |\widehat{\varphi}_n(t, x)| = \|\varphi_n(\cdot, x)\|_\infty, \qquad x \in X_\beta.$$

We claim that

$$\mathcal{H}_n := \Big\{ \sum_{i,j=1}^m \chi_{B_j} \widehat{\varphi}_n(\cdot, x_{ij}) : \{B_1, \ldots, B_m\}\ \Sigma\text{-P of } S, \\ \varrho(B_j) = B_j, w_1((x_{ij})_{i\le m}) \le \beta,\ j \le m\Big\}$$

is a directed set with respect to the pointwise order "$\le$":

Let $f, g \in \mathcal{H}_n$. Then it is possible to find a measurable set A such that $\varrho(A) = A$ and $f \vee g = \chi_A f + \chi_{S\setminus A} g$. Hence, $f \vee g$ belongs to $\mathcal{H}_n$.

We next claim that $\mathcal{H}_n$ is a bounded subset of $L_1(\lambda)$:

Let $g = \sum_{i,j=1}^m \chi_{B_j} \widehat{\varphi}_n(\cdot, x_{ij}) \in \mathcal{H}_n$. Define $g_{ij} := \chi_{B_j} x_{ij}$, $1 \le i, j \le m$. Then $w_1((g_{ij})_{1\le i,j\le m}) \le \beta$, and our assertion follows from

$$\begin{aligned} \beta\pi_{1,\beta}(T) &\ge \sum_{i,j} Tg_{ij} = \sum_{i,j} \int\limits_S \chi_{B_j}(t)\varphi(t, x_{ij})\, d\lambda(t) \\ &\ge \sum_{i,j} \int\limits_S \chi_{B_j}(t)\varphi_n(t, x_{ij})\, d\lambda(t) \\ &= \int\limits_S \sum_{i,j} \chi_{B_j}(t)\widehat{\varphi}_n(t, x_{ij})\, d\lambda(t) = \int\limits_S g\, d\lambda(t). \end{aligned}$$

Finally, we state a triviality:

$$\pi_{1,\beta}(\widehat{\varphi}_n(t))\beta = \sup_{\mathcal{H}_n} g(t), \qquad t \in S.$$

Now, applying 8.4.4 and 11.5.4 of [7] leads to $\pi_{1,\beta}(\widehat{\varphi}_n(t)) \in L_1(\lambda)$ with

$$\int_S \pi_{1,\beta}(\widehat{\varphi}_n(t))\,d\lambda(t) = \beta^{-1} \sup_{g\in\mathcal{H}_n} \int_S g\,d\lambda(t) \le \pi_{1,\beta}(T).$$

We claim that $s_{1,\beta-1}(\widehat{\varphi}_n(\cdot))$ is λ-measurable. This can be seen by applying [7], 11.5.4, to the directed set

$$\mathcal{G}_n := \Big\{ \sum_{i,j=1}^{m} \chi_{B_j}\widehat{\varphi}_n(\cdot, x_{ij}) w_1((x_{ij})_{i\le m})^{-1} : \{B_1,\ldots,B_m\}\ \Sigma\text{-P of } S,$$
$$\varrho(B_j) = B_j,\ 0 < w_1((x_{ij})_{i\le m}) \le \beta/2,\ j \le m\Big\},$$

of λ-measurable functions, where

$$s_{1,\beta/2}(\widehat{\varphi}_n(t)) = \sup_{\mathcal{G}_n} g(t).$$

Hence, by Proposition 1.1, $s_{1,\beta/2}(\widehat{\varphi}_n(t)) \in L_1(\lambda)$ with

$$s_{1,\beta/2}(\widehat{\varphi}_n(t)) \le 2\pi_{1,\beta}(\widehat{\varphi}_n(t)).$$

For Σ-simple functions $g_j = \sum_{i=1}^{m_j} \chi_{B_{ij}} x_{ij} \in B(\Sigma, X)$ with $w_1((g_j)_j) \le \beta/2$, we then derive

$$\begin{aligned}(5)\qquad \sum_j \int_S \varphi_n(t, g_j(t))\,d\lambda(t) &= \int_S \sum_{i,j} \chi_{B_{ij}}\varphi_n(t, x_{ij})\,d\lambda(t)\\ &= \int_S \sum_{i,j} \chi_{B_{ij}}\widehat{\varphi}_n(t, x_{ij})\,d\lambda(t)\\ &\le \int_S s_{1,\beta/2}(\widehat{\varphi}_n(t)) w_1((g_j(t))_j)\,d\lambda(t)\\ &= d_n \int_S w_1((g_j(t))_j)\,d\lambda_n(t),\end{aligned}$$

where $d_n := \int_S s_{1,\beta/2}(\widehat{\varphi}_n(t))\,d\lambda(t)$ and the probability measure λ_n is defined by

$$\lambda_n(A) = d_n^{-1} \int_A s_{1,\beta/2}(\widehat{\varphi}_n(t))\,d\lambda(t).$$

By uniform approximation with Σ-simple functions we can extend inequality (5) to all sequences $(g_j)_j$ in $B(\Sigma, X)$ with $w_1((g_j)_j) \le \beta/2$. Observe that

$$d_n \le 2\int_S \pi_{1,\beta}(\widehat{\varphi}_n(t))\,d\lambda(t) \le 2\pi_{1,\beta}(T).$$

Finally, we take a weak* cluster point $\mu \in C(S)^*$ of the sequence $(\lambda_n)_n$. Then, for any finite sequence $(f_j)_j$ of functions in $C(S, X)$ with $w_1((f_j)_j) \le \beta/2$, we may conclude

$$\begin{aligned}(6)\qquad \sum_j Tf_j = \sum_j \int_S \varphi(t, f_j(t))\,d\lambda(t) &= \lim_{n\to\infty} \sum_j \int_S \varphi_n(t, f_j(t))\,d\lambda(t)\\ &\le 2\pi_{1,\beta}(T) \int_S w_1((f_j(t))_j)\,d\mu(t).\end{aligned}$$

To see that inequality (6) remains valid if we replace $C(S, X)$ by $B(\Sigma, X)$ we have to make a few observations.

We take a compact $K \subseteq S$ and $x \in X_{\beta/2}$. Then, for every open U with $K \subseteq U \subseteq S$, we may find a function $\phi_{K,U} \in C(S)$ with $0 \le \phi_{K,U} \le 1$ and

$$\phi_{K,U}(t) = \begin{cases} 0, & t \in S \setminus U,\\ 1, & t \in K.\end{cases}$$

We conclude from inequality (6) that

$$\begin{aligned}T(\chi_K x) = \int_S \varphi(t, \chi_K(t)x)\,d\lambda(t) &\le \int_S \varphi(t, \phi_{K,U}(t)x)\,d\lambda(t)\\ &\le 2\pi_{1,\beta}(T) \int_S \|\phi_{K,U}(t)x\|\,d\mu(t).\end{aligned}$$

Since μ is a regular measure we can approximate χ_K μ-a.e. with functions $\phi_{K,U}$ as above to obtain

$$(7)\qquad T(\chi_K x) \le 2\pi_{1,\beta}(T) \int_S \|\chi_K(t)x\|\,d\mu(t) = 2\pi_{1,\beta}(T)\mu(K)\|x\|.$$

Exploiting the regularity of λ and μ we can prove inequality (7) for arbitrary $A \in \Sigma$. For this take a sequence of compact sets $K_i \subseteq A$ with $(\lambda + \mu)(A \setminus K_i) \to 0$ as $i \to \infty$. Then

$$\begin{aligned}T(\chi_A x) = \int_S \varphi(t, \chi_A(t)x)\,d\lambda(t) &= \lim_{i\to\infty} \int_S \varphi(t, \chi_{K_i}(t)x)\,d\lambda(t)\\ &\le 2\pi_{1,\beta}(T) \lim_{i\to\infty} \mu(K_i)\|x\| = \pi_{1,\beta}(T)\mu(A)\|x\|.\end{aligned}$$

Hence, we may extend inequality (7) to Σ-simple functions $g \in B(\Sigma, X)_{\beta/2}$:

$$(8)\qquad Tg \le 2\pi_{1,\beta}(T) \int_S \|g(t)\|\,d\mu(t).$$

By uniform approximation with Σ-simple functions and by using the kernels $(\varphi_n)_n$ again, we can extend inequality (8) to all functions in $B(\Sigma, X)_{\beta/2}$.

Now, take a sequence $(g_j)_{j\le m}$ in $B(\Sigma, X)$ with $w_1((g_j)_{j\le m}) \le \beta/2$. Let $H := \operatorname{span}\{g_j : j \le m\}$ and fix $\varepsilon > 0$. Applying Luzin's theorem ([7], III, 8.3) to the sequence $(g_j)_{j\le m}$ and the measure μ provides us with a compact $K \subseteq S$ such that

$$\mu(S \setminus K) \le \frac{\varepsilon}{2c\beta m}, \qquad g_j|_K \in C(K, X),\ j \le m,$$

with $c := \max\{2\pi_{1,\beta}(T), 1\}$. Let $\theta_1 : H \to C(K, X)$ be the restriction map $g \mapsto g|_K$. With an extension theorem of Bombal and Cembranos ([4], p. 138), we obtain an isometric embedding

$$\theta_2 : \theta_1(H) \to C(S, X)$$

such that $\theta_2(f)|_K = f$ for $f \in \theta_1(H)$. We define $\theta := \theta_2 \circ \theta_1$. Then $\|\theta\| \le 1$ and $\theta(g)|_K = g|_K$ for $g \in H$. For $f_j := \theta(g_j)$, $j \le m$, we derive from Lemma 2.1,

$$w_1((f_j)_j) \le \|\theta\| w_1((g_j)_j) \le \beta/2,$$

and obtain

$$(9)\qquad \int_S w_1((f_j(t))_j)\,d\mu(t) \le \int_S w_1((g_j(t))_j)\,d\mu(t) + \int_{S\setminus K} |w_1((f_j(t))_j) - w_1((g_j(t))_j)|\,d\mu(t) \le \int_S w_1((g_j(t))_j)\,d\mu(t) + \frac{\varepsilon}{2cm}.$$

Using inequality (8) we deduce that for every $j \le m$,

$$(10)\qquad |Tg_j - Tf_j| \le \int_{S\setminus K} \varphi(t, g_j(t))\,d\lambda(t) + \int_{S\setminus K} \varphi(t, f_j(t))\,d\lambda(t)$$
$$\le 2\pi_{1,\beta}(T)\Big(\int_{S\setminus K} \|g_j(t)\|\,d\mu(t) + \int_{S\setminus K} \|f_j(t)\|\,d\mu(t)\Big)$$
$$\le 2\pi_{1,\beta}(T)\beta\mu(S\setminus K) \le \frac{\varepsilon}{2m}.$$

Finally, by putting (6), (9) and (10) together we get what we wanted to prove:

$$\sum_j Tg_j \le \sum_j |Tg_j - Tf_j| + \sum_j Tf_j$$
$$\le \varepsilon/2 + 2\pi_{1,\beta}(T)\int_S w_1((f_j(t))_j)\,d\mu(t)$$
$$\le \varepsilon + 2\pi_{1,\beta}(T)\int_S w_1((g_j(t))_j)\,d\mu(t). \ \blacksquare$$

The following theorem provides a characterization of 1-summing operators on $B(\Sigma, X)$-spaces which have weak integral representations in the sense of Lemma 2.2.

THEOREM 2.3. *Let $T : B(\Sigma, X)_\beta \to Y$, $\beta > 0$, have weak integral representations $\langle Tg, y^*\rangle = \int_S u_{y^*}(t, g(t))\,d\omega_{y^*}(t)$, $y^* \in Y^*$, with respect to measures $\omega_{y^*} \in \mathrm{rca}(\Sigma)_+$ and kernels u_{y^*} which satisfy the conditions* (i)–(iii) *of Lemma 2.2 and the additional condition*

(iv) $u_{y^*}(\cdot, 0) = 0$ ω_{y^*}*-a.e.*

If $\pi_{1,\beta}(T) < \infty$, then there exists a probability measure $\mu \in C(S)^*$ such that

$$\|(Tg_j)_j\|_1 \le 4\pi_{1,\beta}(T)\int_S w_1((g_j(t))_j)\,d\mu(t), \qquad w_1((g_j)_j) \le \beta/2.$$

Proof. Let $I = \{y_1^*, \ldots, y_n^*\} \subset B_{Y^*}$. We define $\lambda := \sum_{i=1}^n \omega_{y_i^*}$. Let $h_{y_i^*}$ be the Radon–Nikodým derivative of $\omega_{y_i^*}$ with respect to λ. Then

$$\langle Tg, y_i^*\rangle = \int_S u_{y_i^*}(t, g(t))\,d\omega_{y_i^*}(t) = \int_S \widehat{u}_{y_i^*}(t, g(t))\,d\lambda(t)$$

with $\widehat{u}_{y_i^*}(t,\cdot) = h_{y_i^*}(t)u_{y_i^*}(t,\cdot)$. Note that $h_{y_i^*}(\cdot) \ge 0$ λ-a.e. and so

$$(11)\qquad \int_S |u_{y_i^*}(t, g(t))|\,d\omega_{y_i^*}(t) = \int_S |\widehat{u}_{y_i^*}(t, g(t))|\,d\lambda(t).$$

For the operator $P_I : B(\Sigma, X)_\beta \to \mathbb{R}$ defined by

$$P_I g := \int_S |\widehat{u}_{y_1^*}(t, g(t))| \vee \ldots \vee |\widehat{u}_{y_n^*}(t, g(t))|\,d\lambda(t)$$

we claim that $\pi_{1,\beta}(P_I) \le 2\pi_{1,\beta}(T)$:

For this, we choose functions $(g_j)_{j\le m}$ with $w_1((g_j)_{j\le m}) \le \beta$. We fix $j \le m$ and take pairwise disjoint, measurable sets $A_{j1}, \ldots, A_{jn}$ such that

$$P_I(g_j) = \sum_{i=1}^n \int_{A_{ji}} |\widehat{u}_{y_i^*}(t, g_j(t))|\,d\lambda(t) \le \sum_{i=1}^n \int_S |\widehat{u}_{y_i^*}(t, \chi_{A_{ji}} g_j(t))|\,d\lambda(t)$$
$$= \sum_{i=1}^n \int_S |u_{y_i^*}(t, \chi_{A_{ji}} g_j(t))|\,d\omega_{y_i^*}(t) \qquad \text{by (11)}$$
$$= \sum_{i=1}^n \int_{A_{ji}} |u_{y_i^*}(t, g_j(t))|\,d\omega_{y_i^*}(t) \qquad \text{by (iv)}$$
$$\le \sum_{i=1}^n \int_{A_{ji}} (|\mathrm{Re}\, u_{y_i^*}(t, g_j(t))| + |\mathrm{Im}\, u_{y_i^*}(t, g_j(t))|)\,d\omega_{y_i^*}(t) =: J.$$

Then we may find measurable sets $B_{ji}, C_{ji}, D_{ji}, E_{ji}$ with $A_{ji} = B_{ji} \dot\cup C_{ji} = D_{ji} \dot\cup E_{ji}$, $1 \le i \le n$, such that

$$J = \sum_{i=1}^n \int_{B_{ji}} \mathrm{Re}\, u_{y_i^*}(t, g_j(t))\,d\omega_{y_i^*}(t) - \int_{C_{ji}} \mathrm{Re}\, u_{y_i^*}(t, g_j(t))\,d\omega_{y_i^*}(t)$$
$$+ \sum_{i=1}^n \int_{D_{ji}} \mathrm{Im}\, u_{y_i^*}(t, g_j(t))\,d\omega_{y_i^*}(t) - \int_{E_{ji}} \mathrm{Im}\, u_{y_i^*}(t, g_j(t))\,d\omega_{y_i^*}(t)$$

$$\le \sum_{i=1}^{n} (|\langle T(\chi_{B_{ji}} g_j), y_i^*\rangle| + |\langle T(\chi_{C_{ji}} g_j), y_i^*\rangle| + |\langle T(\chi_{D_{ji}} g_j), y_i^*\rangle| + |\langle T(\chi_{E_{ji}} g_j), y_i^*\rangle|)$$
$$\le \sum_{i=1}^{n} (\|Tg_j\chi_{B_{ji}}\| + \|Tg_j\chi_{C_{ji}}\| + \|Tg_j\chi_{D_{ji}}\| + \|Tg_j\chi_{E_{ji}}\|).$$

From our construction and Lemma 2.1 we see that

$$w_1(\{g_j\chi_{B_{ji}}, g_j\chi_{C_{ji}}\}_{ji}) \le \beta, \quad w_1(\{g_j\chi_{D_{ji}}, g_j\chi_{E_{ji}}\}_{ji}) \le \beta$$

and so

$$\sum_{j=1}^{m} P_I g_j \le \sum_{j=1}^{m}\sum_{i=1}^{n} (\|Tg_j\chi_{B_{ji}}\| + \|Tg_j\chi_{C_{ji}}\| + \|Tg_j\chi_{D_{ji}}\| + \|Tg_j\chi_{E_{ji}}\|)$$
$$\le 2\beta\pi_{1,\beta}(T),$$

which proves our last assertion.

By applying Lemma 2.2 we get a probability measure $\lambda_I \in C(S)^*$ such that

$$\|(P_I g_j)_j\|_1 \le 4\pi_{1,\beta}(T) \int_S w_1((g_j(t))_j)\, d\lambda_I(t)$$

is valid for all sequences $(g_j)_j$ in $B(\Sigma, X)$ with $w_1((g_j)_j) \le \beta/2$. To replace P_I by T in the above inequality we use again a weak* argument. Note that $\mathcal{E} := \{I \subset B_{Y^*} : |I| < \infty\}$ is directed by inclusion and for $I_1, I_2 \in \mathcal{E}$ with $I_1 \subseteq I_2$ we have

$$\|(P_{I_1} g_j)_j\|_1 \le \|(P_{I_2} g_j)_j\|_1.$$

We choose a weak* cluster point μ of the net $(\lambda_I)_{I\in\mathcal{E}}$. Then, for every $I \in \mathcal{E}$ and every finite sequence $(f_j)_j$ in $C(S,X)$ with $w_1((f_j)_j) \le \beta/2$, we may conclude that

$$(12) \qquad \|(P_I f_j)_j\|_1 \le 4\pi_{1,\beta}(T) \int_S w_1((f_j(t))_j)\, d\mu(t).$$

By using the same arguments as in the proof of Lemma 2.2 we see that inequality (12) remains valid if we replace $C(S,X)$ by $B(\Sigma, X)$.

Finally, take a sequence $(g_j)_{j\le n}$ in $B(\Sigma, X)$ with $w_1((g_j)_{j\le n}) \le \beta/2$. Then we find $I = \{y_1^*, \ldots, y_n^*\} \in \mathcal{E}$ with

$$\|(Tg_j)_{j\le n}\|_1 = \|(\langle Tg_j, y_j^*\rangle)_{j\le n}\|_1 \le \|(P_I g_j)_{j\le n}\|_1$$
$$\le 4\pi_{1,\beta}(T) \int_S w_1((g_j(t))_{j\le n})\, d\mu(t). \ \blacksquare$$

The characterization of p-dominated operators on $C(S)$-spaces takes a simpler form if we apply the corresponding theorem for linear p-summing operators on the respective spaces ([11], 17.3.3):

An operator $T : C(S) \to Y$ belongs to $\mathcal{AD}_p(C(S), Y)$ $(1 \le p < \infty)$ iff there exists a probability measure $\lambda \in C(S)^*$ and for all $\beta > 0$ there exist constants $c_\beta \ge 0$ such that

$$(13) \qquad \|Tf\| \le c_\beta \Big(\int_S |f(t)|^p\, d\lambda(t)\Big)^{1/p}, \qquad f \in C(S)_\beta.$$

Computing the infimum of all such constants c_β with respect to probability measures $\lambda \in C(S)^*$ we get $\mathrm{ad}_{p,\beta}(T)$.

Note that (13) remains a sufficient condition for p-dominated operators if we replace $C(S)$ by $B(\Sigma)$ (Theorem 1.2).

From (13) and Theorem 2.3 it follows that every 1-summing operator on a $B(\Sigma)$-space which has weak integral representations in the sense of 2.3, is 1-dominated.

3. Summing Hammerstein operators. First, we recall basic definitions and facts. For detailed information about linear operators and Hammerstein operators on $C(S,X)$-spaces we refer to [2, 3, 1, 10]. We call an operator $T \in \mathcal{M}(C(S,X), Y)$ a *Hammerstein operator*, and write $T \in \mathcal{M}_{\mathrm{HP}}(C(S,X), Y)$, if

$$(14) \qquad \forall f, f_1, f_2 \in C(S,X), \operatorname{supp}(f_1) \cap \operatorname{supp}(f_2) = \emptyset :$$
$$T(f + f_1 + f_2) = T(f + f_1) + T(f + f_2) - T(f).$$

If we are interested in operators with the Hammerstein property (14) in a special subclass $\mathcal{A} \subset \mathcal{M}(C(S,X), Y)$, we write $\mathcal{A}_{\mathrm{HP}}(C(S,X), Y)$.

Let $U : \Sigma \to \mathcal{M}(X, Y)$ be an additive set function and $\alpha > 0$. Then the *semivariation* $\mathrm{sv}(U_\alpha, \cdot) : \Sigma \to [0, \infty]$ is defined by

$$\mathrm{sv}(U_\alpha, A) := \sup\Big\{\Big\|\sum_{j=1}^{r} U(A_j)x_j\Big\| : \{A_1, \ldots, A_r\}\ \Sigma\text{-P of } A,\ x_1, \ldots, x_r \in X_\alpha\Big\}.$$

In [1], Theorem 2, the Riesz representation theorem was extended to Hammerstein operators: every Hammerstein operator $T \in \mathcal{M}_{\mathrm{HP}}(C(S,X), Y)$ has an integral representation

$$Tf = \int_S f(x)\, dU, \qquad f \in C(S,X),$$

for an additive set function $U : \Sigma \to \mathcal{M}(X, Y^{**})$ with bounded semivariation. In many cases (see [10], Theorem 3), we get the existence of a control measure $\lambda \in \mathrm{rca}(\Sigma)_+$ for the family of semivariations $\mathrm{sv}(U_\alpha, \cdot)$, $\alpha > 0$, which means

$$\mathrm{sv}(U_\alpha, \cdot) \to 0 \quad \text{whenever} \quad \lambda(\cdot) \to 0,\ \alpha > 0.$$

A Hammerstein operator $T \in \mathcal{M}_{\mathrm{HP}}(C(S,X),Y)$ which admits a control measure can be represented by an additive set function $U : \Sigma \to \mathcal{M}(X,Y)$ and has an extension ([10], Proposition 2.2) $\widehat{T} : B(\Sigma, X) \to Y$,

$$\widehat{T}g = \int_S g(x)\, dU, \qquad g \in B(\Sigma, X).$$

Let us now pass to summing Hammerstein operators and their extensions.

THEOREM 3.1. *Every p-summing Hammerstein operator has a control measure.*

Proof. Let $T \in \mathcal{P}_{p,\mathrm{HP}}(C(S,X),Y)$. It suffices to verify criterion (v) of [10], Theorem 3. Let $(f_n)_n$ be a sequence in $C(S,X)_\alpha$, $\alpha > 0$, with pairwise disjoint supports. By Lemma 2.1,

$$w_p((f_n)_n) = \sup\{\|(\langle x^*, f_n(t)\rangle)_n\|_p : x^* \in B_{X^*},\ t \in S\} \le \sup_n \|f_n\| \le \alpha.$$

Hence, $\|(Tf_n)_n\|_p \le \pi_{p,\alpha}(T)\alpha$ and therefore $\lim_{n\to\infty} \|Tf_n\| = 0$. ■

THEOREM 3.2. *Let $T \in \mathcal{P}_{p,\mathrm{HP}}(C(S,X),Y)$ and let $\widehat{T} : B(\Sigma,X) \to Y$ be its extension. Then $\widehat{T} \in \mathcal{P}_p(B(\Sigma,X),Y)$ and $\pi_{p,\beta}(T) = \pi_{p,\beta}(\widehat{T})$, $\beta > 0$.*

Proof. According to (3), it suffices to show that $\pi^N_{p,\beta}(T) = \pi^N_{p,\beta}(\widehat{T})$, $\beta > 0$, $N \in \mathbb{N}$. So, let $(g_n)_{n\le N} \in W_p(B(\Sigma,X))_\beta$, $\varepsilon > 0$, with

$$\|(\widehat{T}g_n)_{n\le N}\|_p \ge \pi^N_{p,\beta}(\widehat{T})\beta - \varepsilon.$$

For $H := \mathrm{span}\{g_n : n \le N\}$ we can find a linear map ([10], Proposition 2.4) $\Theta : H \to C(S,X)$ with

$$\|\Theta\| \le 1, \qquad \|T\Theta g - \widehat{T}g\| < \varepsilon N^{-1/p}, \qquad g \in H_\beta.$$

Setting $h_n := \Theta g_n$, we may conclude $w_p((h_n)_{n\le N}) \le w_p((g_n)_{n\le N})\|\Theta\| \le \beta$ and

$$\|(\widehat{T}g_n)_{n\le N}\|_p \le \|(\widehat{T}g_n - Th_n)_{n\le N}\|_p + \|(Th_n)_{n\le N}\|_p \le \varepsilon + \pi^N_{p,\beta}(T)\beta.$$

Finally, we get $\pi^N_{p,\beta}(T)\beta \ge \|(\widehat{T}g_n)_{n\le N}\|_p - \varepsilon \ge \pi^N_{p,\beta}(\widehat{T})\beta - 2\varepsilon$. ■

Given any Hammerstein operator $T \in \mathcal{M}_{\mathrm{HP}}(C(S,X),\mathbb{C})$, according to [1], Theorem 3, there exists an integral representation

$$Tf = \int_S u(t, f(t))\, d\lambda(t), \qquad f \in C(S,X),$$

with a $\lambda \in \mathrm{rca}(\Sigma)_+$ and a kernel $u : S \to \mathcal{M}(X,\mathbb{C})$, λ-a.e., such that $u(\cdot,x) \in \mathcal{L}_\infty(\lambda)$, $x \in X$ and $M_\alpha(u(\cdot))$ is bounded for each $\alpha > 0$. We pointed out before that the extension of every p-summing Hammerstein operator is p-summing again. Hence, Theorem 2.3 applies and together with a weak* argument we get the following characterization of 1-summing Hammerstein operators on $C(S,X)$-spaces.

THEOREM 3.3. *For every $T \in \mathcal{P}_{1,\mathrm{HP}}(C(S,X),Y)$ there exists a probability measure $\mu \in C(S)^*$ such that*

$$\|(Tf_j)_j\|_1 \le 4\pi_{1,2\beta}(T) \int_S w_1((f_j(t))_j)\, d\mu(t), \qquad w_1((f_j)_j) \le \beta, \qquad \beta > 0.$$

The combination of 3.3, 1.1 and inequality (13) leads to an extension of the Grothendieck–Pietsch domination theorem for summing Hammerstein operators on $C(S)$-spaces.

COROLLARY 3.4. *$\mathcal{P}_{1,\mathrm{HP}}(C(S),Y) = \mathcal{AD}_{1,\mathrm{HP}}(C(S),Y)$, and*

$$\tfrac14 \mathrm{ad}_{1,\beta}(T) \le \pi_{1,2\beta}(T) \le s_{1,2\beta}(T) \le \mathrm{ad}_{1,2\beta}(T)$$

for each $T \in \mathcal{P}_{1,\mathrm{HP}}(C(S),Y)$ and $\beta > 0$.

Every 1-dominated operator is p-summing for $1 \le p < \infty$ and therefore

COROLLARY 3.5. *$\mathcal{P}_{1,\mathrm{HP}}(C(S),Y) \subset \mathcal{P}_{p,\mathrm{HP}}(C(S),Y)$, $1 \le p < \infty$.*

The corresponding result concerning 3.4 for linear operators on $C(S)$-spaces can be found in [11], 17.3.3. It is an open problem whether 2.3 and 3.4 can be extended to p-summing integral, respectively Hammerstein operators for $1 < p < \infty$. Note that the operator $T \in \mathcal{M}(l_2, l_1)$,

$$T(x) = \begin{cases} (1 - 4\|x - e_i\|)e_i & \text{if } \|x - e_i\| \le 1/4 \text{ for some } i, \\ 0 & \text{otherwise}, \end{cases}$$

is 1-summing but not 2-summing, where $(e_i)_i$ denotes the canonical basis in l_2 respectively l_1. This reveals that 3.4 and 3.5 do not hold for 1-summing Hammerstein operators on $C(S,X)$-spaces in general.

Using the natural inclusion between p-dominated and p-summing operators, we immediately see that every p-dominated Hammerstein operator has a control measure. Just as for p-summing operators in 3.2, it is also possible to prove that the corresponding extension of a p-dominated Hammerstein operator is p-dominated again.

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On Q-independence, limit theorems and q-Gaussian distribution

by

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Abstract. We formulate the notion of Q-independence which generalizes the classical independence of random variables and free independence introduced by Voiculescu. Here Q stands for a family of polynomials indexed by tiny partitions of finite sets. The analogs of the central limit theorem and Poisson limit theorem are proved. Moreover, it is shown that in some special cases this kind of independence leads to the q-probability theory of Bożejko and Speicher.

1. Introduction. In this paper we are concerned with a certain generalization of the classical notion of independence of random variables. The classical case describes properties of a commutative probability system, i.e. the set of complex measurable functions defined on a measurable space with a normalized positive measure. In [17] D. Voiculescu showed that in order to define a reasonable and essentially different independence one should consider more general concepts of random variables and probability systems.

DEFINITION 1.1. A *probability system* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital C^*-algebra and φ is a state on $\mathcal{A}$.

Here $\mathcal{A}$ plays the role of a noncommutative analog of a set of complex random variables and φ is a "noncommutative" probability measure. One can define the distribution of an element of $\mathcal{A}$.

DEFINITION 1.2. Let $(\mathcal{A}, \varphi)$ be a probability system and $a \in \mathcal{A}$. A functional $\widetilde{\mu}_a$ on the *-algebra $\mathbb{C}[X]$ of complex formal polynomials is called the *distribution* of a if

$$\widetilde{\mu}_a(P) = \varphi(P(a))$$

for every $P \in \mathbb{C}[X]$.

From the well-known Gelfand–Naimark theorem we easily get

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