STUDIA MATHEMATICA 129 (1) (1998)

# Stable elements of Banach and Fréchet algebras

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**Abstract.** We introduce an algebraic notion—*stability*—for an element of a commutative ring. It is shown that the stable elements of Banach algebras, and of Fréchet algebras, may be simply described. Part of the theory of power-series embeddings, given in [1] and [4], is seen to be of a purely algebraic nature. This approach leads to other natural questions.

**1. Introduction.** In [5], we introduced the notion of a *stable inverse-limit sequence* of groups, and gave various applications, mostly in the theory of Fréchet algebras. The main idea of the present paper is a development of the example of §3.4 of [5]. In fact, we may define a purely algebraic notion of a *stable element* of a commutative ring $R$ (see §2 below). It will turn out that, if $A$ is a commutative Banach algebra, then $x \in A$ is stable if and only if $x$ has finite closed descent (see [2], recalled in [4], page 271). If $A$ is a commutative Fréchet algebra, then the property of stability precisely characterizes the elements having *locally* finite closed descent ([4], page 276).

It turns out that a portion of the theory of embedding formal power series (given for Banach algebras in [1], and extended to Fréchet algebras in [4]) uses only the algebraic property of stability. This viewpoint leads often to more illuminating proofs. It also suggests other very natural questions, that may be of interest in the theory of automatic continuity.

We refer to [5], §1, for generalities on inverse-limit sequences (IL-sequences) (of sets, or of groups), the abstract Mittag-Leffler theorem and the basic properties of Fréchet algebras. As in that paper, we write **ILG** for the category of IL-sequences of groups and homomorphisms. One point of notation should be mentioned: we write $L$ for the inverse-limit functor on **ILG**. Thus, in particular, if say

$$\mathcal{G}: \quad G_1 \xleftarrow{g_1} G_2 \xleftarrow{g_2} \ldots \leftarrow G_n \xleftarrow{g_n} G_{n+1} \leftarrow \ldots$$

1991 *Mathematics Subject Classification*: Primary 46J05, 46M15, 46M40; Secondary 46M20.

is a sequence in **ILG**, then

$$L(\mathcal{G}) \equiv \varprojlim(G_n; g_n) \equiv \Big\{(x_n) \in \prod_{n\geq 1} G_n : x_n = g_n(x_{n+1})\ (n \geq 1)\Big\}.$$

Note that $L$ is a left-exact functor from **ILG** to the category of groups and homomorphisms. (See [5], §1.)

For convenience, we shall summarize the content of [5], §2, but omitting proofs and comments. Except in §6, all the groups considered in this paper will be abelian, being the additive groups of rings or algebras, so we shall write the group operation as addition.

Let $G$, $H$ be groups and $f : G \to H$ be a homomorphism. As a temporary notation, for any $\eta \in H$ let $[\eta] + f : G \to H$ be the mapping defined by $([\eta]+f)(\gamma) = \eta + f(\gamma)$ for all $\gamma \in G$. If $\mathcal{G} = (G_n; g_n)_{n\geq 1}$ is a sequence in **ILG** then, for any $\gamma = (\gamma_n) \in \prod_{n\geq 1} G_n$, we define $[\gamma] + \mathcal{G}$ to be the IL-sequence $(G_n; [\gamma_n] + g_n)$; we call $[\gamma] + \mathcal{G}$ a *perturbed sequence* of $\mathcal{G}$.

A sequence $\mathcal{G}$ in **ILG** is *stable* if and only if every perturbed sequence of $\mathcal{G}$ has a non-empty inverse limit. Thus, the sequence

$$G_1 \xleftarrow{g_1} G_2 \xleftarrow{g_2} \ldots \leftarrow G_n \xleftarrow{g_n} G_{n+1} \leftarrow \ldots,$$

in **ILG**, is stable if and only if, for every choice of $\gamma_n \in G_n$ $(n \geq 1)$, we may simultaneously solve the equations $x_n = \gamma_n + g_n(x_{n+1})$ for $x_n \in G_n$ $(n \geq 1)$.

There are two more-or-less trivial classes of examples of stable sequences: Let $\mathcal{G} = (G_n; g_n)$ be a sequence in **ILG**. Then:

(i) If $g_n(G_{n+1}) = G_n$ for each $n$, then also $([\gamma_n] + g_n)(G_{n+1}) = G_n$, for every choice of $\gamma_n \in G_n$. Hence $L([\gamma]+\mathcal{G}) \neq \emptyset$ for every perturbed sequence, i.e. $\mathcal{G}$ is stable.

(ii) If each $g_n$ is the trivial homomorphism $g_n(x) = 0$ $(x \in G_{n+1})$ then, for every choice of $\gamma_n \in G_n$, we solve $x_n = \gamma_n + g_n(x_{n+1})$ $(n \geq 1)$ by putting $x_n = \gamma_n$ for all $n$; so again $\mathcal{G}$ is stable.

These trivial examples are special cases of:

THEOREM 1.1. *Let $\mathcal{G} = (G_n; g_n)$ be a sequence in* **ILG**. *Then $\mathcal{G}$ is stable provided it satisfies either of the following conditions*:

(i) *each $G_n$ is a complete metrizable topological group and each homomorphism $g_n$ is continuous with $g_n(G_{n+1})$ dense in $G_n$*;

(ii) *each $G_n$ is a Hausdorff topological group and each homomorphism $g_n$ is continuous with $g_n(G_{n+1})$ compact.*

By a *Mittag-Leffler sequence*, we shall mean an IL-sequence $(G_n; g_n)$ where each $G_n$ is a complete metrizable topological group and, for each $n \geq 1$, $g_n$ is a continuous homomorphism with $\overline{g_n(G_{n+1})} = G_n$. Thus, part (i) of Theorem 1.1 states that every Mittag-Leffler sequence is stable.



LEMMA 1.2. *Let $\mathcal{G}$ be a sequence in* **ILG**. *Then the following are equivalent*:

(i) *$\mathcal{G}$ is stable*;
(ii) *every subsequence of $\mathcal{G}$ is stable*;
(iii) *some subsequence of $\mathcal{G}$ is stable.*

(The notion of a *subsequence* of an IL-sequence is given in [5], page 279.)

The point of the idea of stability lies in the following theorem.

THEOREM 1.3. *Let $0 \to \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{K} \to 0$ be a short exact sequence in* **ILG**. *If $\mathcal{G}$ is stable, then the sequence*

$$0 \to L(\mathcal{G}) \xrightarrow{L(\alpha)} L(\mathcal{H}) \xrightarrow{L(\beta)} L(\mathcal{K}) \to 0$$

*is also exact.*

REMARK. For sequences of *abelian* groups, there is a converse to Theorem 1.3 ([5], Theorem 16).

LEMMA 1.4. *Let $0 \to \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{K} \to 0$ be a short exact sequence in* **ILG**. *Then*:

(i) *if $\mathcal{H}$ is stable, then $\mathcal{K}$ is stable*;
(ii) *if $\mathcal{G}$ and $\mathcal{K}$ are stable, then $\mathcal{H}$ is stable.*

COROLLARY 1.5. *Let $\mathcal{G} = (G_n; g_n)$ be a sequence in* **ILG**. *Suppose that each $G_n$ has a normal subgroup $H_n$ such that*:

(i) *$g_n(G_{n+1}) \subseteq H_n$ $(n \geq 1)$*;
(ii) *the sequence $\mathcal{H} = (H_n; \overline{g}_n)$ is stable (where $\overline{g}_n = g_n|H_{n+1} : H_{n+1} \to H_n$).*

*Then $\mathcal{G}$ is stable.*

EXAMPLE. Various examples of stable and non-stable sequences are given in [5]. Here is another example of a stable sequence. Let $A$ be any Fréchet algebra, with its topology defined by an increasing sequence $(p_n)_{n\geq 1}$ of submultiplicative seminorms. Let $K_n = \ker p_n$ and consider the IL-sequence $\mathcal{K}$ defined by

$$\mathcal{K}: \quad K_1 \xleftarrow{j_1} K_2 \xleftarrow{j_2} K_3 \xleftarrow{j_3} \ldots,$$

where each $j_n$ is an inclusion mapping. Each ideal $K_n$ is closed, so that $\mathcal{K}$ is not a Mittag-Leffler sequence (except when $p_n$ is actually a *norm* for all $n \geq$ some $n_0$). We shall prove that the sequence $\mathcal{K}$ is stable.

Thus, let $a_n \in K_n$ $(n \geq 1)$. Then, for any given $m \geq 1$ and for all $n \geq m$, we have $p_m(a_n) \leq p_n(a_n) = 0$, i.e. $p_m(a_n) = 0$ for all $n \geq m$. The series $\sum_{n\geq 1} a_n$ is thus certainly convergent in the Fréchet topology of $A$. We define $x_n = \sum_{k \geq n} a_k$ for each $n$. Then $x_n \in K_n$ and, for every $n$, $x_n = j_n(x_{n+1}) + a_n$. The stability of $\mathcal{K}$ is thus proved.

**2. Stable elements of commutative rings.** If $G$ is an (abelian) group and if $T : G \to G$ is an endomorphism, then we say that $T$ *acts stably* on $G$ if and only if

$$G \xleftarrow{T} G \xleftarrow{T} G \xleftarrow{T} \ldots$$

is a stable sequence in **ILG**. We shall sometimes write $[G; T]$ for such a sequence.

Now let $R$ be a commutative ring (not necessarily with an identity); for $x \in R$, let $L_x : R \to R$ be the multiplication mapping, $L_x(y) = xy$ $(y \in R)$. We say that $x$ is a *stable element* of $R$ if and only if $L_x$ acts stably on $R$. In this definition, the group $G$ is the additive group of $R$; we are regarding $L_x$ as being, in particular, an endomorphism of that group.

More explicitly, the element $x$ of $R$ is stable if and only if, for every sequence $(b_n)$ in $R$, there is a sequence $(a_n)$ in $R$ such that

$$a_n = a_{n+1}x + b_n \quad (n = 1, 2, \ldots).$$

LEMMA 2.1. *Let $R$ be a commutative ring and let $x \in R$. Then the following are equivalent*:

(i) *$x$ is stable*;
(ii) *$x^n$ is stable for all $n \geq 1$*;
(iii) *$x^n$ is stable for some $n \geq 1$.*

Proof. This follows immediately from Lemma 1.2, on observing that, for each $n \geq 1$, $[R; L_{x^n}] = [R; L_x^n]$ is a subsequence of $[R; L_x]$.

Lemma 1.4 and Corollary 1.5 have immediate implications for stable elements. Let $R$, $S$ be commutative rings and let $x \in R$. Then any ring homomorphism $\varphi : R \to S$ gives rise, in a natural way, to a morphism $\widetilde{\varphi} : [R; L_x] \to [S; L_{\varphi(x)}]$ in the category **ILG**. Explicitly,

$$\begin{array}{ccccccccc}
[R; L_x]: & & R & \xleftarrow{L_x} & R & \xleftarrow{L_x} & R & \xleftarrow{L_x} & \ldots \\
\downarrow{\scriptstyle\widetilde{\varphi}} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \\
[S; L_{\varphi(x)}]: & & S & \xleftarrow{L_{\varphi(x)}} & S & \xleftarrow{L_{\varphi(x)}} & S & \xleftarrow{L_{\varphi(x)}} & \ldots
\end{array}$$

This correspondence is, in an obvious sense, functorial; it is also *exact*, in the sense that, for any $x \in R$, a sequence of ring homomorphisms $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ is exact if and only if the sequence

$$[R; L_x] \xrightarrow{\widetilde{\varphi}} [S; L_{\varphi(x)}] \xrightarrow{\widetilde{\psi}} [T; L_{\psi\circ\varphi(x)}],$$



in **ILG**, is exact (which is immediate from the definition of exactness in **ILG**).

If $x \in R$ and if $I$ is an ideal of $R$, then $L_x(I) \subseteq I$, so that, in an obvious sense, $L_x$ acts on both $I$ and on $R/I$. If $q : R \to R/I$ is the quotient mapping, then $L_x$ acts stably on $R/I$ if and only if $q(x)$ is a stable element of $R/I$.

LEMMA 2.2. (i) *If $\varphi : R \to S$ is a surjective homomorphism of commutative rings, and if $x$ is stable in $R$, then $\varphi(x)$ is stable in $S$.*

(ii) *Let $I$ be an ideal of the commutative ring $R$ and let $x \in R$. Suppose that $L_x$ acts stably on both $I$ and on $R/I$. Then $x$ is a stable element of $R$.*

Proof. This follows immediately from Lemma 1.4 and the discussion just given.

LEMMA 2.3. *Let $I$ be an ideal of the commutative ring $R$ and let $x \in I$. Then $x$ is stable as an element of $I$ if and only if it is stable as an element of $R$.*

Proof. (i) Let $x \in I$ be stable as an element of $I$. Since $L_x(R) \subseteq I$, the stability of $x$ (as an element of $R$) follows from Corollary 1.5.

(ii) Conversely, let $x \in I$ be stable as an element of $R$ and let $(b_n)$ be a sequence in $I$. Then there is some sequence $(a_n)$ in $R$ such that $a_n = a_{n+1}x + b_n$ for all $n$. But $x \in I$ and each $b_n \in I$, so that $a_n \in I$ for every $n$. Thus $x$ is stable in $I$.

If $R$ is a commutative ring with identity, we can form its *unitization* $R_+$, by putting the obvious multiplication on $R \oplus \mathbb{Z}1$ (where "1" is the adjoined identity); if $R$ is an algebra over the field $k$ then $R_+$ will, instead, be the unital *algebra* $R \oplus k1$. We shall not make any notational distinction between these cases. In either case, $R$ is naturally embedded as an ideal of $R_+$. The corollary below therefore follows at once from Lemma 2.3.

COROLLARY 2.4. *If $R$ is a commutative ring without identity and if $x \in R$, then $x$ is stable in $R$ if and only if $x$ is stable as an element of $R_+$.*

In view of this corollary, we shall usually take rings to be unital.

EXAMPLES. (i) Firstly, remark that, if $x$ is an element of a commutative ring $R$ having the property that $Rx^m = Rx^{m+1}$ for some integer $m$, then $x$ is a stable element of $R$. (We say that such an element has *finite descent*.)

To see this, let $I = Rx^m$; then $L_x^m(I) = I$, so that $L_x^m$ acts stably on $I$ (by the trivial case (i) just before Theorem 1.1). But also, $L_x^m$ acts trivially (and so stably) on $R/I$; so, by Lemma 2.2(ii), $x^m$ is stable. Hence, by Lemma 2.1, $x$ is stable.

In particular, nilpotent elements and units (i.e. invertible elements) in any commutative ring are examples of stable elements. Every element of a field is stable.

(ii) There are non-trivial examples in which the nilpotents and the units are the *only* stable elements of a ring. Let $D$ be an integral domain (with identity) and let $R = D[X]$, the polynomial ring with coefficients in $D$. Recall that $R$ is also an integral domain and that the units of $R$ are just the units of $D$ (embedded in $R$ as the constant polynomials).

PROPOSITION 2.5. *The set of stable elements of $R = D[X]$ consists just of 0 and the set of units (of $D$).*

Proof. Let $f(X) \in R$ be non-zero and not a unit.

CASE (a): $\deg f = 0$, i.e. $f(X) = d$, say, where $d$ is a non-zero, non-unit element of $D$. Then, if $d$ were stable in $R$, we would, in particular, have some sequence $(g_n(X))$ such that, for all $n \geq 0$,

$$g_n(X) = dg_{n+1}(X) - X^n.$$

Let the coefficient of $X^n$ in $g_{n+1}(X)$ be $a_n$. Then a simple calculation shows that the coefficient of $X^n$ in $g_0(X)$ is

$$d^{n+1}a_n - d^n = d^n(da_n - 1) \neq 0,$$

since $D$ is an integral domain in which $d$ is non-zero and not a unit.

So $\deg g_0 \geq n$ for all $n$, a contradiction.

CASE (b): $\deg f \geq 1$. Then, if $f$ were stable, there would be a sequence $(h_n(X))$ in $R$ such that, for all $n \geq 0$,

$$(*) \qquad h_n = fh_{n+1} + 1.$$

Let $N \geq 1$; then there is some $m \geq N$ such that $h_m \neq 0$, since, for example, from $(*)$ we see that $h_N$, $h_{N+1}$ cannot both be zero. But then, by repeated application of $(*)$, it follows that $\deg h_n \geq 1 + \deg h_{n+1}$ for all $n \leq m-1$. In particular, $\deg h_0 \geq N$, for every $N$, and again there is a contradiction.

(iii) Rings of formal power series will be important later in this paper. There is a simple description of the stable elements of such rings.

THEOREM 2.6. *Let $R$ be any commutative ring, and let $F = R[[X]]$. Then an element $f(X)$ of $F$ is stable in $F$ if and only if $f(0)$ is stable in $R$.*

Proof. Firstly, since the mapping $f \mapsto f(0)$ is a surjective homomorphism from $F$ onto $R$, it is immediate from Lemma 2.2 that, if $f(X)$ is stable in $F$, then $f(0)$ is stable in $R$.

Conversely, let $f(X) \in F$ be such that $f(0)$ is stable in $R$. So, writing $f(X) = r + \sum_{k\geq 1} r_k X^k$, we see that $r$ is a stable element of $R$ and $(r_k)_{k\geq 1}$ is some sequence in $R$. Let $(g_n)_{n\geq 1}$ be any given sequence in $F$; we have to find a sequence $(h_n)$ in $F$ such that

$$h_n(X) = h_{n+1}(X)f(X) + g_n(X) \quad (n \geq 1).$$



Write

$$g_n(X) = \sum_{k=0}^{\infty} b_n^{(k)} X^k, \qquad h_n(X) = \sum_{k=0}^{\infty} a_n^{(k)} X^k,$$

where the $b_n^{(k)}$ are given, but the $a_n^{(k)}$ are to be found. Then, in terms of these coefficients, we wish to satisfy the equations

$$\sum_{k=0}^{\infty} a_n^{(k)} X^k = \Big(\sum_{k=0}^{\infty} a_{n+1}^{(k)} X^k\Big)\Big(r + \sum_{k=1}^{\infty} r_k X^k\Big) + \sum_{k=0}^{\infty} b_n^{(k)} X^k,$$

for all $n \geq 1$.

Equating constant terms, a first necessary condition is:

$$a_n^{(0)} = a_{n+1}^{(0)} r + b_n^{(0)}.$$

But the stability of $r$ in $R$ means precisely that we can find a sequence $(a_n^{(0)})_{n\geq 1}$ in $R$ that satisfies these equations.

The induction step is very similar: we suppose that, for some $k \geq 1$, sequences $(a_n^{(i)})_{n\geq 1}$ have been found for $i = 0, \ldots, k-1$. Then, again using the stability of $r$ in $R$, we may find a sequence $(a_n^{(k)})_{n\geq 1}$ such that, for all $n$,

$$a_n^{(k)} = a_{n+1}^{(k)} r + \Big(\sum_{i=0}^{k-1} a_{n+1}^{(i)} r_{k-i} + b_n^{(k)}\Big).$$

Thus, suitable sequences $\{(a_n^{(k)}) : n \geq 1,\ k \geq 0\}$ are definable by induction on $k$, and the proof is complete.

COROLLARY 2.7. *For every commutative ring $R$, $X$ is a stable element of $R[[X]]$. If $k$ is a field, then every element of $k[[X]]$ is stable.*

(iv) Combining examples (ii) and (iii), it follows that it is possible to have a commutative ring $A$ (with identity), a unital subring $B$ and an element $a \in B$ such that $a$ is stable as an element of $A$, but not stable as an element of $B$. In fact, let $A = \mathbb{C}[[X]]$ with $B = \mathbb{C}[X]$ embedded as a subalgebra of $A$ in the usual way. Then $X$ is not stable in $B$ (by Proposition 2.5), but is stable in $A$, by the last corollary.

To see that the opposite is also possible, let $B = \mathbb{C}[[X]]$ and then let $A = B[Y] = \mathbb{C}[[X]][Y]$, where $X$, $Y$ are independent indeterminates. Then $X$ is stable as an element of $B$, but it is not stable as an element of $A$, since $X \neq 0$ but $X$ is not a unit of $B$. (Recall the well-known fact that $B$ is an integral domain.)

Note further that, by the main result of [1], it is possible to define an algebra-norm on $\mathbb{C}[[X]]$, and so also on $\mathbb{C}[[X]][Y]$. Thus, in (iv), all the rings considered may be taken to be complex normed algebras. Later (Corollary 4.8 and the remarks following) we shall see the effect of completeness assumptions.

Let $R$ be a commutative ring and let $x \in R$. There are two ideals, associated with $x$, for which we use a special notation. The first is $I(x) = \bigcap_{n\geq 1} Rx^n$; the second ideal, $I_0(x)$, is the set of all $a \in R$ such that there is a sequence $(a_n)_{n\geq 0}$ in $R$ with $a = a_0$ and $a_n = a_{n+1}x$ for $n = 0, 1, \ldots$ Otherwise expressed, if we let $L_0(x) = \varprojlim[R; L_x]$, then $I_0(x)$ is the projection of $L_0(x)$ onto its first coordinate; i.e. $I_0(x) = \pi_0(L_0(x))$, where $\pi_0((x_0, x_1, \ldots, x_n)) = x_0$, for all $(x_0, x_1, \ldots, x_n) \in L_0(x)$. It is evident that $I(x)$ and $I_0(x)$ are ideals of $R$, and that $I_0(x) \subseteq I(x)$. Also, notice that the projection of $L_0(x)$ onto *any* of its coordinates is, again, the ideal $I_0(x)$; thus, $I_0(x) = 0$ if and only if $L_0(x) = 0$. In cases where there might be doubt about the intended ring, we may write $I^R(x)$, $L_0^R(x)$, $I_0^R(x)$ in place of $I(x)$, $L_0(x)$, $I_0(x)$ respectively.

LEMMA 2.8. *Let $R$ be a commutative ring and let $x \in R$. Then*:

(i) $L_x(I_0(x)) = I_0(x)$; *in particular, $L_x$ acts stably on $I_0(x)$*;

(ii) *$x$ is stable if and only if $L_x$ acts stably on $R/I_0(x)$*;

(iii) *$I_0(x) = I(x)$ if and only if $L_x(I(x)) = I(x)$*;

(iv) *if $x$ is stable, then $L_x$ acts stably on $I(x)$*;

(v) *let $\pi : R \to S \equiv R/I_0(x)$ be the quotient map; then $I_0^S(\pi(x)) = 0$.*

Proof. (i) This is immediate from the definition of $I_0(x)$.

(ii) This follows from (i) and Lemma 2.2.

(iii) One implication is obvious from (i). For the converse, let $L_x(I(x)) = I(x)$ and let $y \in I(x)$. Then $y = y_1 x$, say, for some $y_1 \in I(x)$. A simple induction now shows that $y \in I_0(x)$, so that (iii) is proved.

(iv) Let $x$ be stable and let $(a_n)$ be any sequence in $I(x)$. Then there is a sequence $(b_n)$ in $R$ such that

$$b_n = b_{n+1}x + a_n, \tag{$*$}$$

for all $n \geq 1$. Suppose that, for some $k \geq 0$, we know that $b_n \in Rx^k$ for all $n \geq 1$. Then, from $(*)$, we immediately deduce that, for every $n$, $b_n \in Rx^{k+1} + I(x) \subseteq Rx^{k+1}$. It therefore follows, by induction on $k$, that $b_n \in Rx^k$ for all $n$ and $k$, i.e. $b_n \in I(x)$ for all $n$. Thus $L_x$ acts stably on $I(x)$.

(v) Write $\xi = \pi(x)$ and consider the short exact sequence, in **ILG**,

$$0 \to \mathcal{I}_0 \xrightarrow{J} \mathcal{R} \xrightarrow{Q} \mathcal{R}/\mathcal{I}_0 \to 0,$$

where $\mathcal{R} = [R; L_x]$, $\mathcal{I}_0 = [I_0(x); \overline{L}_x]$ (with $\overline{L}_x = L_x|I_0(x)$) and $\mathcal{R}/\mathcal{I}_0 = [S; L_\xi]$; $J$ is a sequence of inclusions, and $Q$ a sequence of quotient mappings.

By (i), each mapping $\overline{L}_x$ is surjective, so that the sequence $\mathcal{I}_0$ is stable; the inverse limits of the three sequences are, respectively, $L_0(x)$, $L_0(x)$ and $L_0(\xi)$. From Theorem 1.3 we therefore have a short exact sequence

$$0 \to L_0(x) \to L_0(x) \to L_0(\xi) \to 0,$$

from which we deduce that $L_0(\xi) = 0$, and hence also $I_0(\xi) = 0$.



QUESTION. If $R$ is any commutative ring and if $x$ is a stable element of $R$, can it happen that $I_0(x) \neq I(x)$? In view of Lemma 2.8(v), this is equivalent to asking whether there is an example for which $I_0(x) = (0)$ but $I(x) \neq (0)$. We suppose that such an example probably exists, but have not been able to find one.

**3. Stable elements and rings of power series.** Let $R$ be a commutative ring and let $\mathcal{F} \equiv \mathcal{F}_R = R[[X]]$, the ring of all formal power series in the indeterminate $X$, with coefficients from $R$. The algebraic background needed is a very small subset of [11], Chapter 7. In particular, recall that $f(X)$ is a unit of $\mathcal{F}$ if and only if $f(0)$ is a unit of $R$. This is analogous to the result in Theorem 2.6, that $f(X)$ is stable in $\mathcal{F}$ if and only if $f(0)$ is stable in $R$.

LEMMA 3.1. *Let $R$ be a commutative ring and let $x, y \in R$. Then $y \in I_0(x)$ if and only if $y = (X - x)g(X)$, for some $g(X) \in \mathcal{F}$.*

Proof. (i) Let $y \in I_0(x)$; then there is a sequence $(y_n)$ in $R$ such that $y = y_1 x$ and $y_n = y_{n+1}x$ for all $n \geq 1$. Then, setting $g(X) = -\sum_{n\geq 1} y_n X^{n-1}$, a simple calculation shows that $(X - x)g(X) = y_1 x = y$.

(ii) Conversely, suppose that $y = (X - x)(a_0 + a_1 X + a_2 X^2 + \ldots)$, for some sequence $(a_n)$ in $R$. Then comparing coefficients shows that $y = -a_0 x$ and $a_n = a_{n+1}x$ $(n \geq 0)$. Thus $y \in I_0(x)$.

LEMMA 3.2. *Let $R$ be a commutative ring and let $x \in R$. Then $x$ is stable if and only if, for every $f(X) \in \mathcal{F}_R$, there exist $g(X) \in \mathcal{F}_R$ and $r \in R$ such that*

$$f(X) = (X - x)g(X) + r.$$

*Moreover, in that case, the element $r$ is uniquely determined mod $I_0(x)$.*

Proof. Suppose firstly that $x$ is a stable element of $R$. Let $f(X) = \sum_{n\geq 0} a_n X^n$ be an element of $\mathcal{F}$.

By the stability of $x$, there is a sequence $(b_n)$ in $R$ such that $b_n = b_{n+1}x + a_n$ for all $n \geq 1$. Define $g(X) = \sum_{n\geq 1} b_n X^{n-1}$. Then

$$(X - x)g(X) = -b_1 x + \sum_{n\geq 1} a_n X^n = f(X) - r,$$

say, where $r = a_0 + b_1 x \in R$.

Conversely, if the condition on $\mathcal{F}_R$ holds, let $(a_n)$ be any given sequence in $R$ and define $f(X) = \sum_{n\geq 0} a_n X^n$. By the hypothesis, there is a series, say

$g(X) = \sum_{n\geq 0} b_n X^n$, in $\mathcal{F}_R$ and $r \in R$ such that $f(X) = (X - x)g(X) + r$. Then equating coefficients gives

$$a_0 = -b_0 x + r, \quad a_n = b_{n-1} - b_n x \quad (n \geq 1),$$

which proves the stability of $x$.

Moreover, in the case where $x$ is stable, if also $f(X) = (X-x)g_1(X)+r_1$, then $r - r_1 \in (X - x)\mathcal{F}_R$, i.e. $r - r_1 \in I_0(x)$, by Lemma 3.1.

REMARK. Before the next result, we note that both $\mathcal{F}_R$ and $R$ itself have a natural structure as $R$-modules. An *$R$-homomorphism* $T : \mathcal{F}_R \to R$ is a ring homomorphism that is also an $R$-module homomorphism. Otherwise expressed, it is a ring homomorphism such that $T(r) = r$ for every $r \in R$ (regarded as the subring of constant series in $\mathcal{F}_R$).

THEOREM 3.3. *Let $R$ be a commutative ring, let $x \in R$ and suppose that $I_0(x) = 0$. Then $x$ is stable if and only if there is a unital $R$-homomorphism $\theta_x : \mathcal{F}_R \to R$ such that $\theta_x(X) = x$.*

*Moreover, in the case where $x$ is stable, the homomorphism $\theta_x$ is uniquely determined,* $\operatorname{im}\theta_x = R$ *and* $\ker\theta_x = (X-x)\mathcal{F}_R$, *so that* $R \cong R[[X]]/(X-x)$.

Proof. Let $x$ be stable. By Lemma 3.2, for each $f(X) \in \mathcal{F}_R$, we have $f(X) = (X - x)g(X) + r$, for some $g(X) \in \mathcal{F}_R$ and a *uniquely* determined element $r \in R$. We then define $\theta_x(f(X)) = r$. Because of the uniqueness of $r$, it is very simple to see that $\theta_x$ is an $R$-homomorphism, with $\theta_x(X) = x$.

For each $r \in R$ (regarded as a "constant series"), we have $\theta_x(r) = r$, so that $\operatorname{im}\theta_x = R$. If $f(X) \in \ker\theta_x$ then $f(X) = (X - x)g(X)$, i.e. $f(X) \in (X - x)\mathcal{F}_R$, and the converse assertion is clear. Moreover, if $\theta$ is *any* $R$-homomorphism from $\mathcal{F}_R$ to $R$ such that $\theta(X) = x$, then the representation $f(X) = (X - x)g(X) + r$ shows that $\theta(f) = r = \theta_x(f)$; so $\theta_x$ is uniquely determined.

Conversely, suppose that there is an $R$-homomorphism $\theta : \mathcal{F}_R \to R$ such that $\theta(X) = x$. As an $R$-homomorphism, $\theta$ is surjective; but, by Corollary 2.7, $X$ is stable in $\mathcal{F}_R$ and so, by Lemma 2.2, $x = \theta(X)$ is stable in $R$. This completes the proof.

REMARK. If $T : R \to S$ is a homomorphism of commutative rings, there is a corresponding homomorphism, say $f(X) \mapsto (Tf)(X)$, $\mathcal{F}_R \to \mathcal{F}_S$, obtained by applying $T$ to the coefficients of a series. Thus, if $f(X) = \sum a_n X^n$, then $(Tf)(X) = \sum (Ta_n)X^n$. It is clear that this correspondence is an exact functor.

THEOREM 3.4. *Let $R$ be a commutative ring and let $x \in R$. Write $S_0 = R/I_0(x)$, let $\pi : R \to S_0$ be the quotient mapping, and let $\xi = \pi(x)$. Then the following are equivalent*:

(i) *$x$ is stable in $R$;*



(ii) *there is an $R$-homomorphism $\Psi_x : \mathcal{F}_R \to S_0$ such that $\Psi_x(X) = \xi$.*

*Moreover, in the case where $x$ is stable, the homomorphism $\Psi_x$ is uniquely defined,* $\operatorname{im}\Psi_x = S_0$ *and* $\ker\Psi_x = (X - x)\mathcal{F}_R + I_0(x)$.

Proof. (i)⇒(ii). Let $x$ be stable in $R$. Then $\xi$ is stable in $S_0$ (by Lemma 2.8(ii)). By Lemma 2.8(v), $I_0(\xi) = 0$ and so, by Theorem 3.3, there is an $S_0$-homomorphism $\theta : \mathcal{F}_{S_0} \to S_0$ such that $\theta(X) = \xi$.

Then, using the notation explained just before this theorem, we define $\Psi_x : \mathcal{F}_R \to S_0$ by setting $\Psi_x(f(X)) = \theta((\pi f)(X))$, for all $f(X) \in \mathcal{F}_R$. It is clear that then $\Psi_x$ is an $R$-homomorphism with $\Psi_x(X) = \xi$. (We are regarding $S_0$ as an $R$-module, as a quotient of $R$. That $\Psi_x$ is an $R$-homomorphism now means that $\Psi_x$ is a ring homomorphism with $\Psi_x(r) = \pi(r)$ for all $r \in R$.)

The uniqueness of $\Psi_x$ (subject to $\Psi_x(X) = \xi$) follows from Lemmas 3.2 and 3.1, in a very similar way to the uniqueness part of Theorem 3.3, as does the description of $\ker\Psi_x$. The fact that $\Psi_x$ is surjective is clear.

(ii)⇒(i). If an $R$-homomorphism $\Psi$ exists, with $\Psi(X) = \xi$, then, since $\Psi$ is necessarily surjective, it follows as in the proof of Theorem 3.3 that $\xi$ is stable in $S_0$. By Lemma 2.8(ii), this implies that $x$ is stable in $R$. The proof is complete.

**4. Stable elements of Banach and Fréchet algebras.** In view of the ring-theoretic nature of the last two sections, we shall begin with a few simple results in the context of topological rings. By a *topological ring* we shall mean a non-zero ring $A$ equipped with a topology $\tau$, such that (i) $A$ is a topological group under addition, and (ii) the ring multiplication is separately continuous. A *topological algebra* is a non-zero complex algebra which is a topological ring and also a topological vector space. An *$F$-ring* is a complete metrizable topological ring; an *$F$-algebra* is a complete metrizable topological algebra (in which case, the multiplication is necessarily jointly continuous [6]). Normally our interest is only in Hausdorff topological rings and algebras (and, indeed, chiefly in Banach and Fréchet algebras). However, in certain results (especially Theorems 4.6 and 4.7) it is convenient to be able to make use of the non-Hausdorff case. We remark that the few simple results that apply to $F$-rings contain (very simple) algebraic results as special cases: this is because an arbitrary ring may be made into an $F$-ring by giving it the discrete topology.

Let $A$ be a commutative topological ring and let $x \in A$. Then $x$ is said to have *finite closed descent* (FCD) if and only if, for some integer $m \geq 0$, $Ax^{m+1}$ is dense in $Ax^m$. (Conventionally, when $m = 0$ then $Ax^m$ means $A$, even when $A$ has no identity element.) We write $\delta_A(x) \equiv \delta(x)$ for the least integer $m$ with this property, and call $\delta(x)$ the *closed descent* of $x$; we may also write "$\delta(x) < \infty$" to mean that $x$ has FCD, and "$\delta(x) = \infty$" to indicate that $x$ does *not* have FCD. The notion of "finite closed descent" was introduced for topological algebras in [1], but not given a name until [2]; elementary properties are summarized in [4], §1. For the most elementary properties, it makes little difference that we are now dealing with rings rather than algebras.

It is trivial that, in any topological ring $A$, any element $x$ having finite descent—i.e. such that $Ax^{m+1} = Ax^m$ for some $m$ (see Example (i) after Corollary 2.4)—has FCD. In particular, if $x$ is either invertible or nilpotent, then $x$ has FCD.

LEMMA 4.1. *Let $A$ be a commutative topological ring.*

(i) *If $B$ is a subring of $A$ then, for all $x \in B$, $\delta_B(x) < \infty \Rightarrow \delta_A(x) < \infty$. If $\overline{B} = A$, then, for every $x \in B$, $\delta_B(x) < \infty$ if and only if $\delta_A(x) < \infty$.*

(ii) *If $I$ is an ideal of $A$, then, for every $x \in I$, $\delta_I(x) < \infty$ if and only if $\delta_A(x) < \infty$.*

(iii) *If $A$ does not have a 1, and if $A_+$ is the unitization of $A$, then, for every $x \in A$, $\delta_A(x) < \infty$ if and only if $\delta_{A_+}(x) < \infty$.*

Proof. (i) Let $x \in B \subseteq A$ and let $\delta_B(x) = k < \infty$. Then $Bx^{k+2}$ is dense in $Bx^k$ so, in particular, $x^{k+1} \in \overline{Bx^{k+2}} \subseteq \overline{Ax^{k+2}}$. Thus $Ax^{k+1} \subseteq \overline{Ax^{k+2}}$, so that $\delta_A(x) \le k+1 = \delta_B(x) + 1$.

Now suppose that $\overline{B} = A$ and let $x \in B$ with $\delta_A(x) = m < \infty$. Then $Ax^{m+1}$ is dense in $Ax^m$; but $B$ is dense in $A$, so $Bx^{m+1}$ is dense in $Ax^m$. Since $Bx^{m+1} \subseteq Bx^m \subseteq Ax^m$, it follows that $\delta_B(x) \le m$.

(ii) If $I$ is an ideal of $A$ and if $x \in I$ then we show that $\delta_A(x) < \infty \Rightarrow \delta_I(x) < \infty$. So, suppose that $\delta_A(x) = r < \infty$. Then

$$Ax^{r+2} \subseteq Ix^{r+1} \subseteq Ix^r \subseteq Ax^r.$$

Since $Ax^{r+2}$ is dense in $Ax^r$, it follows that $Ix^{r+1}$ is dense in $Ix^r$, so that $\delta_I(x) \le \delta_A(x)$.

(iii) This follows immediately from (ii), since $A$ is embedded as an ideal in $A_+$.

An important property that holds for an element $x$ of FCD is that (with the notation explained just before Lemma 2.8) $I_0(x) = I(x)$. This follows because $L_x$ maps $I(x)$ bijectively onto itself (see [1], Lemma 1, also Lemma 4.2 below). The crucial point here is that, if $x$ has FCD, then the sequence of annhilators, $(\operatorname{Ann} x^n)$, is constant for sufficiently large $n$ (where $\operatorname{Ann} y = \{a \in A : ay = 0\}$). Given the algebraic nature of much of this paper, it seems worth giving the slightly more general result.

LEMMA 4.2. (i) *Let $A$ be a commutative topological ring and let $x \in A$ have $\delta(x) = m < \infty$. Then $\operatorname{Ann} x^n = \operatorname{Ann} x^{m+1}$ for all $n \ge m+1$.*



(ii) *Let $A$ be a commutative ring and let $x \in A$ be such that, for some $k \ge 1$, $\operatorname{Ann} x^{k+1} = \operatorname{Ann} x^k$. Then $L_x$ maps $I(x)$ bijectively onto itself.*

(iii) *Let $A$ be a commutative topological ring and let $x \in A$ have FCD. Then $L_x$ maps $I(x)$ bijectively onto itself.*

Proof. (i) We have $\delta(x) = m$, so that $Ax^n$ is dense in $Ax^m$ for all $n \ge m$. If $a \in \operatorname{Ann} x^n$ for some $n \ge m+1$, then $aAx^n = 0$; but $Ax^n$ is dense in $Ax^m$, so that $aAx^m = 0$. In particular, $ax^{m+1} = 0$ and thus $a \in \operatorname{Ann} x^{m+1}$. Since $(\operatorname{Ann} x^n)$ is certainly an increasing sequence, it follows that $\operatorname{Ann} x^n = \operatorname{Ann} x^{m+1}$ for all $n \ge m+1$ (and that $\operatorname{Ann} x^n \subseteq \operatorname{Ann} x^{m+1}$ for *all* $n$).

(ii) Suppose, firstly, that $y \in Ax^k$ and that $L_x(y) = 0$. Then, say, $y = ax^k$ for some $a \in A$, $ax^{k+1} = 0$. So $a \in \operatorname{Ann} x^{k+1} = \operatorname{Ann} x^k$, whence $y = ax^k = 0$. Thus $L_x$ is injective on $Ax^k$, so certainly also on $I(x) \subseteq Ax^k$.

We must now show that $L_x$ maps $I(x)$ *onto* itself. Thus, let $y \in I(x)$ and, for each $n \ge 1$, let $D_n = \{a \in A : ax^n = y\}$. Then $D_n \ne \emptyset$ and clearly $L_x(D_{n+1}) \subseteq D_n$ for each $n$. So, writing just $L_x$ for each $L_x|D_n$, there is an IL-sequence (of sets and mappings), say $\mathcal{D}$,

$$\mathcal{D}: \quad D_1 \xleftarrow{L_x} D_2 \xleftarrow{L_x} D_3 \xleftarrow{L_x} \cdots$$

Suppose that $a, b \in D_n$ for some $n$; then $a - b \in \operatorname{Ann} x^n \subseteq \operatorname{Ann} x^k$. Thus, for every $n \ge 1$, $L_x^k(D_n)$ is a singleton; i.e. the subsequence $\mathcal{D}'$ of $\mathcal{D}$, where

$$\mathcal{D}': \quad D_1 \xleftarrow{L_x^k} D_{k+1} \xleftarrow{L_x^k} D_{2k+1} \xleftarrow{L_x^k} \cdots,$$

has each mapping with singleton range. Thus $L(\mathcal{D}) \cong L(\mathcal{D}')$ is a singleton—and, in particular, $L(\mathcal{D}) \ne \emptyset$. But if $(a_n)_{n\ge1} \in L(\mathcal{D})$, then $a_1 \in D_1 \cap I(x)$, i.e. there exists $a_1 \in I(x)$ with $L_x(a_1) = y$.

(iii) This follows at once from (i) and (ii).

COROLLARY 4.3. *Let $A$ be a commutative topological ring and let $x \in A$ have FCD. Then $I(x) = I_0(x)$. For $a \in I(x)$ let $(a_n)_{n\ge0}$ be a sequence in $A$ such that*

$$a = a_0, \quad a_0 = a_1 x, \quad a_1 = a_2 x, \quad \ldots, \quad a_n = a_{n+1} x, \quad \ldots$$

*Then the sequence $(a_n)$ lies in $I(x)$ and is uniquely determined by $a$. The mapping $a \mapsto (a_n)$ is an isomorphism between $I(x)$ and $L_0(x)$, the inverse limit of the sequence*

$$A \xleftarrow{L_x} A \xleftarrow{L_x} A \xleftarrow{L_x} \cdots$$

Proof. This is immediate from Lemma 4.2(iii).

We now begin to establish the connections between stability and the property of having FCD.

PROPOSITION 4.4. *Let $A$ be a commutative F-ring and let $x \in A$ have FCD. Then:*

(i) *$x$ is stable in $A$;*
(ii) *if $\delta(x) = m$, then $I(x)$ is dense in $Ax^m$;*
(iii) *$I(x) = 0$ if and only if $x$ is nilpotent.*

Proof. (i) Let $\delta(x) = m$ and set $I = \overline{Ax^m}$. Then, writing $\overline{L}_x = L_x|I$, we see that $\operatorname{im} \overline{L}_x$ is dense in $I$. Thus the IL-sequence, say $\mathcal{I}$,

$$\mathcal{I}: \quad I \xleftarrow{\overline{L}_x} I \xleftarrow{\overline{L}_x} \ldots$$

is a Mittag-Leffler sequence, so is stable by Theorem 1.1(i).

The image of $x$ in the quotient ring $A/I$ is nilpotent—so stable (see Example (i) following Lemma 1.4). The stability of $x$ in $A$ now follows from Lemma 2.2(ii).

(ii) This follows from the Mittag-Leffler theorem (e.g. [4], Theorem 1) applied to the IL-sequence $\mathcal{I}$.

(iii) If $\delta(x) = m$ then, by (ii), $I(x)$ is dense in $Ax^m$. Hence, if $I(x) = 0$ then also $Ax^m = 0$; in particular, $x^{m+1} = 0$, so that $x$ is nilpotent. The converse is clear.

REMARKS. For a commutative topological ring $R$, let $\mathcal{F}(R)$ be the set of elements in $R$ that have FCD. We record a few simple comments about the set $\mathcal{F}(R)$.

(i) If $x, y \in \mathcal{F}(R)$ then $xy \in \mathcal{F}(R)$ (see the argument in Proposition 6.5). It is also clear that, if $x \in \mathcal{F}(R)$, then $\lambda x \in \mathcal{F}(R)$ for every $\lambda \in \mathbb{C}$.

(ii) Even when $R$ is a commutative Banach algebra, $\mathcal{F}(R)$ is not always closed under addition. For example, let $R = L^1(\mathbb{R}^+; e^{-t^2})$. (A convenient reference for a discussion of the basic properties of such algebras is [8], §7.) Let $u$, $v$ be the elements of $R$ defined by

$$u(t) = 1 \quad (t \in \mathbb{R}^+), \qquad v(t) = \begin{cases} -1 & \text{for } 0 \le t \le 1; \\ 0 & \text{for } t > 1. \end{cases}$$

Then $\delta(u) = \delta(v) = 0$, but $u + v = 1_{(1,\infty)}$, which certainly does not have FCD.

(iii) Given a commutative $F$-ring $R$, let $E_n = \{x \in R : x^n \in \overline{x^{n+1}R}\}$; evidently $\mathcal{F}(R) = \bigcup_{n\ge1} E_n$. But clearly

$$E_n = \bigcap_{m\ge1} \{x \in R : \operatorname{dist}(x^n, x^{n+1}R) < 1/m\},$$

so that each $E_n$ is a $G_\delta$-set and then $\mathcal{F}(R)$ is a $G_{\delta\sigma}$-set.

(iv) Generally, $\mathcal{F}(R)$ need not be closed, even when $R$ is a commutative Banach algebra. For example, let $R$ again be the radical Banach algebra $R = L^1(\mathbb{R}^+; e^{-t^2})$, considered in (ii). If $f \in R$ has a non-zero constant value on $[0, \delta]$ (for some $\delta > 0$), then $\overline{R * f} = R$ ([3], Theorem 4). Clearly, such elements form a dense subset of $R$, so that certainly $\mathcal{F}(R)$ is dense in $R$. But, as remarked in (ii), not every element of $R$ has FCD; thus $\mathcal{F}(R)$ is not closed in $R$.

It is a simple remark that $\mathcal{F}(R)$ is only *open* in case $\mathcal{F}(R) = R$. This is because, always, $0 \in \mathcal{F}(R)$, so that if $\mathcal{F}(R)$ were open it would contain a neighbourhood of 0, and so in fact $\mathcal{F}(R) = R$.

(v) If $\Omega$ is any compact Hausdorff space, then the uniform algebra $A = C(\Omega)$ has $\mathcal{F}(A) = A$ (since, for every $f \in C(\Omega)$, $f$ and $f^2$ generate the same closed ideal).

The Volterra algebra $V = L^1[0,1]$ is a commutative radical Banach algebra in which each element either generates a dense principal ideal, or is nilpotent. Thus $\mathcal{F}(V) = V$.

(vi) If $A = A(\Delta)$ is the familiar disc algebra, then it is clear that no function (apart from $f \equiv 0$) in $A$ that has a zero in $\operatorname{int} \Delta$ can have FCD. However, certain functions $f$ with $f^{-1}(0) \subset \partial\Delta$ do have FCD; the simplest example is $f_0(z) = z - 1$. Using well-known properties of $A(\Delta)$, it may be shown that $\overline{Af_0} = \overline{Af_0^2} = \{f \in A : f(1) = 0\}$.

Let $A$ be a commutative $F$-ring, with topology $\tau$, and let $x \in A$ have FCD. From Corollary 4.3, it follows that, although $I(x)$ is not generally closed in $A$, it carries a complete metrizable topology, $\tau_x$ say, as the inverse limit of the sequence

$$A \xleftarrow{L_x} A \xleftarrow{L_x} A \xleftarrow{L_x} \ldots$$

We can describe convergence in $\tau_x$ explicitly as follows. We know that $L_x$ maps $I(x)$ bijectively; write $L_x^{-1}$ for the inverse of this bijection. Then, for a sequence $(y_k)$ in $I(x)$ and element $y \in I(x)$, we have $y_k \to y$ $(\tau_x)$ if and only if, for every $r \ge 0$, $L_x^{-r}(y_k) \to L_x^{-r}(y)$ $(\tau)$, as $k \to \infty$. We have the following lemma; it extends Lemma 6 of [4] to $F$-rings.

LEMMA 4.5. *Let $T : A \to B$ be a continuous homomorphism of commutative $F$-rings. Let $x \in A$ have FCD, and let $y = T(x)$. Then:*

(i) *$y$ has FCD in $B$ and $T(I(x)) \subseteq I(y)$;*
(ii) *$T|I(x) : I(x) \to I(y)$ is continuous for the topologies $\tau_x$, $\tau_y$;*
(iii) *if $T(A)$ is dense in $B$, then $T(I(x))$ is $\tau_y$-dense in $I(y)$.*

Proof. (i) This is immediate.

(ii) Let $z_k \in I(x)$, $z_k \to z \in I(x)$ in the topology $\tau_x$; then, for every $r \ge 0$, $L_x^{-r}(z_k) \to L_x^{-r}(z)$ $(\tau)$. But $T(L_x^{-r}(z_k)) = L_y^{-r}(T(z_k))$; so, in the topology of $B$, $L_y^{-r}(T(z_k)) \to T(L_x^{-r}(z)) = L_y^{-r}(T(z))$, for all $r \ge 0$. Thus $T(z_k) \to T(z)$ $(\tau_y)$.

(iii) First we show that $T(I(x))$ is dense in $I(y)$ for the topology $\sigma$ of $B$. Let $\delta(x) = m$; then $I(x)$ is dense in $Ax^m$. Since $T(A)$ is dense in $B$, it follows that $T(I(x))$ is $\sigma$-dense in $By^m$. But $T(I(x)) \subseteq I(y) \subseteq By^m$, so $T(I(x))$ is $\sigma$-dense in $I(y)$.

Since, for every $z \in I(x)$, $T(L_x^{-r}(z)) = L_y^{-r}(T(z))$, it follows that $T(I(x))$ is dense in $I(y)$ for the topology $\tau_y$.

For the rest of this section we shall be considering *algebras*, rather than more general rings. Although, to be definite, we take the algebras over the complex field, all the results of this section would apply equally to real algebras.

Let $\mathcal{F} = \mathcal{F}_{\mathbb{C}} = \mathbb{C}[[X]]$, the algebra of all formal power series in one variable, with complex coefficients. If $A$ is any algebra with identity, then we regard $\mathbb{C}[[X]]$ as a subalgebra of $A[[X]]$ in the natural way. The following result is fundamental for what follows.

THEOREM 4.6. *Let $p$ be any submultiplicative seminorm on $\mathcal{F}$. Then $X$ has FCD relative to the $p$-topology.*

Proof. See [4], Lemma 4. (The main ingredient in the proof is Theorem 1 of [1].)

THEOREM 4.7. (i) *Let $A$ be a commutative normed algebra and let $x$ be a stable element of $A$. Then $x$ has FCD.*

(ii) *If $A$ is a commutative Banach algebra, then an element $x$ of $A$ is stable if and only if it has FCD.*

Proof. (i) Because of Corollary 2.4 and Lemma 4.1, we may assume that $A$ has an identity. Next, if $x$ has finite descent, i.e. if $Ax^m = Ax^{m+1}$ for some $m \geq 0$, then certainly $x$ has FCD. We may, thus, assume that $x$ does not have finite descent in $A$. Equivalently, we may assume that, for all $m \geq 0$, $x^m \notin I(x)$.

Since $x$ is stable, by Theorem 3.4, there is a surjective homomorphism $\Psi_x : \mathcal{F}_A \to A/I_0(x)$ such that $\Psi_x(X) = \pi(x)$ (where $\pi : A \to A/I_0(x)$ is the quotient mapping). Let $\Theta_x = \Psi_x|\mathcal{F}$; then $\Theta_x$ is a homomorphism.

Suppose that $\Theta_x$ were not injective. Then, since the non-zero proper ideals of $\mathcal{F}$ are just the principal ideals $\mathcal{F}X^m$ ($m \geq 1$), it would follow that $x^m \in I_0(x)$ for some $m \geq 1$, contrary to the assumption that $x^m \notin I(x)$. Thus $\Theta_x$ is injective.

Let $p$ be the quotient seminorm on $A/I_0(x)$, derived from the norm $\|\cdot\|$ of $A$; i.e.

$$p(\pi(a)) = \inf\{\|a+y\| : y \in I_0(x)\} \quad (a \in A).$$

By Theorem 4.6 and Lemma 4.1, $\pi(x) = \Theta_x(X)$ has FCD relative to the $p$-topology on $A/I_0(x)$; let $\delta(\pi(x)) = m$. Then, for every $\varepsilon > 0$, there is $a \in A$ such that $p(\pi(x^m - ax^{m+1})) < \varepsilon$. By the definition of $p$, there is then some $y \in I_0(x)$ such that $\|x^m - ax^{m+1} - y\| < \varepsilon$. But $y \in I_0(x) \subseteq Ax^{m+1}$, and so $x^m \in \overline{Ax^{m+1}}$, i.e. $\delta_A(x) \leq m$.



(ii) This is immediate from (i) and Proposition 4.4.

COROLLARY 4.8. (i) *Let $A$ be a commutative Banach algebra, $B$ a subalgebra of $A$ (not necessarily closed). If $x \in B$ is stable in $B$, then $x$ is stable in $A$.*

(ii) *Let $C$ be any commutative algebra, $A$ a commutative Banach algebra and let $T : C \to A$ be a homomorphism. If $x$ is a stable element of $C$, then $T(x)$ is stable in $A$.*

Proof. (i) Let $x \in B$ be stable in $B$. Then $B$ is a normed algebra so, by Theorem 4.7(i), $x$ has FCD in $B$. By Lemma 4.1, $x$ has FCD in $A$, so that $x$ is stable in $A$ by Theorem 4.7(ii).

(ii) Let $x \in C$ be stable in $C$. By Lemma 2.2(i), $T(x)$ is stable in the subalgebra $\operatorname{im} T$ of $A$. By (i), $T(x)$ is then also stable in $A$.

REMARKS. (i) We have already seen (in Example (iv) following Corollary 2.7) that there exists an incomplete normed algebra $A$, with a subalgebra $B$ and an element $x \in B$, such that $x$ is stable in $B$, but is not stable in $A$.

(ii) Even when $A$ is a Banach algebra and $B$ a closed subalgebra, there may be some $x \in B$ that is stable in $A$ but not stable in $B$. Recall remarks (v), (vi) following Proposition 4.4: let $\Delta$ be the closed unit disc in $\mathbb{C}$, let $A = C(\Delta)$ and let $B = A(\Delta)$, so that $B$ is a closed subalgebra of $A$. Let $u(z) = z$ ($z \in \Delta$); then $u$ is stable in $A$, but not stable in $B$.

(iii) Let $A = C[0,1]$ and let $B$ be the subalgebra of polynomial functions, normed as a dense subalgebra of $A$. Let $u(t) = t$ ($0 \leq t \leq 1$); then $u$ has FCD in $A$, so is stable in $A$. By Lemma 4.1(i), $u$ also has FCD in the dense subalgebra $B$. However, $u$ is not a stable element of $B \cong \mathbb{C}[X]$, by Proposition 2.5. So, in particular, an element of an incomplete normed algebra that has FCD is not necessarily stable.

We now turn to the problem of characterizing stable elements of commutative Fréchet algebras. A *Fréchet algebra* is an $F$-algebra whose topology may be defined by an (increasing) sequence $(p_n)_{n\geq 1}$ of submultiplicative seminorms. The basic theory of Fréchet algebras was introduced in [7] and [10]; a brief summary, with discussion of a number of examples, was given in [4], §2. We remark, in particular, that a (commutative) Fréchet algebra $A$ may be represented (not uniquely) as the inverse limit, $A = \varprojlim(A_n; d_n)$, of a Mittag-Leffler sequence of (commutative) Banach algebras $(A_n)$ and continuous homomorphisms $d_n : A_{n+1} \to A_n$. Such an inverse-limit representation will be called an *Arens–Michael representation* of $A$. Associated with such a representation are continuous homomorphisms $\pi_n : A \to A_n$ such that $\pi_n = d_n \circ \pi_{n+1}$ for all $n$; it is important to recall that, for each $n$, $\pi_n(A)$ is a dense subalgebra of $A_n$.

In [4] (pages 276–277), we made the following definitions. Let $A$ be a commutative Fréchet algebra and let $x$ in $A$. Then $x$ is said to have *locally finite closed descent* (LFCD) if and only if, for every continuous, submultiplicative seminorm $p$ on $A$, the element $x$ has FCD relative to the $p$-topology. Equivalently, if $A = \varprojlim(A_n; d_n)$ is an Arens–Michael representation of $A$, then $x \in A$ has LFCD if and only if, for each $n$, $x_n \equiv \pi_n(x)$ has FCD in the Banach algebra $A_n$. (Remark that, by Lemma 4.1(i), the element $x_n$ has FCD in the normed algebra $\pi_n(A)$ if and only if it has FCD in $A_n$.) In particular, if $A$ is a commutative Banach algebra, then $x \in A$ has LFCD if and only if it has FCD.

Again, let $A$ be a commutative Fréchet algebra and let $x$ in $A$. Then $x$ is called *locally nilpotent* if and only if for every continuous, submultiplicative seminorm $p$ on $A$, there is some positive integer $N$ (which may depend on $p$) such that $p(x^N) = 0$. Again, if $A = \varprojlim(A_n; d_n)$ is an Arens–Michael representation of $A$, then $x \in A$ is locally nilpotent if and only if, for each $n$, $x_n$ is nilpotent. If $x$ is locally nilpotent, then $x$ has LFCD and also $x \in \operatorname{rad} A$. Also, if $A$ is a Banach algebra, then $x \in A$ is locally nilpotent if and only if it is nilpotent. (See [4], Proposition 3; also examples on pages 277–279 of [4]).

THEOREM 4.9. *Let $A$ be a commutative Fréchet algebra, and let $x \in A$. Then $x$ is stable if and only if it has LFCD.*

Proof. (i) Let $A = \varprojlim(A_n; d_n)$ be an Arens–Michael representation of $A$, with $\pi_n : A \to A_n$ the canonical homomorphism (for each $n$). If $x \in A$ is stable then, for every $n$, by Corollary 4.8(ii), $x_n = \pi_n(x)$ is stable in $A_n$, i.e. $x_n$ has FCD in $A_n$, by Theorem 4.7. So $x$ has LFCD.

(ii) Conversely, let $x \in A$ have LFCD, i.e. suppose that, for each $n$, $x_n$ has FCD in $A_n$. We wish to prove that $x$ is stable in $A$. One quick method would be to use Theorem 4 of [4] to show that there is a homomorphism $\Psi_x : \mathcal{F} \to A/I(x)$ with $\Psi_x(X) = \pi(x)$ and then Corollary 4 of [4], which implies that $I(x) = I_0(x)$. We would then observe that the method of [4] could, with minimal changes, have defined $\Psi_x$ from $\mathcal{F}_A$, rather than just from $\mathcal{F}$. We could then use Theorem 3.4 ((ii)$\Rightarrow$(i)) to deduce the stability of $x$.

However, our aim in this paper is to show that the stability results of §3 (especially Theorem 3.4(i)$\Rightarrow$(ii)) may be used to give an alternative proof of [4], Theorem 4 (and Corollary 4). We thus wish to give a proof of the stability of $x$ that does not use the theory of power-series embeddings.

To do that we shall, in the next section, extend the theory of IL-sequences to a theory of "IL-squares". This extension has a certain elegance and interest in its own right. The completion of the proof of Theorem 4.9 will be a by-product (§6.1).



**5. Inverse-limit squares.** Our aim is to prove a result that will say that "a stable limit of stable sequences" is itself a stable sequence. To this end we define an *IL-square* to be a *commutative* diagram, say $\mathcal{G}$, of the form,

$$\begin{tikzcd}
G_{11} & G_{12} \arrow[l, "f_{11}"'] & G_{13} \arrow[l, "f_{12}"'] & \cdots \arrow[l, "f_{13}"'] \\
G_{21} \arrow[u, "g_{11}"'] & G_{22} \arrow[l, "f_{21}"'] \arrow[u, "g_{12}"'] & G_{23} \arrow[l, "f_{22}"'] \arrow[u, "g_{13}"'] & \cdots \arrow[l, "f_{23}"'] \\
G_{31} \arrow[u, "g_{21}"'] & G_{32} \arrow[l, "f_{31}"'] \arrow[u, "g_{22}"'] & G_{33} \arrow[l, "f_{32}"'] \arrow[u, "g_{23}"'] & \cdots \arrow[l, "f_{33}"'] \\
\vdots \arrow[u, "g_{31}"'] & \vdots \arrow[u, "g_{32}"'] & \vdots \arrow[u, "g_{33}"'] &
\end{tikzcd}$$

In this diagram, each row and each column is an IL-sequence of groups and homomorphisms; the groups are not required to be abelian. Let $\mathcal{G}_{i,*}$ denote the $i$th row of $\mathcal{G}$, regarded as an element of **ILG**. Let $G_i \equiv L(\mathcal{G}_{i,*})$, where $L$ is the inverse-limit functor. Then, for each $i$ (because of the commutativity of the diagram), the mappings $(g_{i,j})_{j\ge1}$ define a morphism $g_{i,*} : \mathcal{G}_{i+1,*} \to \mathcal{G}_{i,*}$ which, in turn, via the inverse-limit functor, defines a group homomorphism, say $g_i : G_{i+1} \to G_i$. We refer to the IL-sequence $\mathcal{G}_{\mathrm{col}} \equiv (G_i; g_i)$ as the *limit column* of the square $\mathcal{G}$.

Explicitly, an element of $G_{i+1}$ is a sequence, say $x_{i+1,*} = (x_{i+1,j})_{j\ge1}$, where $x_{i+1,j} \in G_{i+1,j}$ and $x_{i+1,j} = f_{i+1,j}(x_{i+1,j+1})$ for all $j$. Then

$$g_i(x_{i+1,*}) = g_{i,j}(x_{i+1,j})_{j\ge1}.$$

In a precisely analogous manner, there is defined the *limit row*, $\mathcal{G}_{\mathrm{row}}$, of the square, which is an IL-sequence $(H_j; f_j)$, where, for each $j$, $H_j$ is the inverse limit of the $j$th column, say $\mathcal{G}_{*,j} = (G_{i,j})_{i\ge1}$, and each homomorphism $f_j : H_{j+1} \to H_j$ is obtained by applying the inverse-limit functor to the sequence of columns.

There is also a natural way to define an inverse-limit of the square $\mathcal{G}$ itself, namely we define $L(\mathcal{G})$ to be $(x_{i,j})_{i,j\ge1}$ where $x_{i,j} \in G_{i,j}$ and $x_{i,j} = f_{i,j}(x_{i,j+1}) = g_{i,j}(x_{i+1,j})$ for each $i, j$.

A little reflection on the definitions will make the following proposition clear:

PROPOSITION 5.1. *Let $\mathcal{G}$ be an IL-square of groups and homomorphisms. Then there are natural isomorphisms*

$$L(\mathcal{G}) \cong L(\mathcal{G}_{\mathrm{row}}) \cong L(\mathcal{G}_{\mathrm{col}}).$$

The reference to "natural isomorphisms" in the last proposition can be understood in a technical way. There is, in a rather obvious sense, a category of IL-squares of groups and homomorphisms that we shall denote by **ILSG**. It does, we think, add clarity to consider, briefly, a more general situation. Let $\mathcal{G} = (G_\lambda; g_{\lambda\mu})_{\lambda\in\Lambda}$ be an IL-system, of groups and homomorphisms, indexed by a directed set $(\Lambda, \le)$; the homomorphism $g_{\lambda\mu} : G_\mu \to G_\lambda$ is defined whenever $\lambda \le \mu$ in $\Lambda$. It is required that, whenever $\lambda \le \mu \le \nu$, then $g_{\lambda\nu} = g_{\lambda\mu} \circ g_{\mu\nu}$. There is an extensive discussion of such systems in, for example, Chapter 8 of [9].

For fixed $\Lambda$, there is a category of IL-systems indexed by $\Lambda$. If $\mathcal{G} = (G_\lambda; g_{\lambda\mu})$ and $\mathcal{H} = (H_\lambda; h_{\lambda\mu})$ are two such systems, then a morphism $\phi : \mathcal{G} \to \mathcal{H}$ is a system $(\phi_\lambda)_{\lambda\in\Lambda}$ of group homomorphisms $\phi_\lambda : G_\lambda \to H_\lambda$ such that, whenever $\lambda \le \mu$ in $\Lambda$, the square

$$\begin{tikzcd} G_\lambda \arrow[d, "\phi_\lambda"'] & G_\mu \arrow[l, "g_{\lambda\mu}"'] \arrow[d, "\phi_\mu"] \\ H_\lambda & H_\mu \arrow[l, "h_{\lambda\mu}"] \end{tikzcd}$$

commutes. If $\mathcal{G}$ is a system indexed by $\Lambda$, then its *inverse limit* is the group

$$L(\mathcal{G}) = \Big\{(x_\lambda) \in \prod_{\lambda\in\Lambda} G_\lambda : x_\lambda = g_{\lambda\mu}(x_\mu)\ (\lambda \le \mu)\Big\}.$$

We shall consider only a directed set $\Lambda$ that has a *totally ordered cofinal subsequence*, say $C = \{\lambda_1 < \lambda_2 < \ldots\}$ (so that, for every $\lambda \in \Lambda$, there is some $n$ with $\lambda \le \lambda_n$). Then there is a natural isomorphism between $L(\mathcal{G})$ and the sequential inverse limit $\varprojlim(G_{\lambda_n}; g_n)$, where $g_n \equiv g_{\lambda_n\lambda_{n+1}}$. Moreover, there is a natural notion of stability for systems indexed by $\Lambda$, which turns out to be equivalent to the stability of any cofinal subsequence. We shall now explain this in more detail for the category **ILSG** of inverse-limit squares.

In the case of **ILSG** the index set is, of course, $\mathbb{N}\times\mathbb{N}$, with the directed ordering $(m_1, n_1) \le (m_2, n_2)$ if and only if both $m_1 \le m_2$ and $n_1 \le n_2$. Let $\mathcal{G}$ be an IL-square of groups and homomorphisms. The simplest choice of totally ordered cofinal subsequence is the *diagonal sequence* $(G_{n,n}; d_n)$, where $d_n : G_{n+1,n+1} \to G_{n,n}$ is defined by $d_n = f_{i,j}g_{i,j+1} = g_{i,j}f_{i+1,j}$; note that the commutativity of the square has been used.

It is convenient to have an explicit description of what it means for an IL-square to be stable. To make the account reasonably self-contained, this may be taken as a *definition*. Since we are now giving general theory, we do not wish to assume that the groups are abelian, and we shall write the group operations as multiplication.

We need the notion of an *allowable perturbation* of $\mathcal{G}$. Let elements $a_{i,j}$, $b_{i,j} \in G_{i,j}$ be given, for each $i, j$. The elements are required to satisfy the conditions

$$(*) \qquad a_{i,j}\cdot f_{i,j}(b_{i,j+1}) = b_{i,j}\cdot g_{i,j}(a_{i+1,j})$$

for all $i, j$. We then define the perturbed mappings

$$f'_{i,j} = a_{i,j}.f_{i,j}, \qquad g'_{i,j} = b_{i,j}.g_{i,j},$$

for all $i, j$. The *perturbed square* $\mathcal{G}'$ is then the IL-square (of sets and mappings) obtained from $\mathcal{G}$ by replacing each $f_{i,j}$ (respectively, each $g_{i,j}$) by the corresponding mapping $f'_{i,j}$ (respectively, by the mapping $g'_{i,j}$). It is readily checked that the conditions $(*)$ that define an *allowable* perturbation are precisely the conditions needed to ensure that the perturbed square $\mathcal{G}'$ is still a *commutative* diagram.

The IL-square $\mathcal{G}$ is called *stable* if and only if, for every allowable perturbation, the perturbed square $\mathcal{G}'$ has non-empty inverse limit. Even more explicitly, the IL-square $\mathcal{G}$ is stable if and only if, for every choice of $a_{i,j}, b_{i,j} \in G_{i,j}$ (for all $i, j$) that satisfies the condition $(*)$, there exist $x_{i,j} \in G_{i,j}$ (for all $i, j$) such that

$$x_{i,j} = a_{i,j}f_{i,j}(x_{i,j+1}) = b_{i,j}g_{i,j}(x_{i+1,j}),$$

for all $i, j$. It follows from the general discussion that:

PROPOSITION 5.2. *An IL-square $\mathcal{G}$ is stable if and only if its diagonal sequence $(G_{i,i}; d_i)$ is a stable IL-sequence.*

REMARKS. (i) The interested reader may care to write a direct proof of Proposition 5.2, without reference to more general IL-systems.

(ii) The diagonal sequence is merely the simplest choice of totally ordered cofinal subsequence. The proposition is equally true with any other choice.

(iii) The reduction to sequences means that we do not need to give separate proofs in the category **ILSG** for results corresponding to Lemma 1.4 and Corollary 1.5, already given for IL-sequences. We also have, corresponding to Theorem 1.1:

THEOREM 5.3. *Let $\mathcal{G} \in$ **ILSG**. Then $\mathcal{G}$ is stable provided that either of the following conditions holds:*

(i) *each $G_{i,j}$ is a complete, metrizable topological group and all the homomorphisms $f_{i,j}$, $g_{i,j}$ are continuous with dense range;*

(ii) *each $G_{i,j}$ is a Hausdorff topological group, all the homomorphisms $f_{i,j}$, $g_{i,j}$ are continuous and* either *every $f_{i,j}$* or *every $g_{i,j}$ has compact range.*

Proof. We simply note that either of the alternative conditions implies that the diagonal sequence has the corresponding sequential property. We then apply the corresponding part of Theorem 1.1 to deduce the stability of the diagonal sequence. The proof is then completed by applying Proposition 5.2.

REMARK. An IL-square $\mathcal{G}$ that satisfies condition (i) of Theorem 5.3 may be called a *Mittag-Leffler square.*

As explained at the beginning of the section, our main interest in IL-squares is concerned with connections (if any) between the stability of the individual rows and columns and stability of the limit row and limit column. First, there is the following simple result.

THEOREM 5.4. *Let $\mathcal{G} \in$ **ILSG** be a stable square. Then the limit row and limit column are stable sequences.*

Proof. It clearly suffices to prove stability of the limit row; we use the notation set out before Proposition 5.2.

Consider, then, the limit row

$$\mathcal{G}_{\text{row}}: \quad H_1 \xleftarrow{f_1} H_2 \xleftarrow{f_2} \dots \xleftarrow{f_{j-1}} H_j \xleftarrow{f_j} H_{j+1} \dots,$$

where $H_j = L(\mathcal{G}_{*,j})$ and $f_j : H_{j+1} \to H_j$ is given by

$$f_j((x_{i,j+1})_{i\geq 1}) = (f_{i,j}(x_{i,j+1}))_{i\geq 1},$$

for all $x_{*,j+1} \equiv (x_{i,j+1})_{i\geq 1} \in H_{j+1}$.

Now let $a_{*,j} \in H_j$ $(j \geq 1)$ be given. Then, say, $a_{*,j} = (a_{i,j})_{i\geq 1}$, where, for all $i$, $j$,

$$a_{i,j} = g_{i,j}(a_{i+1,j}).$$

Define $b_{i,j} = 1_{i,j}$, the identity of $G_{i,j}$, for all $i$, $j$. Then the sets of elements $(a_{i,j})$, $(b_{i,j})$ define an allowable perturbation of $\mathcal{G}$. Since $\mathcal{G}$ is stable, there exist elements $x_{i,j} \in G_{i,j}$ (for all $i$, $j$) such that,

$$x_{i,j} = a_{i,j} f_{i,j}(x_{i,j+1}) = b_{i,j} g_{i,j}(x_{i+1,j}) = g_{i,j}(x_{i+1,j}).$$

Let $x_{*,j} = (x_{i,j})_{i\geq 1}$ for all $j$. Then $x_{*,j} \in H_j$ and $x_{*,j} = a_{*,j} f_j(x_{*,j+1})$.

The stability of $\mathcal{G}_{\text{row}}$ is therefore proved.

We shall next give examples to show that the connection between the stability of an IL-square and the stability of its rows and columns is not quite straightforward. The first example will show that the converse to Theorem 5.4 is false.

EXAMPLE 5.5: *A non-stable IL-square in which the limit row and the limit column are both stable.* Let $A$ be a commutative Banach algebra, with an element $x \in A$ such that $x$ is not nilpotent, but with $I(x) = 0$. For example, $A$ could be the disc algebra $A = A(\Delta)$, with $x$ being the coordinate function, $x(z) = z$ $(z \in \Delta)$. It follows, from Proposition 4.4(iii), that $x$ does not have FCD, so is not stable (Theorem 4.7(ii)).



Now consider the IL-square

$$\begin{tikzcd}
A & A \arrow[l, "L_x"'] & A \arrow[l, "L_x"'] & \dots \arrow[l, "L_x"'] \\
A \arrow[u, "L_x"] & A \arrow[l, "L_x"'] \arrow[u, "L_x"] & A \arrow[l, "L_x"'] \arrow[u, "L_x"] & \dots \arrow[l, "L_x"'] \\
A \arrow[u, "L_x"] & A \arrow[l, "L_x"'] \arrow[u, "L_x"] & A \arrow[l, "L_x"'] \arrow[u, "L_x"] & \dots \arrow[l, "L_x"'] \\
\vdots \arrow[u, "L_x"] & \vdots \arrow[u, "L_x"] & \vdots \arrow[u, "L_x"] &
\end{tikzcd}$$

in which every arrow is $L_x$. Then the diagonal sequence is

$$A \xleftarrow{L_x^2} A \xleftarrow{L_x^2} A \xleftarrow{L_x^2} \dots,$$

which is non-stable, being a subsequence of

$$A \xleftarrow{L_x} A \xleftarrow{L_x} A \xleftarrow{L_x} \dots$$

So, by Proposition 5.2, the square $\mathcal{S}$ is also non-stable.

But, since $I(x) = 0$, the limit row and limit column are both isomorphic to the trivial stable sequence, $0 \leftarrow 0 \leftarrow 0 \leftarrow \dots$

EXAMPLE 5.6: *A stable IL-square that has all its rows and all its columns non-stable.* Suppose that we have a commutative Banach algebra $A$ containing elements $x$, $y$ such that $xy$ has FCD, but neither $x$ nor $y$ has FCD. Then, by Theorem 4.7(ii), as elements of $A$, $xy$ is stable, but neither $x$ nor $y$ is stable.

Then let $\mathcal{S}$ be the following IL-square:

$$\begin{tikzcd}
A & A \arrow[l, "L_x"'] & A \arrow[l, "L_x"'] & \dots \arrow[l, "L_x"'] \\
A \arrow[u, "L_y"] & A \arrow[l, "L_x"'] \arrow[u, "L_y"] & A \arrow[l, "L_x"'] \arrow[u, "L_y"] & \dots \arrow[l, "L_x"'] \\
A \arrow[u, "L_y"] & A \arrow[l, "L_x"'] \arrow[u, "L_y"] & A \arrow[l, "L_x"'] \arrow[u, "L_y"] & \dots \arrow[l, "L_x"'] \\
\vdots \arrow[u, "L_y"] & \vdots \arrow[u, "L_y"] & \vdots \arrow[u, "L_y"] &
\end{tikzcd}$$

in which every horizontal arrow is $L_x$ and every vertical arrow is $L_y$. Then, by definition of stable element, each row and each column is non-stable. But the diagonal sequence is

$$A \xleftarrow{L_{xy}} A \xleftarrow{L_{xy}} A \xleftarrow{L_{xy}} \ldots,$$

which is stable. It follows from Proposition 5.2 that the square $\mathcal{S}$ is also stable.

To complete the example, we must give a suitable Banach algebra $A$. Let $\Delta^2 = \{(z,w) \in \mathbb{C}^2 : |z| \le 1,\ |w| \le 1\}$, the unit bi-disc in $\mathbb{C}^2$, and let $B = P(\Delta^2)$ be the *bi-disc algebra*, i.e. the uniform closure in $C(\Delta^2)$ of the set of complex polynomials in $z$, $w$. We also write $z$, $w$ for the coordinate functions, considered as elements of $B$. Now let $I = B(zw)^2$; then $I$ is a closed ideal of $B$, so that $A = B/I$ is a commutative, unital Banach algebra. Let $q : B \to A$ be the quotient mapping, and define $x = q(z)$, $y = q(w)$.

Then $(xy)^2 = 0$, so $xy$ has FCD in $A$. Using elementary consideration of Taylor series it is simple to show that (a) neither $x$ nor $y$ is nilpotent, but (b) $I(x) = 0 = I(y)$. It then follows from Proposition 4.4 that neither $x$ nor $y$ has FCD. The example is therefore complete.

EXAMPLE 5.7: *An IL-square in which every row and every column is stable, but with neither the square, nor its limit row, nor its limit column being stable.* Let

$$\mathcal{A}:\quad A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} A_3 \xleftarrow{f_3} \ldots,$$

be *any* non-stable IL-sequence of groups and homomorphisms. For each $n \ge 1$, let $j_n : A_n \to A_n$ be an identity mapping. Then we "extend" $\mathcal{A}$ to an IL-square $\mathcal{S}$ as in the following diagram (with the sequence $\mathcal{A}$ shown in bold type):

$$\begin{array}{ccccccccc}
\mathbf{A_1} & \xleftarrow{\mathbf{f_1}} & \mathbf{A_2} & \xleftarrow{j_2} & A_2 & \xleftarrow{j_2} & A_2 & \xleftarrow{j_2} & \ldots \\
\uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle \mathbf{f_2}} & & \uparrow{\scriptstyle f_2 f_3} & & \uparrow{\scriptstyle f_2 f_3} & & \\
A_1 & \xleftarrow{f_1 f_2} & \mathbf{A_3} & \xleftarrow{\mathbf{f_3}} & \mathbf{A_4} & \xleftarrow{j_4} & A_4 & \xleftarrow{j_4} & \ldots \\
\uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle j_3} & & \uparrow{\scriptstyle \mathbf{f_4}} & & \uparrow{\scriptstyle f_4 f_5} & & \\
A_1 & \xleftarrow{f_1 f_2} & A_3 & \xleftarrow{f_3 f_4} & \mathbf{A_5} & \xleftarrow{\mathbf{f_5}} & \mathbf{A_6} & \xleftarrow{j_6} & \ldots \\
\uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle j_3} & & \uparrow{\scriptstyle j_5} & & \uparrow{\scriptstyle \mathbf{f_6}} & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Then every row, and every column, is eventually just a sequence of identity mappings, so is trivially stable.



But the diagonal sequence is, say,

$$\mathcal{A}':\quad A_1 \xleftarrow{f_1 f_2} A_3 \xleftarrow{f_3 f_4} A_5 \xleftarrow{f_5 f_6} \ldots,$$

which is a subsequence of $\mathcal{A}$, and therefore non-stable, by Lemma 1.2. Hence also the square $\mathcal{S}$ is non-stable (Proposition 5.2).

Moreover, it is also simple to see that the limit row $\mathcal{S}_{\mathrm{row}} \cong \mathcal{A}'$, and is therefore again non-stable. The limit column is isomorphic to the "even index" subsequence, say $\mathcal{A}''$, of $\mathcal{A}$, namely

$$\mathcal{A}'':\quad A_2 \xleftarrow{f_2 f_3} A_4 \xleftarrow{f_4 f_5} A_6 \xleftarrow{f_6 f_7} \ldots,$$

so it is also non-stable.

A useful criterion for stability of a square is as follows.

THEOREM 5.8. *Let $\mathcal{G} \in$ **ILSG** have the properties*:

(i) *every row is stable*;
(ii) *the limit column is stable.*

*Then $\mathcal{G}$ is stable and so, in particular, the limit row is stable.*

Proof. Let $a_{i,j}, b_{i,j} \in G_{i,j}$ (all $i$, $j$) be elements defining an allowable perturbation of $\mathcal{G}$, i.e. such that, for all $i$, $j$,

$$a_{i,j} f_{i,j}(b_{i,j+1}) = b_{i,j} g_{i,j}(a_{i+1,j}).$$

Then, for each $i$, by the stability of the $i$th row, there are $y_{i,j} \in G_{i,j}$ $(j \ge 1)$ such that $y_{i,j} = a_{i,j} f_{i,j}(y_{i,j+1})$.

Define $d_{i,j} = y_{i,j}^{-1} b_{i,j} g_{i,j}(y_{i+1,j})$. Then a simple calculation shows that $d_{i,j} = f_{i,j}(d_{i,j+1})$, for all $i$, $j$.

Thus, for each $i$, $d_{i,*} \equiv (d_{i,j})_{j \ge 1} \in L(\mathcal{G}_{i,*}) \equiv G_i$. By the stability of the limit column, there are $z_{i,*} \in G_i$ $(i \ge 1)$ such that $z_{i,*} = d_{i,*} \cdot g_i(z_{i+1,*})$. So, writing $z_{i,*} \equiv (z_{i,j})_{j \ge 1}$, we find that, firstly, since $z_{i,*} \in G_i$,

$$z_{i,j} = f_{i,j}(z_{i,j+1}),$$

and also,

$$z_{i,j} = d_{i,j}\, g_{i,j}(z_{i+1,j}) = y_{i,j}^{-1} b_{i,j} g_{i,j}(y_{i+1,j} z_{i+1,j}).$$

If we now set $x_{i,j} = y_{i,j} z_{i,j}$, then we deduce that

$$x_{i,j} = y_{i,j} f_{i,j}(z_{i,j+1}) = a_{i,j} f_{i,j}(x_{i,j+1}) \quad \text{and} \quad x_{i,j} = b_{i,j} g_{i,j}(x_{i+1,j}).$$

This proves the stability of $\mathcal{S}$.

The final remark follows from Theorem 5.4.

## 6. Some applications of inverse-limit squares

**6.1.** *Completion of the proof of Theorem 4.9.* We have the commutative Fréchet algebra $A$, with Arens–Michael representation $A \cong \varprojlim(A_n; d_n)$, and

$x \in A$ that has LFCD, i.e. such that, for each $n$, $x_n = \pi_n(x)$ has FCD, and so is stable in $A_n$. We wish to prove that $x$ is stable in $A$. We shall, in fact, give the following more general result.

THEOREM 6.1. *Let the commutative $F$-ring $R$ be the inverse limit of a Mittag-Leffler sequence of $F$-rings, say, $R = \varprojlim(R_n; d_n)$. Let $x = (x_n)_{n\ge 1}$ in $R$ be such that, for each $n$, $x_n$ has FCD in $R_n$. Then $x$ is stable in $R$.*

Proof. Consider the IL-square

$$\begin{tikzcd}
R_1 & R_1 \arrow[l, "L_{x_1}"'] & R_1 \arrow[l, "L_{x_1}"'] & \cdots \arrow[l, "L_{x_1}"'] \\
R_2 \arrow[u, "d_1"] & R_2 \arrow[l, "L_{x_2}"'] \arrow[u, "d_1"] & R_2 \arrow[l, "L_{x_2}"'] \arrow[u, "d_1"] & \cdots \arrow[l, "L_{x_2}"'] \\
R_3 \arrow[u, "d_2"] & R_3 \arrow[l, "L_{x_3}"'] \arrow[u, "d_2"] & R_3 \arrow[l, "L_{x_3}"'] \arrow[u, "d_2"] & \cdots \arrow[l, "L_{x_3}"'] \\
\vdots \arrow[u, "d_3"] & \vdots \arrow[u, "d_3"] & \vdots \arrow[u, "d_3"] &
\end{tikzcd}$$

Then each row is stable. The limit column is (using Lemma 4.2(iii)) isomorphic to, say,

$$\mathcal{I}\colon\quad I(x_1) \xleftarrow{\overline{d}_1} I(x_2) \xleftarrow{\overline{d}_2} I(x_3) \xleftarrow{\overline{d}_3} \ldots,$$

where $\overline{d}_n = d_n | I(x_{n+1})$. Using Lemma 4.5, if each $I(x_n)$ is given its Fréchet topology $\tau_{x_n}$, then $\mathcal{I}$ becomes a Mittag-Leffler sequence; in particular, $\mathcal{I}$ is stable. By Theorem 5.8, the limit row is then stable. But the limit row is isomorphic to

$$R \xleftarrow{L_x} R \xleftarrow{L_x} R \xleftarrow{L_x} \ldots$$

Hence $x$ is a stable element of $R$.

REMARK. The reader is referred to [4], §3, for further discussion of elements of LFCD in a commutative Fréchet algebra. It may be useful to give a summarizing proposition:

PROPOSITION 6.2. *Let $A$ be a commutative Fréchet algebra, and let $x \in A$ have LFCD. Then, with standard notation as above, the Arens–Michael representation induces isomorphisms:*

(i) $I(x) \cong \varprojlim(I(x_n); \overline{d}_n)$;

(ii) $A/I(x) \cong \varprojlim(A_n/I(x_n); \widetilde{d}_n)$.



*Also,*

(iii) *$L_x$ maps $I(x)$ bijectively onto itself;*

(iv) *$I(x) = 0$ if and only if $x$ is locally nilpotent.*

Proof. This is contained in [4], Lemma 7 (for (i) and (ii)) and Corollary 4 (for (iii) and (iv)). We remark that the deduction of (ii) from (i) is, in the terminology of the present paper, a direct application of the stability of $\mathcal{I}$, applying Theorem 1.3. (That is not mathematically different from the proof in [4]—but, in that paper, the author had not isolated the notion of "stability".)

The results of Corollary 4.8, true for Banach algebras, but not for general commutative rings, extend to Fréchet algebras.

PROPOSITION 6.3. (i) *Let $A$ be a commutative Fréchet algebra, $B$ a subalgebra of $A$ (not necessarily closed) and let $x \in B$ be stable as an element of $B$. Then $x$ is stable in $A$.*

(ii) *Let $C$ be any commutative algebra, let $A$ be a commutative Fréchet algebra, and let $T : C \to A$ be a homomorphism. If $x \in C$ is stable, then $T(x)$ is stable in $A$.*

Proof. (i) Let $A = \varprojlim(A_n; d_n)$ be an Arens–Michael representation of $A$, with $\pi_n : A \to A_n$ $(n \ge 1)$ the canonical homomorphisms.

Let $x \in B$ be stable in $B$. By Corollary 4.8(ii), $\pi_n(x)$ is stable in $A_n$, for each $n$. Hence $x$ has LFCD in $A$, so is stable in $A$ by Theorem 4.9.

(ii) This follows from (i), just as part (ii) of Corollary 4.8 followed from part (i).

**6.2.** *Joint stability.* Let $R$ be a commutative ring and let $x, y \in R$. We say that the pair $(x, y)$ is *jointly stable* if and only if the IL-square, say $\mathcal{S}(x, y)$,

$$\begin{tikzcd}
A & A \arrow[l, "L_x"'] & A \arrow[l, "L_x"'] & \cdots \arrow[l, "L_x"'] \\
A \arrow[u, "L_y"] & A \arrow[l, "L_x"'] \arrow[u, "L_y"] & A \arrow[l, "L_x"'] \arrow[u, "L_y"] & \cdots \arrow[l, "L_x"'] \\
A \arrow[u, "L_y"] & A \arrow[l, "L_x"'] \arrow[u, "L_y"] & A \arrow[l, "L_x"'] \arrow[u, "L_y"] & \cdots \arrow[l, "L_x"'] \\
\vdots \arrow[u, "L_y"] & \vdots \arrow[u, "L_y"] & \vdots \arrow[u, "L_y"] &
\end{tikzcd}$$

is stable. We immediately have the following simple criterion.

PROPOSITION 6.4. *Let $R$ be a commutative ring and let $x, y \in R$. Then the pair $(x, y)$ is jointly stable if and only if $xy$ is a stable element.*

Proof. In the square $\mathcal{S}(x, y)$, the diagonal sequence is

$$R \xleftarrow{L_{xy}} R \xleftarrow{L_{xy}} R \xleftarrow{L_{xy}} \dots$$

The result now follows from Proposition 5.2.

REMARKS. (i) We saw, in Example 5.6, that, even if $R$ is a commutative Banach algebra, it may happen that $xy$ is stable, but neither $x$ nor $y$ is stable; i.e. we may have $(x, y)$ jointly stable, but neither $x$ nor $y$ stable.

(ii) Although, in Example 5.7, we had a non-stable square in which every row and column is stable, we do not know whether this can happen for the square $\mathcal{S}(x, y)$. Thus, we have:

QUESTION. *If $x$, $y$ are stable elements of a commutative ring $R$, is $xy$ necessarily stable (i.e. is $(x, y)$ a stable pair)?*

We note that the answer to the last question is "Yes" in case $R$ is a commutative Fréchet algebra:

PROPOSITION 6.5. *Let $A$ be a commutative Fréchet algebra and let $x$, $y$ be stable elements of $A$. Then $xy$ is stable.*

Proof. By Theorem 4.9, $x$ and $y$ have LFCD, and we have to show that so does $xy$. By considering an Arens–Michael representation of $A$, the question is reduced to the case where $A$ is a Banach algebra.

But then $x$ and $y$ have FCD in $A$; let $m = \max(\delta(x), \delta(y))$. Then $Ax^{m+1}$ is dense in $Ax^m$ and $Ay^{m+1}$ is dense in $Ay^m$. So

$$\overline{A(xy)^{m+1}} \supseteq \overline{Ax^{m+1}}y^{m+1} \supseteq (Ay^{m+1})x^m,$$

so that also $\overline{A(xy)^{m+1}} \supseteq A(xy)^m$. Thus $\delta(xy) \le \max(\delta(x), \delta(y)) < \infty$. This concludes the proof.

**6.3.** *Pseudonilpotents.* There is one other case in which we can prove closure under forming products. We have already recalled (just before Theorem 4.9) the definition of a *locally nilpotent* element of a commutative Fréchet algebra. It is a very simple remark that the set of locally nilpotent elements of a commutative Fréchet algebra is an ideal (included in the Jacobson radical). Also, every locally nilpotent element has LFCD, so is stable. In fact, if $x$ is a stable element of the commutative Fréchet algebra $A$, then $x$ is locally nilpotent if and only if $I(x) = 0$ (see [4], Corollary 4).

At least part of this survives in the context of a general commutative ring. Let $R$ be a commutative ring; we say that an element $x \in R$ is *pseudonilpotent* if and only if (i) $x$ is a stable element of $R$, and (ii) $I_0(x) = 0$.



REMARKS. (i) If $R$ is a commutative Banach algebra, then $x \in R$ is pseudonilpotent if and only if it is nilpotent. If $R$ is a commutative Fréchet algebra, then $x \in R$ is pseudonilpotent if and only if it is locally nilpotent.

(ii) Since, for a stable element $x$ of a commutative Fréchet algebra, we have $I_0(x) = I(x)$ (see Lemma 2.8(iii) and Proposition 6.2(iii)), it is not clear whether, in the definition of "pseudonilpotent", the condition $I_0(x) = 0$ should be replaced by $I(x) = 0$. (But note the remark at the end of §2: it could be that always $I_0(x) = I(x)$.) We shall, therefore, give the following proposition in a way that makes clear that the result would remain true if the definition were modified.

PROPOSITION 6.6. *Let $R$ be a commutative ring and let $x \in R$ be pseudonilpotent. Then, for every $y \in R$, $xy$ is also pseudonilpotent.*

Proof. The reader is left to make the simple verification that, for any elements $x, y \in R$, we have both $I(xy) \subseteq I(x)$ and $I_0(xy) \subseteq I_0(x)$. In particular, if $I_0(x) = 0$ (respectively, if $I(x) = 0$) then also $I_0(xy) = 0$ (respectively, $I(xy) = 0$).

It thus remains to show that if $x$ is pseudonilpotent, then $xy$ is stable, for every $y \in R$. For this, we consider again the IL-square given at the beginning of §6.2. Under the present hypotheses, each row is stable; further, since $I_0(x) = 0$, also $L_0(x) = 0$ (see remarks before Lemma 2.8), so that the limit column is isomorphic to the trivial IL-sequence,

$$0 \leftarrow 0 \leftarrow 0 \leftarrow \dots,$$

which is certainly stable. The stability of the square follows from Theorem 5.8; but the diagonal sequence is $[R; L_{xy}]$, so the stability of $xy$ follows from Proposition 5.2.

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*Received November 20, 1997* (3941)

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