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On quasipositive elements in ordered Banach algebras

by

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Abstract. Let a real Banach algebra A with unit be ordered by an algebra cone K. We study the elements $a \in A$ with $\exp(ta) \in K$, $t \geq 0$.

1. Introduction. Let $(A, \|\cdot\|)$ be a real Banach algebra with unit 1. A *wedge* K is a closed convex subset of A with $\lambda K \subset K$, $\lambda \geq 0$, and K is called a *cone* if in addition $K \cap (-K) = \{0\}$. A cone K is called *normal* if there exists $\gamma \geq 1$ with $0 \leq x \leq y \Rightarrow \|x\| \leq \gamma\|y\|$, and K is called *solid* if $\operatorname{Int} K \neq \emptyset$. A cone K is called an *algebra cone* if $1 \in K$ and $a, b \in K \Rightarrow ab \in K$. If $K \subset A$ is an algebra cone, we consider A as an ordered Banach algebra. As usual $x \leq y :\Leftrightarrow y - x \in K$.

Let A^* denote the dual Banach space of A and let K^* denote the *dual wedge* of K, i.e.

$$K^* = \{\varphi \in A^* : \varphi(a) \geq 0,\ a \in K\}.$$

The cone K is called *polyhedral* if there exist $\psi_1, \ldots, \psi_n \in A^*$ with $K = \{x \in A : \psi_k(x) \geq 0,\ k = 1, \ldots, n\}$. Of course in this case $\dim A \leq n$.

The most common examples of ordered real Banach algebras are generated in the following way: Let E be a real Banach space ordered by a solid cone K_E. The Banach algebra $L(E)$ (the linear continuous endomorphisms of E) can be ordered by the algebra cone

$$K = \{T \in L(E) : Tx \geq 0,\ x \geq 0\}.$$

The operators in K are called *positive*. For a survey on positive operators see e.g. [1], [3], [7], and the references given there.

Now let $A_c = A \times A$ denote the complexification of A (see e.g. [2]), and identify $a \in A$ with $(a, 0) \in A_c$. The spectrum of $a \in A$ is denoted by $\sigma(a) := \sigma((a, 0))$, and $r(a) := r((a, 0))$ denotes its spectral radius. Moreover, we define

$$\tau(a) := \max\{\operatorname{Re} \lambda : \lambda \in \sigma(a)\}.$$

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We call $a \in A$ *positive* if $a \geq 0$, and *quasipositive* if $\exp(ta) \geq 0$, $t \geq 0$. Now let

$$Q := \{a \in A : a \text{ is quasipositive}\}, \qquad -Q := \{a \in A : -a \in Q\},$$
$$H := \{a \in A : \exists \lambda \in \mathbb{R} : a + \lambda 1 \text{ is positive}\}, \qquad -H := \{a \in A : -a \in H\}.$$

Moreover, for $\varphi \in K^*$ let

$$H_\varphi := \{x \in A : \varphi(x) \geq 0\}.$$

REMARKS. 1. We have $K \subset H \subset Q$.

2. If $1 \in \operatorname{Int} K$, then we get the trivial case $Q = H = A$.

3. Since K is closed under multiplication we have

$$a \in Q \Leftrightarrow \exists \varepsilon > 0\ \forall t \in [0, \varepsilon] : \exp(ta) \geq 0.$$

4. We have $a \in (-Q) \cap Q \Leftrightarrow \exp(ta) \geq 0$, $t \in \mathbb{R}$.

A function $f : A \to A$ is called *quasimonotone increasing* (in the sense of Volkmann, see [8]) if

$$a \leq b,\ \varphi \in K^*,\ \varphi(a) = \varphi(b) \Rightarrow \varphi(f(a)) \leq \varphi(f(b)).$$

We will prove the following assertions:

THEOREM 1. *Let A be ordered by an algebra cone. Then*

(1) *Q is a wedge;*
(2) *$(-Q) \cap Q$ is a closed subspace of A;*
(3) *$(-H) \cap H$ is a subalgebra of A;*
(4) *$a \in Q$, $\tau(a) < 0 \Rightarrow a^{-1} \leq 0$;*
(5) *$a \in Q \Leftrightarrow$ the function $f : A \to A$ defined by $f(x) = ax$ is quasimonotone increasing;*
(6) *$Q = \overline{H} = \bigcap_{\varphi \in K^*, \varphi(1) = 0} H_\varphi$;*
(7) *$a \in (-H) \cap H \Rightarrow \pm a + r(a)1 \geq 0$;*
(8) *$(-H) \cap H$ is closed;*
(9) *If K is polyhedral then $Q = H$;*
(10) *$a \in (-Q) \cap Q \Rightarrow a^2 \in Q$;*
(11) *$a \in Q \Leftrightarrow \lim_{h \to 0+} h^{-1} \operatorname{dist}(1 + ha, K) = 0$;*
(12) *If K is normal then $a \in Q \Rightarrow \tau(a) \in \sigma(a)$.*

REMARKS. 1. With a similar proof to that in Section 2 it can be shown that $a \in Q$ if and only if the function $f : A \to A$ defined by $f(x) = xa$ is quasimonotone increasing.

2. In general $H \neq Q$ (even if $\dim A < \infty$) (see Examples 2 and 3).

3. In general $(-Q) \cap Q$ is not a subalgebra of A, since if $a, b \in (-Q) \cap Q$ then ab is in general not in Q (see Example 2). Moreover, if $a \in (-Q) \cap Q$ then a^2 is in general not in $-Q$ (see Examples 2 and 3).

To prove (12) of Theorem 1 we will use the following proposition (see [5]).

PROPOSITION 1. *Let A be ordered by a normal algebra cone. Then*

(1) *$a, b \in K$, $0 \leq a \leq b \Rightarrow r(a) \leq r(b)$,*
(2) *$a \in K \Rightarrow r(a) \in \sigma(a)$ (comp. Theorem of Perron–Frobenius).*

2. Proofs

Proof of Theorem 1. (1) Obviously Q is closed, and $a \in Q \Rightarrow \lambda a \in Q$, $\lambda \geq 0$. Moreover, $a, b \in Q \Rightarrow a + b \in Q$, according to Trotter's product formula:

$$\lim_{n \to \infty} (\exp(a/n) \exp(b/n))^n = \exp(a + b).$$

Hence Q is a wedge.

(2) Since Q is a wedge, $-Q$ is also a wedge. Hence $(-Q) \cap Q$ is a closed subspace of A.

(3) Obviously $(-H) \cap H$ is a subspace of A. Now let $a, b \in (-H) \cap H$. There exists $\lambda \geq 0$ with $\pm a + \lambda 1 \geq 0$ and $\pm b + \lambda 1 \geq 0$. Hence

$$\lambda^2 1 \geq \pm \lambda a, \qquad \lambda^2 1 \geq \pm \lambda b,$$

and

$$\pm ab + \lambda(a \pm b) + \lambda^2 1 \geq 0.$$

Now, since

$$ab + 3\lambda^2 1 \geq ab + \lambda(a + b) + \lambda^2 1 \geq 0$$

and

$$-ab + 3\lambda^2 1 \geq -ab + \lambda(a - b) + \lambda^2 1 \geq 0,$$

we see that $(-H) \cap H$ is closed under multiplication.

(4) Let $a \in Q$ with $\tau(a) < 0$. According to the spectral mapping theorem (see e.g. [2]) we get $r(\exp(a)) < 1$ and therefore $\exp(ta) \to 0$ as $t \to \infty$. Hence

$$a^{-1} = -\int_0^\infty \exp(ta)\, dt \leq 0.$$

(5) Let $f(x) = ax$ $(x \in A)$ be quasimonotone increasing. The function $x(t) = \exp(ta)$, $t \geq 0$, is the solution of the initial value problem

$$x'(t) = ax(t), \qquad x(0) = 1, \qquad t \geq 0.$$

According to a theorem of Volkmann on differential inequalities in ordered Banach spaces (see [9]), we have $x(t) \geq 0$ for $t \geq 0$. Hence $a \in Q$.

Now fix $a \in Q$ and let $x \geq 0$ and $\varphi \in K^*$ with $\varphi(x) = 0$. We have $\exp(ta) \geq 0$ for $t \geq 0$. Therefore $\exp(ta)x \geq 0$ for $t \geq 0$, which implies

$$\frac{\varphi(\exp(ta)x)}{t} \geq 0, \qquad t > 0,$$

and for $t \to 0+$ we obtain $\varphi(ax) \geq 0$.

(6) We first show that

$$Q \subset \bigcap_{\varphi \in K^*,\, \varphi(1)=0} H_\varphi.$$

Fix $a \in Q$. Now let $\varphi \in K^*$ with $\varphi(1) = 0$. According to (5) the function $f : A \to A$ defined by $f(x) = ax$ is quasimonotone increasing. Since $1 \geq 0$ we have $\varphi(a) = \varphi(a1) \geq 0$, i.e. $a \in H_\varphi$.

Obviously $H \subset Q$, hence $\overline{H} \subset Q$. Now it is sufficient to show that

$$\bigcap_{\varphi \in K^*,\, \varphi(1)=0} H_\varphi \subset \overline{H}.$$

Let $a \notin \overline{H}$. Since $\overline{H}$ is a closed convex subset of A there exists $\psi \in A^*$ such that

$$\psi(a) < \psi(x), \qquad x \in \overline{H}.$$

Since $K \subset \overline{H}$, we have $\psi(\lambda x) \geq \psi(a)$ for $x \in K$ and $\lambda \geq 0$. So $\psi(x) \geq 0$ for $x \geq 0$, i.e. $\psi \in K^*$. Since $\mathbb{R}1 \subset \overline{H}$ we have $\psi(1) = 0$ and $\psi(a) < 0$. Hence $a \notin H_\psi$.

(7) Let $a \in (-H) \cap H$, $\varepsilon > 0$ and $\lambda_0 = r(a) + \varepsilon$. We have $\tau(\pm a - \lambda_0 1) < 0$. Hence $\pm a - \lambda_0 1$ are invertible and $\tau((\pm a - \lambda_0 1)^{-1}) < 0$. Since $(-H) \cap H$ is an algebra we have

$$(\pm a - \lambda_0 1)^{-1} = -\frac{1}{\lambda_0}\left(1 \mp \frac{a}{\lambda_0}\right)^{-1} = -\frac{1}{\lambda_0}\sum_{k=0}^{\infty}\left(\pm\frac{a}{\lambda_0}\right)^k \in (-Q) \cap Q.$$

Hence according to (4) we have $\pm a - \lambda_0 1 \leq 0$, and so $\pm a + \lambda_0 1 \geq 0$. Letting $\varepsilon \to 0+$ we obtain $\pm a + r(a)1 \geq 0$.

(8) Assume that $(a_k)_{k=1}^\infty$ is a sequence in $(-H) \cap H$ with limit a. According to (7) we have

$$0 \leq \pm a_k + \|a_k\|1 \to \pm a + \|a\|1, \qquad k \to \infty.$$

Therefore $a \in (-H) \cap H$.

(9) We have $K = \{x \in A : \psi_k(x) \geq 0,\ k = 1, \ldots, n\}$. Let $a \in Q$. For $k = 1, \ldots, n$ we have

$$\psi_k(1) = 0 \Rightarrow \psi_k(a) \geq 0,$$
$$\psi_k(1) > 0 \Rightarrow \psi_k\left(a - \frac{\psi_k(a)}{\psi_k(1)}1\right) = 0.$$

Hence for

$$\lambda = \max\left\{-\frac{\psi_k(a)}{\psi_k(1)} : k = 1, \ldots, n,\ \psi_k(1) > 0\right\},$$

we have $a + \lambda 1 \in K$. Hence $a \in H$.

(10) Let $a \in (-Q) \cap Q$ and consider $\varphi \in K^*$ with $\varphi(1) = 0$. Since $-a, a \in Q$ we have $\varphi(a) = 0$. Since $\exp(at) \geq 0$ for $t \in \mathbb{R}$, we have

$$2\lim_{t \to 0} \frac{\varphi(\exp(ta))}{t^2} = \varphi(a^2) \geq 0.$$

Hence

$$a^2 \in \bigcap_{\varphi \in K^*,\, \varphi(1)=0} H_\varphi = Q.$$

(11) If $a \in Q$ then

$$\lim_{h \to 0+} h^{-1} \operatorname{dist}(1 + ha, K) = 0,$$

according to a theorem of Redheffer and Walter (see [6]). Now let for $a \in A$ the limit above be 0 and assume that $a \notin Q$. Then there exists $\varphi \in K^*$ with $\varphi(1) = 0$ and $\varphi(a) < 0$. Moreover, there exists a sequence $(b_n)_{n=1}^\infty$ in K such that

$$\lim_{n \to \infty} \|n1 + a - nb_n\| = 0.$$

Hence $0 \leq \lim_{n \to \infty} \varphi(nb_n - n1) = \varphi(a)$, which is a contradiction.

(12) Let $a \in Q$. Since Q is a wedge and since $\tau(\mu a) = \mu\tau(a)$ for $\mu \geq 0$, as well as $\sigma(\mu a) = \mu\sigma(a)$ for $\mu \geq 0$, we can assume without loss of generality that

$$\sigma(a) \subset \{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \pi/2\}.$$

We have $\exp(a) \geq 0$. According to the spectral mapping theorem and Proposition 1 we have

$$r(\exp(a)) \in \sigma(\exp(a)) = \exp(\sigma(a)).$$

Now there exists $\lambda_0 \in \sigma(a)$ with $\exp(\lambda_0) = r(\exp(a))$. Hence $\operatorname{Re} \lambda_0 = \tau(a)$ and since $|\operatorname{Im} \lambda_0| < \pi/2$ we have $\lambda_0 = \tau(a)$. ∎

3. Examples. We illustrate our results by some examples.

1. Let $E = \mathbb{R}^n$, $K_E = \{x \in \mathbb{R}^n : x_k \geq 0,\ k = 1, \ldots, n\}$, and let $L(E)$ be ordered by the induced cone K. Obviously K is solid, normal and polyhedral. It is well known that

$$Q = H = \{T \in L(E) : t_{ij} \geq 0,\ i \neq j\}.$$

Hence $(-Q) \cap Q$ are exactly the diagonal matrices.

2. Let $E = \mathbb{R}^3$, $K_E = \{x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\}$ (note that $K_E^* = K_E$), and let $L(E)$ be ordered by the induced cone K. For a characterization and

properties of K see [4]. We do not know an easy representation of Q in this case, but it is easy to check that

$$T=\begin{pmatrix}\alpha_1 & \beta & \gamma\\ -\beta & \alpha_2 & \delta\\ \gamma & \delta & \alpha_3\end{pmatrix}\in Q$$

if $\alpha_3\ge\max\{\alpha_1,\alpha_2\}$. Hence $T\in(-Q)\cap Q$ if $\alpha_1=\alpha_2=\alpha_3$. For $\gamma=1$ and all other entries 0 we have $T\in Q$ and $T\notin H$. Now generate T_1 by setting $\gamma=1$ and all other entries 0, and T_2 by setting $\beta=1$ and all other entries 0. Then

$$T_1T_2=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}\notin Q$$

since $\exp(T_1T_2)=I+T_1T_2\notin K$. Hence $(-Q)\cap Q$ is not a subalgebra of $L(E)$. Moreover, consider $T_3=T_1+T_2$. We have $T_3\in(-Q)\cap Q$. According to Theorem 1(10) we have $T_3^2\in Q$, but $T_3^2\notin -Q$ since

$$\exp(-T_3^2)=\begin{pmatrix}1&0&0\\0&2&1\\0&-1&0\end{pmatrix}\notin K.$$

3. Let l^∞ denote the Banach space of all real bounded sequences $(c_n)_{n=0}^\infty$ endowed with the supremum norm $\|\cdot\|_\infty$. Let

$$E=\left\{x\in C^\infty(\mathbb{R},\mathbb{R}):x(s)=\sum_{n=0}^{\infty}\frac{c_n}{n!}s^n,\ (c_n)_{n=0}^\infty\in l^\infty\right\},$$

endowed with the norm

$$\|x\|=\sup_{n\in\mathbb{N}_0}\sup_{s\in\mathbb{R}}(e^{-|s|}|x^{(n)}(s)|).$$

Note that $\|x\|=\|(c_n)_{n=0}^\infty\|_\infty$. Now let $K_E=\{x\in E:x(s)\ge 0,\ s\in\mathbb{R}\}$, and let $L(E)$ be ordered by the induced cone K (note that K_E is solid, for example the function $x\in C^\infty(\mathbb{R},\mathbb{R})$ defined by $x(s)=\cosh(s)$ is in $\operatorname{Int}K_E$). Let $D\in L(E)$ denote the operator $Dx=x'$. We have

$$(\exp(tD)x)(s)=x(s+t),\qquad t\in\mathbb{R},$$

hence $D\in(-Q)\cap Q$. Moreover, $\pm D\notin H$: Let x be defined as $x(s)=1+\sin(s)$. Then $x\in K_E$ but $(\pm D+\lambda I)x\notin K_E$ for $\lambda\in\mathbb{R}$. Next by Theorem 1(10) we have $D^2\in Q$. Of course, from the theory of the Cauchy problem for the heat equation on an infinite strip we have

$$(\exp(tD^2)x)(s)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{-(s-\tau)^2/(4t)}x(\tau)\,d\tau,\quad t>0,$$

which also shows that $D^2\in Q$. On the other hand, $D^2\notin -Q$: Let $x(s)=1+\sin(s)$; then $x\in K_E$ and $\exp(-tD^2)x\notin K_E$ for $t>0$, since

$$(\exp(-tD^2)x)(s)=1+e^t\sin(s),\quad t>0.$$

The same function x shows that $D^4\notin Q$.

It would be interesting to know whether D^2 is in H or not. Numerical experiments indicate that maybe $D^2+I\ge 0$.

Next, since D^2 corresponds to the double left shift in l^∞ we have $\sigma(D^2)=\{z\in\mathbb{C}:|z|\le 1\}$. Hence according to Theorem 1 (4) we have

$$(D^2-\lambda I)^{-1}\le 0,\quad \lambda>1.$$

Finally, according to Theorem 1(6), D^2 can be approximated by elements of H. For example, set

$$D_h^2:=\frac{\exp(hD)+\exp(-hD)-2I}{h^2}.$$

Then $D_h^2\in H$, $h>0$ and $\lim_{h\to 0+}D_h^2=D^2$ in $L(E)$.

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