and one can easily prove that the sequences $c_{i,k-1}$ satisfy the required properties.

The proof ends by combining this result with the decomposition obtained in the previous lemma for a finite linear combination of translates of the sequence a with strictly positive coefficients. ∎

THEOREM 3.14. *Let $0 < p \le 1$. Then $H^p(\mathbb{Z})$ is continuously embedded in $H^p_{\mathrm{at}}(\mathbb{Z})$.*

Proof. This follows immediately from the previous theorem for $k = [1/p]$. ∎

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A constructive proof of the Beurling–Rudin theorem

by

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Abstract. A constructive proof of the Beurling–Rudin theorem on the characterization of the closed ideals in the disk algebra $A(\mathbb{D})$ is given.

Introduction. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, $\overline{\mathbb{D}}$ its closure and let $A(\mathbb{D})$ be the algebra of all functions continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$. Endowed with the supremum norm, $A(\mathbb{D})$ becomes a commutative, complex Banach algebra with unit element, the so-called disk algebra.

In 1957 Rudin [Ru] gave a complete characterization of the closed ideals in $A(\mathbb{D})$. Later, a similar but somewhat simpler and more functional analytic proof was given by Srinivasan and Wang [SrWa]. The proofs were based on Beurling's invariant subspace theorem for the shift operator on the Hilbert space H^2 of all square summable power series in $\mathbb{D}$, the Riesz theorem on the structure of analytic measures on the unit circle $\mathbb{T}$, the Hahn–Banach theorem and the Riesz representation theorem for bounded linear functionals on $C(\overline{\mathbb{D}})$.

In this paper we present an elementary and constructive proof of this theorem. For background material, the reader is referred to the books of J. Garnett [Ga] and K. Hoffman [Ho].

1. A Frostman type theorem for the sum of two inner functions. Let u be an inner function. By Frostman's well known result the inner function $(a-u)/(1-\overline{a}u)$ is a Blaschke product for all $a \in \mathbb{D}$ outside a possibly empty set E of logarithmic capacity zero, denoted by $\operatorname{cap} E = 0$ (see [Ga, p. 79]). Walter Rudin [Rud] extended this result by showing that for every analytic function f of bounded characteristic in $\mathbb{D}$ the inner factor of $f - a$ is a Blaschke product for all $a \in \mathbb{D} \setminus E$, where $\operatorname{cap} E = 0$. Here we have the following result of Donald Sarason (unpublished):

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THEOREM 1.1 (Sarason). *Let u and v be two inner functions having no common factor. Then for every $\varrho > 0$ the inner factor of $u + \varrho e^{it} v$ is a Blaschke product for almost all t in $\mathbb{R}$.*

Proof (Sarason). In view of [Ga, p. 56], it suffices to show that

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta}) + \varrho e^{it} v(re^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|u(e^{i\theta}) + \varrho e^{it} v(e^{i\theta})|\,d\theta$$

for almost all t. Since the integrands on the left side are subharmonic, the integral means increase to a real number not exceeding the right side of the equation above (Fatou theorem). Hence it will suffice to show that

$$\lim_{r\to 1} \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log|u(re^{i\theta}) + \varrho e^{it} v(re^{i\theta})|\,d\theta\,dt = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log|u(e^{i\theta}) + \varrho e^{it} v(e^{i\theta})|\,d\theta\,dt.$$

Because

$$\frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta}) + \varrho e^{it} v(re^{i\theta})|\,dt = \log\max(|u(re^{i\theta})|, \varrho|v(re^{i\theta})|)$$

we are, by Fubini, done if

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log\max(|u(re^{i\theta})|, \varrho\,|v(re^{i\theta})|)\,d\theta = \max(0, \log\varrho).$$

Now the limit on the left is the value at the origin of the least harmonic majorant in $\mathbb{D}$ of the subharmonic function $\max(\log|u|, \log|\varrho v|)$. Denote this majorant by h. So it remains to show that h is the constant function $\max(0, \log\varrho)$.

Without loss of generality let $0 \le \varrho \le 1$. Then $\log|u| \le h \le 0$. This implies that h has radial limits 0 almost everywhere on $\mathbb{T}$. So, if h is not identically zero, then h is the Poisson integral of a negative singular measure on $\mathbb{T}$. Hence $\varphi = \exp(h + i\widetilde{h})$ is a singular inner function (here $\widetilde{h}$ denotes the harmonic conjugate of h in $\mathbb{D}$). Since $|u| \le |\exp(h + i\widetilde{h})|$, that inner function φ divides u. But $|v| \le \frac{1}{\varrho}|\exp(h + i\widetilde{h})|$ implies that φ also divides v, producing a counterexample. Hence $h \equiv 0$. ■

2. Closed ideals in $A(\mathbb{D})$. For a closed set $E \subseteq \mathbb{T}$ of Lebesgue measure zero we denote by $I(E, A(\mathbb{D}))$ the ideal of all functions in $A(\mathbb{D})$ vanishing

on E. A function $p_E \in A(\mathbb{D})$ satisfying $p_E(z) = 1$ for $z \in E$ and $|p_E(z)| < 1$ otherwise is called a *peak function* associated with E (for the construction see [Ho, pp. 80–81]). If $f \in A(\mathbb{D})$, we let $Z(f) = \{z \in \overline{\mathbb{D}} : f(z) = 0\}$ denote its zero set. The *hull* or *zero set* of an ideal I in $A(\mathbb{D})$ is the set $Z(I) = \bigcap_{f\in I} Z(f)$.

If $u = BS_\mu$ is an inner function, then $\operatorname{Sing} u$ is the set of all boundary singularities of u. It is clear that $\operatorname{Sing} u$ equals the union of the support of the measure μ associated with the singular inner part S_μ of u and the set of all cluster points of the zeros of u in $\mathbb{D}$. If μ is a Borel measure on $\mathbb{T}$, then μ_E denotes its restriction to E, where $E \subseteq \mathbb{T}$.

We call an inner function u *normalized* if $u(0) > 0$. The term g.c.d. $\mathcal{F}$ means the greatest common divisor of a set $\mathcal{F}$ of normalized inner functions (see [Ho, p. 85] and [Ga, p. 84]). The main tools of our constructive approach to the Beurling–Rudin theorem are, besides the theorem of Chapter 1, results on divisibility in closed ideals of $A(\mathbb{D})$. The proofs depend on a refinement of ideas appearing in [Mo] for the case of the algebra H^∞ of all bounded analytic functions on $\mathbb{D}$.

The proof of the Beurling–Rudin theorem itself is done in several steps. First we show that the g.c.d., denoted by φ, of the inner parts of the functions in the ideal I is already determined by a countable set of functions in I. Then we construct functions f_n in I with the g.c.d. of their inner parts φ_n being φ, but such that φ_n converges uniformly on compact subsets of $\overline{\mathbb{D}} \setminus (Z(I) \cap \mathbb{T})$ to φ and that $\varphi_n(1 - p_E) \in I$ for a peak function p_E associated with $E = Z(I) \cap \mathbb{T}$. This is done by using the facts that if $f = BS_\mu F \in I$, then $Z(I) \cap \mathbb{D} = \emptyset$ implies that, without leaving the ideal, one can split off the Blaschke factor B, the singular inner function $S_{\mu_{\mathbb{T}\setminus E}}$ and one can replace the outer part F by a fixed outer function vanishing exactly on E.

LEMMA 2.1. *Let (u_n) be a sequence of normalized inner functions without a common factor and let $v_n = \text{g.c.d.}\{u_1, \ldots, u_n\}$. Then*

(1) *v_{n+1} divides v_n for every $n \in \mathbb{N}$,*
(2) *g.c.d.$\{v_n : n \in \mathbb{N}\} = 1$,*
(3) *(v_n) converges locally uniformly to the constant function 1.*

Proof. Note that the first two assertions are trivial. Because (v_n) is a normal family, there exists a locally uniformly converging subsequence (v_{n_j}). Let v be any such limit point. Because v_n divides v_k for every $1 \le k \le n$, there exist inner functions $f_{n,k}$ such that $v_k = v_{n_j} f_{n_j,k}$ $(1 \le k \le n_j,\ j \in \mathbb{N})$. Because for fixed k the set $\{f_{n_j,k} : j \in \mathbb{N}\}$ is a normal family, we can choose a converging subsequence. Without loss of generality let $f_k = \lim_j f_{n_j,k}$. Then $v_k = v f_k$. Hence v divides v_k for every k. Thus $v \equiv 1$. ■

LEMMA 2.2. *Let u be an inner function and let $f_n = uh_n$ be a sequence in $A(\mathbb{D})$ converging in norm to f. Then $f = uh$ for some $h \in A(\mathbb{D})$.*

Proof. Because $\|f_n\| = \|h_n\|$ is bounded, by a normal family argument there exists a locally uniformly converging subsequence of h_n; say $h_{n_k} \to h$, where $h \in H^\infty$. Then uh_{n_k} converges to uh. Since pointwise limits are unique, $f = uh$. By [Ga, p. 78], $f \in A(\mathbb{D})$ implies $h \in A(\mathbb{D})$. ∎

LEMMA 2.3. *Let I be a closed ideal in $A(\mathbb{D})$ such that $Z(I) \cap \mathbb{D} = \emptyset$ and let $g = Bf \in I$ for a Blaschke product B. Then $f \in I$.*

Proof. Let

$$B_N(z) = \prod_{n=N+1}^{\infty} \frac{\overline{a}_n}{|a_n|} \cdot \frac{a_n - z}{1 - \overline{a}_n z}$$

be the Nth tail of the Blaschke product B with zero sequence (a_n). Because $Z(I) \cap \mathbb{D} = \emptyset$, we can choose for every a_j a function $g_j \in I$ such that $g_j(a_j) \neq 0$. The formula

$$\frac{g(z)}{z - a_j} = -\frac{1}{g_j(a_j)}\left(g(z)\,\frac{g_j(z) - g_j(a_j)}{z - a_j} - g_j(z)\frac{g(z)}{z - a_j}\right) \in I,$$

applied for each $j \in \{1, \ldots, n\}$ successively, implies that $B_n f \in I$ for each n. Because B_n converges uniformly to 1 on each compact set in $\overline{\mathbb{D}} \setminus \operatorname{Sing} B$, $B_n f$ tends uniformly to f on $\overline{\mathbb{D}}$. (Note that $\operatorname{Sing} B \subseteq Z(f) \cap \mathbb{T}$). Since I is closed we conclude that $f \in I$. ∎

The following lemma is an immediate consequence of the Nullstellensatz for $A(\mathbb{D})$, for which there exists a constructive proof (see [vR] and [MoRu]).

LEMMA 2.4. *Let I be a closed ideal in $A(\mathbb{D})$, $g \in A(\mathbb{D})$ and let $f \in A(\mathbb{D})$ satisfy $Z(f) \cap Z(I) = \emptyset$. Then $fg \in I$ implies that $g \in I$.*

Proof. By compactness there exist finitely many functions $f_j \in I$ so that

$$\bigcap_{j=1}^{n} Z(f_j) \cap Z(f) = \emptyset.$$

The Nullstellensatz now yields functions $h, h_j \in A(\mathbb{D})$ so that

$$1 = \sum_{j=1}^{n} h_j f_j + hf.$$

Hence $g = (\sum_{j=1}^{n} h_j f_j)g + h(fg) \in I + I \subseteq I$. ∎

LEMMA 2.5. *Let I be a closed ideal in $A(\mathbb{D})$, $E = Z(I) \cap \mathbb{T}$, and let S_μ be a singular inner function such that $S_\mu f \in I$. Then $f \in I$ whenever $\mu(E) = 0$.*

Proof. Because μ is a regular measure and $\mu(E) = 0$, there exist open neighborhoods U_n of E (in $\mathbb{T}$) such that

$$\mu(U_n) < \frac{1}{n}, \quad U_{n+1} \subseteq U_n, \quad \bigcap_{n=1}^{\infty} U_n = E. \tag{1}$$

Since E is compact, we may assume that U_n is a finite union of arcs with disjoint closures. Let

$$G_n(z) = \exp\left(-\frac{1}{2\pi}\int_{U_n} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right) \tag{2}$$

and

$$H_n(z) = \exp\left(-\frac{1}{2\pi}\int_{\mathbb{T}\setminus U_n} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right). \tag{3}$$

It is easy to see that G_n and H_n are inner functions satisfying $G_n H_n = S_\mu$ and that G_n converges to 1 uniformly on every compact set of $\overline{\mathbb{D}} \setminus E$. Let p_n be a peak function associated with the boundary of U_n in $\mathbb{T}$ and let V_n be a finite union of open arcs, slightly bigger than those of U_n but still satisfying (1). By taking suitable powers m_n, we get $|p_n^{m_n}| \leq (1/2)^n$ on $\mathbb{T} \setminus V_n$. Hence $p_n^{m_n}$ tends uniformly to zero on every compact set of $\mathbb{T} \setminus E$. It is even a weakly null sequence in $A(\mathbb{D})$.

Because $\operatorname{Sing} G_n \subseteq \operatorname{supp} \mu \cup \partial U_n \subseteq Z(f) \cup \partial U_n$, we conclude that $(1 - p_n^{m_n})G_n f \in A(\mathbb{D})$ and that $(1 - p_n^{m_n})G_n f$ converges uniformly to f on $\overline{\mathbb{D}}$. Let $f_n = (1 - p_n^{m_n})G_n f$. It remains to show that $f_n \in I$. In fact, since $\operatorname{Sing} H_n \subseteq \overline{\mathbb{T} \setminus U_n}$, we see that $E \cap \operatorname{Sing} H_n = \emptyset$. Moreover, $\operatorname{Sing} H_n \subseteq Z(f) \cup \partial U_n$ implies that $\operatorname{Sing} H_n$ has Lebesgue measure zero. Thus there exists a peak function p_{E_n} in $A(\mathbb{D})$ associated with $E_n = \operatorname{Sing} H_n$. Hence $H = H_n(1 - p_{E_n}) \in A(\mathbb{D})$ and $Z(H) \cap Z(I) = \emptyset$. By the Nullstellensatz for $A(\mathbb{D})$ there exist $\alpha \in A(\mathbb{D})$ and $h \in I$ so that $1 = \alpha H + h$. Hence

$$f_n = (f_n \alpha)H + f_n h = (S_\mu f)\alpha(1 - p_{E_n})(1 - p_n^{m_n}) + f_n h \in I.$$

Since I is closed, we obtain $\lim f_n = f \in I$. ∎

LEMMA 2.6. *Let I be a closed ideal in $A(\mathbb{D})$ such that $Z(I) \cap \mathbb{D} = \emptyset$ and let $E = Z(I)$. Suppose that $f = BS_\sigma F_\mu \in I$, where B is a Blaschke product, S_σ a singular inner function and F_μ the outer part of f. Then $S_{\sigma_E}(1 - p_E) \in I$, where σ_E is the restriction of the measure σ to E and p_E is a peak function in $A(\mathbb{D})$ associated with E.*

Proof. We first note that $\operatorname{Sing} S_\sigma = \operatorname{supp}\sigma \subseteq Z(f)$ and that E has Lebesgue measure zero. Moreover, $f = BS_{\sigma_E}S_{\sigma_{\mathbb{T}\setminus E}}F_\mu$. By Lemmas 2.3 and 2.5 we have $g := S_{\sigma_E}F_\mu \in I$. Since μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$, we obviously have $\mu(E) = 0$. By exactly the same reasoning as in the proof of Lemma 2.5, we obtain bounded analytic functions G_n and H_n defined as in (2), (3) and satisfying $G_nH_n = F_\mu$. Note that

$$\max\{\|G_n\|, \|H_n\|\} \le \max\{1, \|F_\mu\|\}.$$

Because $H_nG_n = F_\mu$ is continuous on $\overline{\mathbb{D}}$ and H_n analytic on U_n with $|H_n| = 1$ on U_n, we see that G_n is continuous on $\mathbb{D}\cup U_n$. Since G_n is analytic on $\mathbb{T}\setminus\overline{U}_n$, the only points of discontinuity of G_n are at the boundary points of U_n.

Let $p_n^{m_n}$ be the peak functions constructed in the proof of Lemma 2.5. We then deduce that $f_n := (1-p_n^{m_n})G_n(1-p_E) \in A(\mathbb{D})$ and that f_n tends uniformly to $1-p_E$ on $\overline{\mathbb{D}}$.

In the last step we show that $S_{\sigma_E}f_n \in I$. Note that $S_{\sigma_E}f_n \in A(\mathbb{D})$. By the same reasoning as for G_n, H_n is a bounded analytic function continuous outside the boundary of U_n, so that $h_n := H_n(1-p_n^{m_n}) \in A(\mathbb{D})$.

Since $g \in I$, we see that

$$\begin{aligned} S_{\sigma_E}f_nh_n &= S_{\sigma_E}f_n(1-p_n^{m_n})H_n = (1-p_n^{m_n})^2(1-p_E)S_{\sigma_E}G_nH_n \\ &= (1-p_n^{m_n})^2(1-p_E)g \in I. \end{aligned}$$

Now $Z(h_n)\cap Z(I) = \emptyset$. By Lemma 2.4 we obtain $S_{\sigma_E}f_n \in I$. By the closedness of I, we conclude that $S_{\sigma_E}(1-p_E) = \lim S_{\sigma_E}f_n \in I$. ∎

THEOREM 2.7 (Beurling–Rudin). *Let I be a nontrivial closed ideal in $A(\mathbb{D})$ such that the greatest common inner divisor of the normalized inner factors of the elements in I is the constant function 1. Then $I = I(E, A(\mathbb{D}))$, where $E = Z(I)\cap\mathbb{T}$. Moreover, I is the closure of the principal ideal generated by $1-p_E$, where p_E is a peak function for E.*

Proof. STEP 1. Since $A(\mathbb{D})$ is a separable Banach algebra (e.g. the polynomials with rational coefficients are dense), every subset, in particular our closed ideal I has this property. Let $\{f_n : n \in \mathbb{N}\}$ be a dense subset of I. Then the g.c.d. of the normalized inner factors of the f_n is, by Lemma 2.2, also a common divisor of all limit points of the f_n. Hence, by our hypothesis, this is the constant function 1.

STEP 2. Since the inner factors of the functions in I have no common inner factor, they do not vanish simultaneously at any common point in $\mathbb{D}$. Hence $Z(I)\cap\mathbb{D} = \emptyset$. Therefore $E = Z(I) \subseteq \mathbb{T}$ and E has Lebesgue measure zero. Let p_E be a peak function associated with E and let $f = \varphi h \in I$, where φ is an inner and h an outer function. By Lemmas 2.6 and 2.3 there exists a singular inner factor u of φ with $\operatorname{Sing} u \subseteq E$ such that $g := u(1-p_E) \in I$. Taking for f our f_n's, we get singular inner functions u_n such that $\operatorname{Sing} u_n \subseteq E$ and $u_n(1-p_E) \in I$. Moreover, because the inner factors of the f_n have no common divisor, the same obviously holds for the u_n.

STEP 3. Now let $v_n = \text{g.c.d.}\{u_1,\ldots,u_n\}$. We claim that $v_n(1-p_E) \in I$.

In fact, let $a \in \mathbb{D}$ be chosen so that by Theorem 1.1 the inner factor of $u_1 + au_2$ is a Blaschke product B times $\text{g.c.d.}\{u_1,u_2\} = v_2$. Hence $u_1(1-p_E) + au_2(1-p_E) = v_2BF \in I$ for some outer function F. By Lemma 2.3 we get $v_2F \in I$. Because $\operatorname{Sing} v_2 \subseteq E$ (note that both u_1 and u_2 are analytic on $\mathbb{T}\setminus E$) we see by Lemma 2.6 that $v_2(1-p_E) \in I$. Now we repeat the same step, replacing u_1 with v_2 and u_2 with u_3. Because $\text{g.c.d.}\{v_{n-1},u_n\} = v_n$, we obtain via induction a proof of our claim that $v_n(1-p_E) \in I$.

STEP 4. Applying now Lemma 2.1, we conclude from $\text{g.c.d.}\{v_1,v_2,\ldots\} = 1$ that (v_n) converges uniformly on compact subsets of $\mathbb{D}$ to the constant function 1. But actually, we have more. In fact, v_n is analytically extendable to $\mathbb{C}\setminus E$. Because $v_n(0)$ is bounded, we see that the family (v_n) is uniformly bounded on every compact set of $\mathbb{C}\setminus E$. Hence by Vitali's theorem, v_n converges uniformly on every compact set of $\mathbb{C}\setminus E$ to 1. In particular, v_n converges uniformly to 1 on $\overline{\mathbb{D}}\setminus E_\varrho$, where E_ϱ is the ϱ-neighborhood of E in $\mathbb{T}$. Thus $v_n(1-p_E)$ converges uniformly to $1-p_E$ on $\overline{\mathbb{D}}$. Since $v_n(1-p_E) \in I$, we conclude by the closedness of I that $1-p_E \in I$.

STEP 5. If $f \in I(E, A(\mathbb{D}))$, then $k_n = (1-p_E^n)f$ converges uniformly to f. But $k_n \in (1-p_E)A(\mathbb{D})$. Hence $f \in \overline{(1-p_E)A(\mathbb{D})}$. Thus

$$I \subseteq I(E, A(\mathbb{D})) \subseteq \overline{(1-p_E)A(\mathbb{D})} \subseteq I. \ \blacksquare$$

REMARK. It is immediately clear from the proof that if I is a nontrivial closed ideal in $A(\mathbb{D})$ with inner factor u, then $I = \overline{(uF)A(\mathbb{D})} = u\overline{FA(\mathbb{D})} = uI(E, A(\mathbb{D}))$ for every outer function F such that $Z(F) = Z(I)\cap\mathbb{T}$.

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On quasipositive elements in ordered Banach algebras

by

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Abstract. Let a real Banach algebra A with unit be ordered by an algebra cone K. We study the elements $a \in A$ with $\exp(ta) \in K$, $t \geq 0$.

1. Introduction. Let $(A, \|\cdot\|)$ be a real Banach algebra with unit 1. A *wedge* K is a closed convex subset of A with $\lambda K \subset K$, $\lambda \geq 0$, and K is called a *cone* if in addition $K \cap (-K) = \{0\}$. A cone K is called *normal* if there exists $\gamma \geq 1$ with $0 \leq x \leq y \Rightarrow \|x\| \leq \gamma\|y\|$, and K is called *solid* if $\operatorname{Int} K \neq \emptyset$. A cone K is called an *algebra cone* if $1 \in K$ and $a, b \in K \Rightarrow ab \in K$. If $K \subset A$ is an algebra cone, we consider A as an ordered Banach algebra. As usual $x \leq y :\Leftrightarrow y - x \in K$.

Let A^* denote the dual Banach space of A and let K^* denote the *dual wedge* of K, i.e.

$$K^* = \{\varphi \in A^* : \varphi(a) \geq 0,\ a \in K\}.$$

The cone K is called *polyhedral* if there exist $\psi_1, \ldots, \psi_n \in A^*$ with $K = \{x \in A : \psi_k(x) \geq 0,\ k = 1, \ldots, n\}$. Of course in this case $\dim A \leq n$.

The most common examples of ordered real Banach algebras are generated in the following way: Let E be a real Banach space ordered by a solid cone K_E. The Banach algebra $L(E)$ (the linear continuous endomorphisms of E) can be ordered by the algebra cone

$$K = \{T \in L(E) : Tx \geq 0,\ x \geq 0\}.$$

The operators in K are called *positive*. For a survey on positive operators see e.g. [1], [3], [7], and the references given there.

Now let $A_c = A \times A$ denote the complexification of A (see e.g. [2]), and identify $a \in A$ with $(a, 0) \in A_c$. The spectrum of $a \in A$ is denoted by $\sigma(a) := \sigma((a, 0))$, and $r(a) := r((a, 0))$ denotes its spectral radius. Moreover, we define

$$\tau(a) := \max\{\operatorname{Re} \lambda : \lambda \in \sigma(a)\}.$$

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