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Discrete Hardy spaces

by

SANTIAGO BOZA and MARÍA J. CARRO (Barcelona)

Abstract. We study various characterizations of the Hardy spaces $H^p(\mathbb{Z})$ via the discrete Hilbert transform and via maximal and square functions. Finally, we present the equivalence with the classical atomic characterization of $H^p(\mathbb{Z})$ given by Coifman and Weiss in [CW]. Our proofs are based on some results concerning functions of exponential type.

1. Introduction. In [CW], Coifman and Weiss extended the usual definition of Hardy spaces $H^p(\mathbb{R}^N)$ (see [FS]) to the more general context of spaces of homogeneous type. Their results are based on the atomic characterization of these spaces. Since then, the theory has been widely developed by many authors. Let us mention the work of Macías and Segovia [MS], where they prove an equivalent characterization of the Hardy spaces via a grand maximal function.

A particular case of space of homogeneous type is the set of integers $\mathbb{Z}$ and hence we have two equivalent definitions of the spaces $H^p(\mathbb{Z})$. In this paper we shall deal with this particular case.

We have to mention other works related to this theory, for example [U] where a maximal characterization of the Hardy spaces is given for spaces of homogeneous type, and [H] where the author obtains an atomic decomposition for Triebel–Lizorkin spaces on spaces of homogeneous type. However, in these two works, the hypothesis assumed on the space X of homogeneous type excludes the case of points of positive measure and hence $\mathbb{Z}$.

From a different point of view, Q. Sun in [Su] gives a characterization of $H^p(\mathbb{Z})$ in terms of discrete square functions.

Also, the space $H^1(\mathbb{Z})$ is defined by Coifman and Weiss in [CW] as the space consisting of all sequences $a = \{a(n)\}_n$ belonging to $\ell^1(\mathbb{Z})$ such that

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$$\sum_k \left| \sum_{n\neq k} \frac{a(n)}{k-n} \right| < \infty;$$

that is, those whose discrete Hilbert transform is in $\ell^1(\mathbb{Z})$. In [E], this definition is extended to the case $0<p<1$ and it is proved that the resulting space is isomorphic to the Paley–Wiener space of functions of exponential type belonging to $L^p(\mathbb{R})$. In that paper, the author left as an open question the equivalence between this characterization in terms of the discrete Hilbert transform and the atomic one in [CW]. We shall prove this equivalence in the range $0<p\le 1$.

The paper is organized as follows: in Section 2 we study some sampling results concerning functions of exponential type. This section is a follow up of some results in [AC] where the authors use some special properties of functions of exponential type to prove that the maximal operator

$$\{a(n)\}_n \to \left\{ \sup_{t>0} \left| \sum_{m\neq 0} \frac{1}{\pi}\cdot\frac{t}{t^2+m^2} a(n-m) \right| \right\}_n$$

is bounded on $\ell^p(\mathbb{Z})$ for $p>1$. From this fact, we see that if we define $H^p(\mathbb{Z})$ as the subspace of $\ell^p(\mathbb{Z})$ consisting of those sequences $a=\{a(n)\}_n$ such that the above maximal sequence is in $\ell^p(\mathbb{Z})$, then $H^p(\mathbb{Z})=\ell^p(\mathbb{Z})$ for $p>1$, as in the classical case.

This leads us to Section 3 where we study various characterizations of the norm in $H^p(\mathbb{Z})$:

a) via the discrete Hilbert transform,

b) via a maximal characterization in terms of the discrete Poisson kernel,

c) via other maximal operators, and

d) via square functions.

In particular, we shall prove that they all agree with the original one of [CW].

As usual, we shall write $f\sim g$ to indicate the existence of two positive constants A and B so that $Af\le g\le Bf$, and constants such as C may change from one occurrence to the next.

Also, for a function F defined in $\mathbb{R}$ we will use the notation F^{d} to indicate the sequence $(F(n))_n$ whenever a different definition is not explicitly written.

We shall write $\star$ for convolution of sequences.

2. Some results on functions of exponential type. The results of this section hold in $\mathbb{R}^N$ and $\mathbb{Z}^N$. However, we shall present them in $\mathbb{R}$ and $\mathbb{Z}$ which is the only case we shall use in the next section.

Let E_R be the set of slowly increasing C^∞ functions f with $\operatorname{supp}\widehat{f}\subset[-R,R]$. The elements of E_R are functions of exponential type R. We recall a well-known sampling theorem (Shannon's formula) for such functions (see [S]):

If $g\in E_R$, then

$$g(x)=\sum_n g\left(\frac{n}{2R}\right)\operatorname{sinc}(2Rx-n),$$

where $\operatorname{sinc} x=\frac{\sin\pi x}{\pi x}$ for $x\neq 0$ and 1 if $x=0$. Also, if $1\le p<\infty$, then (see [B])

$$\|g\|_p \sim \left(\sum_n \left| g\left(\frac{n}{2R}\right)\right|^p\right)^{1/p}. \tag{1}$$

If $R<1/2$, then the above equivalence also works for $0<p\le 1$.

The following lemma will be useful in the sequel (see [FJW]).

Lemma 2.1. *Let $g,h\in E_R$, $R<1/2$, with $h\in\mathcal{S}(\mathbb{R})$ and $\widehat{h}\equiv 1$ on $\operatorname{supp}\widehat{g}$. Then, for every $x\in\mathbb{R}$,*

$$g(x)=(g*h)(x)=\sum_{k\in\mathbb{Z}} g(k)h(x-k).$$

In [AC], the following generalization of the Shannon formula was proved:

Theorem 2.2. *Let $1\le p\le\infty$ and $0<q\le\infty$. Then there exists a constant $C=C(p,q)$ such that*

$$\sum_{n\in\mathbb{Z}}\left(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\right)^{p/q}\le C^p\max(1,R)\int_{\mathbb{R}}\left(\int_0^\infty |g_t(x)|^q\,\frac{dt}{t}\right)^{p/q}dx$$

for every family g_t, $t>0$, of functions in E_R.

Using similar arguments we can show that the previous lemma can be extended to the general case $0<p,q\le\infty$.

Theorem 2.3. *Let $0<p,q\le\infty$. Let $\{g_t(\cdot)\}_{t>0}$ be a family of functions in E_R. Then there exists a constant $C=C(p,q)$ such that*

$$\sum_{n\in\mathbb{Z}}\left(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\right)^{p/q}\le C^p\max(1,R)\int_{\mathbb{R}}\left(\int_0^\infty |g_t(x)|^q\,\frac{dt}{t}\right)^{p/q}dx.$$

Proof. We shall only prove the case $0<p\le\min(1,q)$. The other cases follow as in [AC].

Consider the conjugate exponent of q/p, that is, $q/(q-p)$. Then it is enough to show that, for every family $\{h_t(n)\}_{n\in\mathbb{Z}}$ satisfying

$$\int_0^\infty |h_t(n)|^{pq/(q-p)}\,\frac{dt}{t}\le 1$$

for every $n\in\mathbb{Z}$, we have

$$\sum_{n\in\mathbb{Z}}\int_0^\infty |g_t(n)|^p|h_t(n)|^p\,\frac{dt}{t}\le C^p\max(1,R)\int_{\mathbb{R}}\Big(\int_0^\infty |g_t(x)|^q\,\frac{dt}{t}\Big)^{p/q}dx.$$

Define

$$h_t(x)=\sum_{n\in\mathbb{Z}}h_t(n)\psi(x-n),$$

where ψ is of exponential type m (for some $m\in\mathbb{N}$), $\psi(0)=1$, $\psi(k)=0$ for every $k\in\mathbb{Z}\setminus\{0\}$, and

$$\sup_{x\in\mathbb{R}}\Big(\sum_{n\in\mathbb{Z}}|\psi(x-n)|^p\Big)\le C<\infty.$$

Then, by Minkowski's integral inequality,

$$\begin{aligned}\int_0^\infty |h_t(x)|^{pq/(q-p)}\,\frac{dt}{t}&=\int_0^\infty\Big|\sum_{n\in\mathbb{Z}}h_t(n)\psi(x-n)\Big|^{pq/(q-p)}\frac{dt}{t}\\&\le\int_0^\infty\Big(\sum_{n\in\mathbb{Z}}|h_t(n)|^p|\psi(x-n)|^p\Big)^{q/(q-p)}\frac{dt}{t}\\&\le\sum_{n\in\mathbb{Z}}|\psi(x-n)|^p\Big(\int_0^\infty|h_t(n)|^{pq/(q-p)}\,\frac{dt}{t}\Big)^{(q-p)/q}\le C.\end{aligned}$$

Therefore, since $g_th_t(\cdot)\in E_{R+m}$, we get

$$\begin{aligned}&\int_0^\infty\sum_{n\in\mathbb{Z}}|g_t(n)h_t(n)|^p\,\frac{dt}{t}\\&\quad\le C\max(1,R)\int_0^\infty\int_{\mathbb{R}}|g_t(x)h_t(x)|^p\,dx\,\frac{dt}{t}\\&\quad\le C\max(1,R)\int_{\mathbb{R}}\Big(\int_0^\infty|h_t(x)|^{pq/(q-p)}\,\frac{dt}{t}\Big)^{(q-p)/q}\Big(\int_0^\infty|g_t(x)|^q\,\frac{dt}{t}\Big)^{p/q}dx\\&\quad\le C\max(1,R)\int_{\mathbb{R}}\Big(\int_0^\infty|g_t(x)|^q\,\frac{dt}{t}\Big)^{p/q}dx.\ \blacksquare\end{aligned}$$

Let us now formulate the converse inequality.

THEOREM 2.4. *Let $0<p,q\le\infty$ and $\{g_t(\cdot)\}_{t>0}$ be in E_R with $R<1/2$. Then*

$$\int_{\mathbb{R}}\Big(\int_0^\infty|g_t(x)|^q\,\frac{dt}{t}\Big)^{p/q}dx\le C\sum_{n\in\mathbb{Z}}\Big(\int_0^\infty|g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}$$

for some constant $C=C(p,q)>0$.

Proof. Let $\Psi\in\mathcal{S}(\mathbb{R})\cap E_R$ satisfy $\widehat{\Psi}\equiv 1$ on $\operatorname{supp}\widehat{g}_t$ for every $t>0$. Then by Lemma 2.1 we have, for every $x\in\mathbb{R}$,

$$g_t(x)=\sum_{n\in\mathbb{Z}}g_t(n)\Psi(x-n).$$

We shall consider four cases:

(i) If $0<p\le 1$ and $1\le q\le\infty$, then by Minkowski's integral inequality in q we get

$$\begin{aligned}\int_{\mathbb{R}}\Big(\int_0^\infty|g_t(x)|^q\,\frac{dt}{t}\Big)^{p/q}dx&=\int_{\mathbb{R}}\Big(\int_0^\infty\Big|\sum_n g_t(n)\Psi(x-n)\Big|^q\frac{dt}{t}\Big)^{p/q}\\&\le\int_{\mathbb{R}}\Big(\sum_n|\Psi(x-n)|\Big(\int_0^\infty|g_t(n)|^q\,\frac{dt}{t}\Big)^{1/q}\Big)^p dx\\&\le\int_{\mathbb{R}}\sum_n|\Psi(x-n)|^p\Big(\int_0^\infty|g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}dx\\&=\|\Psi\|_p^p\sum_{n\in\mathbb{Z}}\Big(\int_0^\infty|g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}.\end{aligned}$$

(ii) If $1<p\le\infty$ and $1\le q\le\infty$, then by Minkowski's integral inequality in q and Hölder's inequality we get

$$\begin{aligned}&\Big(\int_0^\infty|g_t(x)|^q\,\frac{dt}{t}\Big)^{1/q}\\&\quad\le\sum_n|\Psi(x-n)|\Big(\int_0^\infty|g_t(n)|^q\,\frac{dt}{t}\Big)^{1/q}\\&\quad\le\Big[\sum_n\Big(\int_0^\infty|g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}|\Psi(x-n)|\Big]^{1/p}\Big(\sum_n|\Psi(x-n)|\Big)^{1/p'}\\&\quad\le C\Big[\sum_n\Big(\int_0^\infty|g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}|\Psi(x-n)|\Big]^{1/p}.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}}\Big(\int_0^\infty |g_t(x)|^q\,\frac{dt}{t}\Big)^{p/q} dx \le C\int_{\mathbb{R}}\sum_n \Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}|\Psi(x-n)|\,dx$$
$$= C\|\Psi\|_1\sum_n\Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}.$$

(iii) If $0<q<1$ and $p/q\le 1$, then

$$\int_0^\infty |g_t(x)|^q\,\frac{dt}{t} = \int_0^\infty \Big|\sum_n g_t(n)\Psi(x-n)\Big|^q\,\frac{dt}{t}$$
$$\le \int_0^\infty \sum_n |g_t(n)|^q|\Psi(x-n)|^q\,\frac{dt}{t}$$
$$= \sum_n |\Psi(x-n)|^q\Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big).$$

From this, we get

$$\int_{\mathbb{R}}\Big(\int_0^\infty |g_t(x)|^q\,\frac{dt}{t}\Big)^{p/q} dx \le \int_{\mathbb{R}}\Big(\sum_n |\Psi(x-n)|^q\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q} dx$$
$$\le \int_{\mathbb{R}}\sum_n |\Psi(x-n)|^p\Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q} dx$$
$$= \sum_n\int_{\mathbb{R}} |\Psi(x-n)|^p\,dx\Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}$$
$$= \|\Psi\|_p^p\sum_n\Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}.$$

(iv) Finally, if $p/q>1$, we can apply Hölder's inequality with exponents p/q, $(p/q)'=p/(p-q)$ to get

$$\int_0^\infty |g_t(x)|^q\,\frac{dt}{t}$$
$$\le \sum_n |\Psi(x-n)|^q\Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)$$
$$\le \Big(\sum_n |\Psi(x-n)|^q\Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}\Big)^{q/p}\Big(\sum_n|\Psi(x-n)|^q\Big)^{(p-q)/p}.$$

Hence,

$$\int_{\mathbb{R}}\Big(\int_0^\infty |g_t(x)|^q\,\frac{dt}{t}\Big)^{p/q} dx \le C\int_{\mathbb{R}}\sum_n |\Psi(x-n)|^q\Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q} dx$$
$$= C\sum_n\Big(\int_{\mathbb{R}}|\Psi(x-n)|^q\,dx\Big)\Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}$$
$$= C\|\Psi\|_q^q\sum_n\Big(\int_0^\infty |g_t(n)|^q\,\frac{dt}{t}\Big)^{p/q}. \ \blacksquare$$

As a consequence of Theorems 2.3 and 2.4 and the usual embeddings of sequence spaces, we get the following result, which is an extension of the case $p\ge 1$ for the Lebesgue spaces L^p.

COROLLARY 2.5. *Let $0<p\le 1$ and $f\in H^p(\mathbb{R})\cap E_R$. If $0<q\le p$, then $f\in H^q(\mathbb{R})$ and there exists a constant $C=C(p,q,R)>0$ such that*

$$\|f\|_{H^q(\mathbb{R})}\le C\|f\|_{H^p(\mathbb{R})}.$$

3. Discrete Hardy spaces. As mentioned in the introduction, $H^1(\mathbb{Z})$ is characterized as the space consisting of all sequences $a=\{a(n)\}_n$ belonging to $\ell^1(\mathbb{Z})$ for which if H^{d} is the discrete Hilbert transform defined by

$$(H^{\mathrm{d}}a)(m)=\sum_{n\ne m}\frac{a(n)}{m-n},$$

then $H^{\mathrm{d}}a\in\ell^1(\mathbb{Z})$ (see [CW]). Hence, it is very natural to give the following definition (see also [E]).

DEFINITION 3.1. Let $0<p\le 1$ and define

$$H^p_{\mathrm{Hilb}}(\mathbb{Z})=\{a\in\ell^p(\mathbb{Z}): H^{\mathrm{d}}a\in\ell^p(\mathbb{Z})\},$$

with the p-norm

$$\|a\|_{H^p_{\mathrm{Hilb}}(\mathbb{Z})}=\|a\|_p+\|H^{\mathrm{d}}a\|_p.$$

DEFINITION 3.2. Let $0<p\le 1$ and let

$$P_t^{\mathrm{d}}(n)=\frac{t}{t^2+n^2},\quad n\ne 0,\qquad P_t^{\mathrm{d}}(0)=0.$$

Then we define

$$H^p_{\max}(\mathbb{Z})=\{a\in\ell^p(\mathbb{Z}):\sup_{t>0}|P_t^{\mathrm{d}}\star a|\in\ell^p(\mathbb{Z})\}$$

with the p-norm

$$\|a\|_{H^p_{\max}(\mathbb{Z})}=\|a\|_p+\|\sup_{t>0}|P_t^{\mathrm{d}}\star a|\|_p.$$

Theorem 3.4 below shows that the above spaces are equivalent, but first we need the following lemma.

LEMMA 3.3. *Let $k_0 \in \mathbb{N}$ and $\Phi \in \mathcal{S}(\mathbb{R}) \cap E_R$ be an even function such that $\widehat{\Phi}(0) = 1$ and $\int_{\mathbb{R}} x^k \Phi(x)\,dx = 0$ for $1 \le k \le k_0 - 1$, $k \in \mathbb{Z}$. If $P_t^\Phi = P_t * \Phi$, then, for every $n \in \mathbb{Z} \setminus \{0\}$,*

$$|P_t^\Phi(n) - P_t(n)| = O(1/|n|^{k_0}),$$

uniformly in $t > 0$.

Proof. For every $t > 0$, P_t^Φ and $\widehat{P_t^\Phi}$ are in $L^1(\mathbb{R})$, and since P_t^Φ is continuous we can apply the inversion theorem for the Fourier transform at every point to get

$$P_t^\Phi(n) = \int_{-R}^{R} \widehat{\Phi}(\xi) e^{-2\pi t|\xi|} e^{-2\pi i n\xi}\,d\xi = 2\int_0^R \widehat{\Phi}(\xi) e^{-2\pi t\xi} \cos 2\pi n\xi\,d\xi$$
$$= \int_0^R \widehat{\Phi}(\xi)(e^{2\pi(in-t)\xi} + e^{-2\pi(in+t)\xi})\,d\xi.$$

Using integration by parts,

$$P_t^\Phi(n) = \left[\widehat{\Phi}(\xi)\left(\frac{e^{2\pi(in-t)\xi}}{2\pi(in-t)} - \frac{e^{-2\pi(in+t)\xi}}{2\pi(in+t)}\right)\right]_0^R$$
$$- \frac{1}{2\pi(in-t)}\int_0^R (\widehat{\Phi})'(\xi) e^{2\pi(in-t)\xi}\,d\xi$$
$$+ \frac{1}{2\pi(in+t)}\int_0^R (\widehat{\Phi})'(\xi) e^{-2\pi(in+t)\xi}\,d\xi$$
$$= \frac{1}{\pi}\cdot\frac{t}{t^2+n^2} + (\mathrm{I}) + (\mathrm{II}).$$

Using the hypothesis on Φ we obtain

$$(\mathrm{I}) = \frac{(-1)^{k_0}}{(2\pi(in-t))^{k_0}}\int_0^R \frac{d^{k_0}\widehat{\Phi}}{d\xi^{k_0}}(\xi) e^{2\pi(in-t)\xi}\,d\xi$$

and

$$(\mathrm{II}) = \frac{(-1)^{k_0+1}}{(2\pi(in+t))^{k_0}}\int_0^R \frac{d^{k_0}\widehat{\Phi}}{d\xi^{k_0}}(\xi) e^{-2\pi(in+t)\xi}\,d\xi.$$

Therefore, for every $n \ne 0$,

$$|P_t^\Phi(n) - P_t(n)| = |(\mathrm{I}) + (\mathrm{II})| \le \frac{C(\widehat{\Phi})}{(2\pi)^{k_0}}\left|\frac{1}{(in-t)^{k_0}} - \frac{1}{(in+t)^{k_0}}\right|$$
$$\le \frac{C}{(t^2+n^2)^{k_0/2}} \le \frac{C}{|n|^{k_0}},$$

with C independent of $t > 0$. ■

THEOREM 3.4. *Let $0 < p \le 1$. Then*

$$H^p_{\mathrm{Hilb}}(\mathbb{Z}) = H^p_{\max}(\mathbb{Z}),$$

with equivalent norms.

Proof. Let $0 < R < 1/2$ and let $\Phi \in E_R$ be as in the previous lemma. Choose $a \in H^p_{\max}(\mathbb{Z})$ and set

$$g(x) = \sum_{n\in\mathbb{Z}} a(n)\Phi(x-n).$$

Since $a \in \ell^p(\mathbb{Z}) \subset \ell^1(\mathbb{Z})$, it follows that $g \in L^1(\mathbb{R}) \cap E_R \subset L^2(\mathbb{R})$, and

$$\widehat{g}(\xi) = \sum_{n\in\mathbb{Z}} a(n) e^{-2\pi i n\xi}\widehat{\Phi}(\xi) \in L^1(\mathbb{R}).$$

Also, if $m \in \mathbb{Z}$, then

$$Hg(m) = (-\pi i\,\mathrm{sign}(\xi)\widehat{g}(\xi))^\vee(m) = -\pi i\int_0^R \widehat{g}(\xi) e^{2\pi i m\xi}\,d\xi \tag{2}$$
$$+ \pi i\int_{-R}^0 \widehat{g}(\xi) e^{2\pi i m\xi}\,d\xi$$
$$= -\pi i\int_0^R \Big(\sum_{n\in\mathbb{Z}} a(n)\widehat{\Phi}(\xi)e^{-2\pi i n\xi}\Big)e^{2\pi i m\xi}\,d\xi$$
$$+ \pi i\int_{-R}^0 \Big(\sum_{n\in\mathbb{Z}} a(n)\widehat{\Phi}(\xi)e^{-2\pi i n\xi}\Big)e^{2\pi i m\xi}\,d\xi$$
$$= 2\pi\sum_{n\ne m} a(n)\int_0^R \widehat{\Phi}(\xi)\sin(2\pi(m-n)\xi)\,d\xi.$$

Using integration by parts one can show that, for every $k_0 \ge 1$ and $m \ne n$,

$$\int_0^R \widehat{\Phi}(\xi)\sin(2\pi(m-n)\xi)\,d\xi = \frac{1}{2\pi(m-n)} + I_{k_0}, \tag{3}$$

where

$$I_{k_0}=\frac{1}{(2\pi)^{k_0}(m-n)^{k_0}}\int_0^R\frac{d^{k_0}\widehat{\Phi}}{d\xi^{k_0}}(\xi)\sin\left(2\pi(m-n)\xi+\frac{\pi}{2}k_0\right)d\xi.$$

To see this, we observe that

$$\int_0^R\widehat{\Phi}(\xi)\sin(2\pi(m-n)\xi)\,d\xi=\left[-\widehat{\Phi}(\xi)\frac{\cos(2\pi(m-n)\xi)}{2\pi(m-n)}\right]_0^R+\frac{1}{2\pi(m-n)}\int_0^R(\widehat{\Phi})'(\xi)\cos(2\pi(m-n)\xi)\,d\xi,$$

which is (3) for $k_0=1$. But, by the conditions assumed on Φ,

$$I_{k_0}=I_{k_0+1},$$

and hence, we get the result. Substituting (3) in (2), we deduce that there exists a positive constant $C=C(\widehat{\Phi},k_0)$ such that, for every $m\in\mathbb{Z}$, $m\neq n$,

$$\Big|Hg(m)-\sum_{n\neq m}\frac{a(n)}{m-n}\Big|\le C\sum_{n\neq m}\frac{|a(n)|}{|m-n|^{k_0}}.$$

Taking now k_0 so that $pk_0>1$, we obtain

$$\|(Hg)^{\mathrm d}-H^{\mathrm d}a\|_p^p\le C\bigg(\sum_m\Big|\sum_{n\neq m}a(n)\frac{1}{(m-n)^{k_0}}\Big|^p\bigg)\le C\|a\|_p^p. \tag{4}$$

Also,

$$(P_t*g)(x)=\sum_{n\in\mathbb{Z}}a(n)(P_t*\Phi)(x-n)=\sum_{n\in\mathbb{Z}}a(n)P_t^{\Phi}(x-n),$$

and by Theorem 2.4,

$$\|\sup_{t>0}|P_t*g|\|_p\le C\|(\sup_{t>0}|P_t*g|)^{\mathrm d}\|_p=C\|\sup_{t>0}|P_t^{\Phi}\star a|\|_p. \tag{5}$$

By (1), (4), (5) and the known corresponding equivalence in $\mathbb{R}$ (see [M]), we get

$$\begin{aligned}\|H^{\mathrm d}a\|_p&\le\|(Hg)^{\mathrm d}\|_p+C\|a\|_p\le C(\|Hg\|_p+\|a\|_p)\\&\le C(\|\sup_{t>0}|P_t*g|\|_p+\|a\|_p)\le C(\|(\sup_{t>0}|P_t*g|)^{\mathrm d}\|_p+\|a\|_p)\\&=C(\|\sup_{t>0}|P_t^{\Phi}\star a|\|_p+\|a\|_p)\le C(\|\sup_{t>0}|P_t^{\mathrm d}\star a|\|_p+\|a\|_p).\end{aligned}$$

For the other embedding, we have to show that

$$\|\sup_{t>0}|P_t^{\mathrm d}\star a|\|_p\le C(\|a\|_p+\|H^{\mathrm d}a\|_p).$$

But, by Lemma 3.3, (5) and the equivalence in $\mathbb{R}$,

$$\begin{aligned}\|\sup_{t>0}|P_t^{\mathrm d}\star a|\|_p&\le\|\sup_{t>0}|P_t^{\Phi}\star a|\|_p+C\|a\|_p\\&\le\|\sup_{t>0}|P_t*g|\|_p+C\|a\|_p\le C(\|g\|_p+\|Hg\|_p+\|a\|_p).\end{aligned}$$

Now, if $0<p\le1$, then

$$\begin{aligned}\|g\|_p&=\Big(\int_{\mathbb{R}}\Big|\sum_{n\in\mathbb{Z}}a(n)\Phi(x-n)\Big|^p\,dx\Big)^{1/p}\\&\le\Big(\sum_{n\in\mathbb{Z}}|a(n)|^p\int_{\mathbb{R}}|\Phi(x-n)|^p\,dx\Big)^{1/p}=\|\Phi\|_p\|a\|_p.\end{aligned}$$

On the other hand, since Hg is in E_R, $R<1/2$,

$$\|Hg\|_p\le C\|(Hg)^{\mathrm d}\|_p\le C(\|H^{\mathrm d}a\|_p+\|a\|_p),$$

and the result follows from (4) and (5). ∎

From now on, we simply write $H^p(\mathbb{Z})$ for both spaces. Similarly to Lemma 3.3 we have the following result.

LEMMA 3.5. *Let $\Phi\in\mathcal{S}(\mathbb{R})$ and let φ be in $\mathcal{S}\cap E_R$ such that $\widehat{\varphi}\equiv1$ in $(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$. Set $\Phi_t^{\varphi}=(\Phi_t*\varphi)$.*

(a) *If $n\in\mathbb{Z}\setminus\{0\}$, then for every $M\ge0$,*

$$|\Phi_t^{\varphi}(n)-\Phi_t(n)|=O(1/|n|^M),$$

uniformly in $t>0$.

(b) *If $\int_{\mathbb{R}}\Phi=0$, then there exists $\phi(t)\in L^2([0,\infty),dt/t)$ such that, for every $n\in\mathbb{Z}\setminus\{0\}$ and $M\ge2$,*

$$|\Phi_t^{\varphi}(n)-\Phi_t(n)|=\phi(t)O(1/|n|^M).$$

Proof. We restrict our attention to the proof of part (b). We can obtain (a) in a similar way.

As in Lemma 3.3 we find that, for every $n\in\mathbb{Z}\setminus\{0\}$ and every $M\in\mathbb{N}$,

$$\begin{aligned}\Phi_t^{\varphi}(n)-\Phi_t(n)&=\frac{1}{(2\pi n)^M}\int_{\varepsilon\le|\xi|\le R}\frac{d^M(\widehat{\Phi}(t\cdot)(\widehat{\varphi}(\cdot)-1))}{d\xi^M}(\xi)e^{\pi i(2n\xi+M/2)}\,d\xi\\&\quad-\frac{1}{(2\pi n)^M}\int_{|\xi|>R}\frac{d^M(\widehat{\Phi}(t\cdot))}{d\xi^M}(\xi)e^{\pi i(2n\xi+M/2)}\,d\xi=I_M.\end{aligned} \tag{6}$$

In order to estimate the above integrals we observe that if we apply Leibniz's formula to

$$\frac{d^M(\widehat{\Phi}(t\cdot)(\widehat{\varphi}(\cdot)-1))}{d\xi^M}(\xi),$$

we obtain as one of its factors the term

$$\widehat{\Phi}(t\xi)\frac{d^M(\widehat{\varphi}(\cdot)-1)}{d\xi^M}(\xi),$$

which can be bounded, for $\varepsilon \le |\xi| \le R$, by $At|\xi|$ if $0<t<1$, and by $C/(1+t|\xi|)^M$ if $t \ge 1$.

On the other hand, the term containing the first derivative of $\widehat{\Phi}(t\cdot)$ can be bounded as follows:

$$\left| t\frac{d\widehat{\Phi}}{d\xi}(t\xi)\frac{d^{M-1}(\widehat{\varphi}(\cdot)-1)}{d\xi^{M-1}}(\xi)\right| \le \begin{cases} Ct & \text{if } 0<t<1,\\ \dfrac{Ct}{(1+t|\xi|)^M} & \text{if } t\ge 1.\end{cases}$$

Finally, if $M \ge 2$, the remaining terms in (6) contain derivatives of order $k \ge 2$ and they can be bounded as

$$\left|\frac{d^k\widehat{\Phi}(t\cdot)}{d\xi^k}\right|(\xi) \le \frac{Ct^k}{(1+t|\xi|)^N},$$

with N large enough. From all these estimates the result follows easily. ■

From this lemma and using the same techniques as in Theorem 3.4, we can show other characterizations of the space $H^p(\mathbb{Z})$. Namely, $\{P_t^{\mathrm{d}}(n)\}_n$ can be substituted by $\{\Phi_t^{\mathrm{d}}(n)\}_{n\in\mathbb{Z}}$, where $\Phi_t^{\mathrm{d}}(n)=t^{-1}\Phi(n/t)$ if $n\ne 0$, $\Phi_t^{\mathrm{d}}(0)=0$ with Φ a function in the Schwartz class so that $\int\Phi=1$ (as is done in the real case [FS]).

THEOREM 3.6. *Let $0<p\le 1$ and let $\Phi\in\mathcal{S}$ be such that $\int_{\mathbb{R}}\Phi=1$. Then*

$$\|a\|_p+\|\sup_{t>0}|\Phi_t^{\mathrm{d}}\star a|\|_p \sim \|a\|_{H^p(\mathbb{Z})}$$

for every $a\in H^p(\mathbb{Z})$.

We may also have another definition in terms of area functions (see also [Su]):

DEFINITION 3.7. Let $0<p\le 1$ and define

$$H_{\mathrm{A}}^p(\mathbb{Z})=\{a\in\ell^p(\mathbb{Z}):\{\|(\Psi_t^{\mathrm{d}}\star a)(n)\|_{L^2(dt/t)}\}_{n\in\mathbb{Z}}\in\ell^p(\mathbb{Z})\},$$

with the p-norm

$$\|a\|_{H_{\mathrm{A}}^p(\mathbb{Z})}=\|a\|_p+\|\,\|\Psi_t^{\mathrm{d}}\star a\|_{L^2(dt/t)}\|_p,$$

where Ψ_t^{d} denotes the restriction to $\mathbb{Z}\setminus\{0\}$ of $\Psi_t(\cdot)=t^{-1}\Psi(\cdot/t)$, with $\Psi\in\mathcal{S}(\mathbb{R})$ such that $\int_{\mathbb{R}}\Psi=0$, $\Psi_t^{\mathrm{d}}(0)=0$.

Using part (b) of Lemma 3.5 and Theorems 2.3 and 2.4 for $q=2$, we can obtain the following equivalence for $H^p(\mathbb{Z})$ in terms of square functions.

THEOREM 3.8. *Let $0<p\le 1$. Then $H_{\mathrm{A}}^p(\mathbb{Z})=H^p(\mathbb{Z})$ with equivalent norms.*

Next, we want to show the connection with the atomic version of the $H^p(\mathbb{Z})$ space introduced in [CW].

DEFINITION 3.9. Let $0<p\le 1$. We say that $a=\{a(n)\}_{n\in\mathbb{Z}}$ is an *H^p-atom* in $\mathbb{Z}$ if the following conditions hold:

(i) $\operatorname{supp} a$ is contained in a ball in $\mathbb{Z}$ of cardinality $2n+1$, $n\ge 1$.
(ii) $\|a\|_\infty \le (2n+1)^{-1/p}$.
(iii) $\sum_n n^\alpha a(n)=0$ for every $\alpha\in\mathbb{N}$ such that $\alpha\le p^{-1}-1$.

The atomic $H_{\mathrm{at}}^p(\mathbb{Z})$ space is defined as the space of all sequences $a=\{a(n)\}_{n\in\mathbb{Z}}$ such that

$$a=\sum_{j=0}^{\infty}\lambda_j a_j,$$

where a_j are H^p-atoms and

$$\|a\|_{H_{\mathrm{at}}^p(\mathbb{Z})}=\inf\Big\{\Big(\sum_j|\lambda_j|^p\Big)^{1/p}\Big\},$$

where the infimum is taking over all possible representations of a.

The standard proof in the setting of homogeneous type spaces shows the following:

THEOREM 3.10. *Let $0<p\le 1$. Then $H_{\mathrm{at}}^p(\mathbb{Z})$ is continuously embedded in $H^p(\mathbb{Z})$.*

We can also prove the converse. For this purpose, we need first the following proposition (see also [Su]).

PROPOSITION 3.11. *Let $0<p\le 1$ and $a\in H^p(\mathbb{Z})$. If $\phi\in L^2(\mathbb{R})$ with $\operatorname{supp}\phi\subset\{|x|\le N\}$, then*

$$f(x)=\sum_{n\in\mathbb{Z}}a(n)\phi(x-n)\in H^p(\mathbb{R}),$$

and there exists a constant $C=C(p)$ such that

$$\|f\|_{H^p(\mathbb{R})}\le C\|a\|_{H^p(\mathbb{Z})}.$$

Proof. Since $a\in H^p(\mathbb{Z})\subset\ell^1(\mathbb{Z})$ and $\phi\in L^2(\mathbb{R})$, it follows that $f\in L^2(\mathbb{R})$, and therefore it is enough to estimate $\|f\|_p+\|Hf\|_p$.

First we observe that

$$\|f\|_{L^p(\mathbb{R})}^p = \int_{\mathbb{R}} \Big|\sum_{n\in\mathbb{Z}} a(n)\phi(x-n)\Big|^p\,dx \le \int_{\mathbb{R}}\sum_{n\in\mathbb{Z}} |a(n)|^p|\phi(x-n)|^p\,dx \le \sum_{n\in\mathbb{Z}} |a(n)|^p\|\phi\|_p^p.$$

On the other hand,

$$\begin{aligned}\|Hf\|_p^p &= \int_{-\infty}^{\infty}\Big|H\Big(\sum_{n\in\mathbb{Z}} a(n)\phi(x-n)\Big)\Big|^p\,dx\\
&= \sum_{m\in\mathbb{Z}}\int_m^{m+1}\Big|\int_{\mathbb{R}}\frac{\sum_{n\in\mathbb{Z}} a(n)\phi(y-n)}{x-y}\,dy\Big|^p\,dx\\
&\le \sum_{m\in\mathbb{Z}}\int_m^{m+1}\Big|\int_{\mathbb{R}}\frac{1}{x-y}\Big(\sum_{|n-m|\le 2N}+\sum_{|n-m|>2N}\Big)a(n)\phi(y-n)\,dy\Big|^p\,dx\\
&\le \sum_{m\in\mathbb{Z}}\int_m^{m+1}\Big|\sum_{|n-m|\le 2N} a(n)\int_{\mathbb{R}}\frac{\phi(y-n)}{x-y}\,dy\Big|^p\,dx\\
&\quad+\sum_{m\in\mathbb{Z}}\int_m^{m+1}\Big|\int_{\mathbb{R}}\sum_{|n-m|>2N} a(n)\frac{\phi(y-n)}{x-y}\Big(1-\frac{x-y}{m-n}\Big)^{N_0}dy\Big|^p\,dx\\
&\quad+\sum_{m\in\mathbb{Z}}\int_m^{m+1}\Big|\int_{\mathbb{R}}\sum_{|n-m|>2N} a(n)\phi(y-n)\\
&\qquad\times\Big(\sum_{k=1}^{N_0}\frac{(m-n-x+y)^{k-1}}{(m-n)^k}\Big)\,dy\Big|^p\,dx\\
&= \text{(I)}+\text{(II)}+\text{(III)},\end{aligned}$$

where N_0 is the integer part of $1/p$, and we have used the fact that

$$\frac{1}{x-y}-\frac{1}{x-y}\Big(1-\Big(\frac{x-y}{m-n}\Big)\Big)^{N_0}=\sum_{k=1}^{N_0}\frac{(m-n-x+y)^{k-1}}{(m-n)^k}.$$

Since $\phi\in L^2(\mathbb{R})$, $H\phi$ is locally integrable, and we can estimate (I) by

$$\begin{aligned}\text{(I)} &= \sum_{m\in\mathbb{Z}}\int_m^{m+1}\Big|\sum_{|n-m|\le 2N} a(n)H\phi(x-n)\Big|^p\,dx\\
&\le \sum_{m\in\mathbb{Z}}\sum_{|m-n|\le 2N}|a(n)|^p\int_{m-n}^{m-n+1}|H\phi(x)|^p\,dx\end{aligned}$$

$$= C\sum_{n\in\mathbb{Z}}|a(n)|^p\int_{-2N}^{2N+1}|H\phi(x)|^p\,dx = C\|a\|_p^p.$$

To estimate (II), we observe that we can assume $N\ge 2$ and then for $|y-n|\le N$, $m\le x\le m+1$ and $|m-n|>2N$, we get

$$\begin{aligned}|m-n| &\le |y-n|+|x-y|+|x-m|\le N+1+|x-y|\\
&\le \frac{3}{2}N+|x-y|\le\frac{3}{4}|m-n|+|x-y|.\end{aligned}$$

Thus, $|x-y|\ge\frac14|m-n|$ and therefore,

$$\Big|\frac{1}{x-y}\Big(1-\frac{x-y}{m-n}\Big)^{N_0}\Big| = \frac{|m-n-x+y|^{N_0}}{|x-y|\cdot|m-n|^{N_0}}\le\frac{C}{|m-n|^{N_0+1}}.$$

Since $(N_0+1)p>1$, we can deduce

$$\begin{aligned}\text{(II)} &= \sum_{m\in\mathbb{Z}}\Big(\int_m^{m+1}\Big|\int_{\mathbb{R}}\sum_{|n-m|>2N} a(n)\frac{\phi(y-n)}{x-y}\Big(1-\frac{x-y}{m-n}\Big)^{N_0}dy\Big|^p\,dx\Big)\\
&\le C\sum_{n\in\mathbb{Z}}\Big(\sum_{|n-m|>2N}\frac{|a(n)|}{|m-n|^{N_0+1}}\int_{\mathbb{R}}|\phi(y)|\,dy\Big)^p\le C\|a\|_{H^p(\mathbb{Z})}^p.\end{aligned}$$

Similarly,

$$\begin{aligned}\text{(III)} &= \sum_{m\in\mathbb{Z}}\int_m^{m+1}\Big|\int_{\mathbb{R}}\sum_{|n-m|>2N} a(n)\phi(y-n)\\
&\qquad\times\Big(\sum_{k=1}^{N_0}\frac{(m-n-x+y)^{k-1}}{(m-n)^k}\Big)\,dy\Big|^p\,dx\\
&\le \sum_{k=1}^{N_0}\sum_{m\in\mathbb{Z}}\int_m^{m+1}\Big|\sum_{|n-m|>2N}\frac{a(n)}{(m-n)^k}\int_{\mathbb{R}}\phi(y)(y+m-x)^{k-1}\,dy\Big|^p\,dx\\
&= \sum_{k=1}^{N_0}\Big(\int_0^1\Big|\int_{\mathbb{R}}\phi(y)(y-x)^{k-1}\,dy\Big|^p\,dx\Big)\sum_{m\in\mathbb{Z}}\Big|\sum_{|n-m|>2N}\frac{a(n)}{(m-n)^k}\Big|^p\\
&\le C\|a\|_{H^p(\mathbb{Z})}^p,\end{aligned}$$

where the last inequality follows because, for $k\in\mathbb{N}$, the sequences $\{1/n^k\}_{n\in\mathbb{Z}\setminus\{0\}}$ are discrete convolution kernels from $H^p(\mathbb{Z})$ to $l^p(\mathbb{Z})$ and hence,

$$\sum_{m\in\mathbb{Z}}\Big|\sum_{|n-m|>2N}\frac{a(n)}{(m-n)^k}\Big|^p$$
$$\le \sum_{m\in\mathbb{Z}}\Big|\sum_{n\neq m}\frac{a(n)}{(m-n)^k}\Big|^p+\sum_{m\in\mathbb{Z}}\Big|\sum_{|n-m|\le 2N,\, m\neq n}\frac{a(n)}{(m-n)^k}\Big|^p$$
$$\le C\|a\|^p_{H^p(\mathbb{Z})}+C\|a\|^p_p.$$

Therefore,

$$\|Hf\|_p\le C\|a\|_{H^p(\mathbb{Z})},$$

and

$$\|f\|_{H^p(\mathbb{R})}\le C(\|f\|_p+\|Hf\|_p)\le C\|a\|_{H^p(\mathbb{Z})}. \ ■$$

In order to prove the continuous embedding of $H^p(\mathbb{Z})$ into the atomic space $H^p_{\mathrm{at}}(\mathbb{Z})$, we need the following auxiliary functions and lemmas.

Let $B_1(x)=\chi_{(-1/2,1/2]}(x)$ and, for any positive integer $k\ge 2$, consider

$$B_k(x)=(B_1 * \overset{(k-1)}{\cdots} * B_1)(x)$$

For these functions, we shall use the following two properties which can be easily proved by induction:

(i) For k even, B_k is a polynomial of degree $k-1$ over any interval of the form $[m,m+1]$ with m an integer, and, for k odd, B_k is a polynomial of degree $k-1$ over any interval of the form $[m-1/2,m+1/2]$.

(ii) For $0\le j\le k-1$,

$$\sum_{m\in\mathbb{Z}} m^j B_k(y-m)=P_{j,k}(y), \tag{7}$$

where $P_{j,k}$ denotes a polynomial of degree j.

LEMMA 3.12. *Let $k\in\mathbb{Z}$ and let $a\in H^p(\mathbb{Z})$ with $0<p\le 1/k$. Consider, for k even,*

$$c_k(m)=\sum_{j=1+[(-2k-1)/4]}^{[(2k-3)/4]} a(m-j)\int_{j+1/4}^{j+3/4} B_k(y)\,dy,\qquad m\in\mathbb{Z},$$

and, for k odd,

$$c_k(m)=\sum_{j=1+[(-2k+1)/4]}^{[(2k-1)/4]} a(m-j)\int_{j-1/4}^{j+1/4} B_k(y)\,dy,\qquad m\in\mathbb{Z}.$$

Then

$$c_k(m)=\sum_{i=0}^{\infty}\lambda_i a_{i,k}(m)$$

where

$$\sum_{i=0}^{\infty}|\lambda_i|^p\le C\|a\|_{H^p(\mathbb{Z})}$$

and $a_{i,k}$ satisfies the following conditions:

(a) *There exists a ball B_i in $\mathbb{Z}$ so that $\operatorname{supp} a_{i,k}\subseteq B_i$.*
(b) $\|a_{i,k}\|_\infty\le 1/(\#B_i)^{1/p}$.
(c) $\sum_{m\in\mathbb{Z}} m^j a_{i,k}(m)=0$, $0\le j\le k-1$.

REMARK. Observe that if $k=[1/p]$, the above sequences $a_{i,k}$ are H^p-atoms in $\mathbb{Z}$.

Proof (of Lemma 3.12). First assume that $k=[1/p]$ is odd. Let $a\in H^p(\mathbb{Z})$ and set

$$f(x)=\sum_{n\in\mathbb{Z}} a(n)\chi_{[n-1/4,n+1/4)}(x).$$

By Proposition 3.11, $f\in H^p(\mathbb{R})$ and

$$\|f\|_{H^p(\mathbb{R})}\le C\|a\|_{H^p(\mathbb{Z})}.$$

Since f is also in $L^2(\mathbb{R})$, it can be decomposed in terms of H^p-atoms $\{b_i\}_{i=0}^{\infty}$; that is,

$$f(x)=\sum_{i=0}^{\infty}\lambda_i b_i(x)\qquad\text{for a.e. } x\in\mathbb{R},$$

where $\sum_i|\lambda_i|^p\le C\|a\|^p_{H^p(\mathbb{Z})}$.

Let I_i be the support of the atom b_i and consider the sets

$$J_1=\{i\in\mathbb{N}:|I_i|>1/8\}\quad\text{and}\quad J_2=\{i\in\mathbb{N}:|I_i|\le 1/8\}.$$

If $i\in J_1$, then

$$\|b_i\|_\infty\le 1/|I_i|^{1/p}\le 8^{1/p},$$

and hence the series $\sum_{i\in J_1}\lambda_i b_i(x)$ converges for a.e. $x\in\mathbb{R}$ and in the distribution sense to a function in $L^2(\mathbb{R})$.

Thus, for each $m\in\mathbb{Z}$, we get

$$\begin{aligned}(f*B_k)(m)&=\Big[\Big(\sum_{i=0}^{\infty}\lambda_i b_i(\cdot)\Big)*B_k\Big](m)\\&=\int_{-k/2}^{k/2}\Big(\sum_{i=0}^{\infty}\lambda_i b_i(m-y)\Big)B_k(y)\,dy\\&=\int_{-k/2}^{k/2}\Big(\sum_{i\in J_1}\lambda_i b_i(m-y)\Big)B_k(y)\,dy\\&\quad+\int_{-k/2}^{k/2}\Big(\sum_{i\in J_2}\lambda_i b_i(m-y)\Big)B_k(y)\,dy.\end{aligned}\tag{8}$$

For the first term, using the dominated convergence theorem, we have

$$\int_{-k/2}^{k/2}\Big(\sum_{i\in J_1}\lambda_i b_i(m-y)\Big)B_k(y)\,dy=\sum_{i\in J_1}\lambda_i(b_i*B_k)(m).$$

Let us now see that the second term in (8) vanishes. If we analyze how the atomic decomposition is obtained for our function f (see [C]), we see that we can assume that $\operatorname{supp} b_i\cap\operatorname{supp} f\neq\emptyset$. Therefore, if $i\in J_2$ and $l\in\mathbb{Z}$, then either $\operatorname{supp} b_i\subset[l-3/8,l+3/8]$ or $[l-1/2,l+1/2]\cap\operatorname{supp} b_i=\emptyset$ and thus

$$\sum_{i\in J_2}\lambda_i b_i(y)=0\quad\text{ for a.e. } y\in(l-1/2,l-3/8)\cup(l+3/8,l+1/2). \tag{9}$$

Given now $j\in\mathbb{Z}$ such that $|j|\le(k-1)/2$, let $\varphi_j\in\mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\varphi_j\subset[m+j-1/2,m+j+1/2]$ and $\varphi_j\equiv1$ on $[m+j-3/8,m+j+3/8]$. By (9), we get

$$\int_{-k/2}^{k/2}\Big(\sum_{i\in J_2}\lambda_i b_i(m-y)\Big)B_k(y)\,dy$$
$$=\sum_{j=-(k-1)/2}^{(k-1)/2}\Big(\int_{\mathbb{R}}\Big(\sum_{i\in J_2}\lambda_i b_i(y)\Big)\varphi_j(y)B_k(m-y)\,dy\Big).$$

Using property (i) of B_k, we see that $\varphi_j(\cdot)B_k(m-\cdot)\in\mathcal{S}$ and hence the above expression equals

$$\sum_{j=-(k-1)/2}^{(k-1)/2}\sum_{i\in J_2}\lambda_i\Big(\int_{\mathbb{R}}b_i(y)\varphi_j(y)B_k(m-y)\,dy\Big).$$

Since, by the cancellation property of the atom b_i, $\int_{\mathbb{R}}b_i(y)B_k(m-y)\,dy=0$, we can easily deduce that, for every i and j,

$$\int_{\mathbb{R}}b_i(y)\varphi_j(y)B_k(m-y)\,dy=0,$$

and therefore

$$\int_{-k/2}^{k/2}\Big(\sum_{i\in J_2}\lambda_i b_i(m-y)\Big)B_k(y)\,dy=0.$$

Consequently, if we write $a_{i,k}(m)=(b_i*B_k)(m)$, we have proved that

$$(f*B_k)(m)=\sum_{i\in J_1}\lambda_i a_{i,k}(m).$$

Let us now prove that $a_{i,k}$ satisfies (a), (b) and (c):

- $\operatorname{supp} a_{i,k}\subseteq\operatorname{supp}(b_i*B_k)\cap\mathbb{Z}\subset(I_i+[-k/2,k/2])\cap\mathbb{Z}\subseteq B_{i,k}$, where $B_{i,k}$ is a ball in $\mathbb{Z}$.
- $\|a_{i,k}\|_\infty\le\|b_i\|_\infty\int_{\mathbb{R}}|B_k(x)|\,dx\le C(k)/|I_i|^{1/p}\le C(k,p)/(\#B_{i,k})^{1/p}$, where the last inequality follows since $|I_i|>1/8$.
- For $0\le j\le k-1$, we use (7) and the cancellation properties for the atom b_i to obtain

$$\sum_{m\in\mathbb{Z}}m^j a_{i,k}(m)=\sum_{m\in\mathbb{Z}}m^j(b_i*B_k)(m)=\int_{\mathbb{R}}b_i(y)P_{j,k}(y)\,dy=0.$$

On the other hand, since

$$(f*B_k)(m)=\sum_{j=1+[(-2k+1)/4]}^{[(2k-1)/4]}a(m-j)\int_{j-1/4}^{j+1/4}B_k(y)\,dy=c_k(m),$$

we obtain the result.

Finally, if $k=[1/p]$ is even, we replace the function f above by

$$f(x)=\sum_{n\in\mathbb{Z}}a(n)\chi_{[1/4,3/4]}(x-n),\qquad a\in H^p(\mathbb{Z}),$$

and we argue as before. ∎

THEOREM 3.13. *Let $k\in\mathbb{Z}$, $k\ge1$ and let $a\in H^p(\mathbb{Z})$ with $0<p\le1/k$. Then*

$$a(n)=\sum_{i=0}^{\infty}\lambda_i a_{i,k}(n),$$

where $\sum_{i=0}^\infty|\lambda_i|^p\le C\|a\|^p_{H^p(\mathbb{Z})}$ and $a_{i,k}$ satisfies (a), (b) *and* (c) *of the previous lemma.*

Proof. We proceed by induction on k. If $k=1$, the result follows by the previous lemma. Assume that the result is true for $k-1$, and let us prove it for k.

By hypothesis, if $a\in H^p(\mathbb{Z})$ with $0<p\le1/k<1/(k-1)$, then

$$a(m)=\sum_{i=0}^{\infty}\mu_i a_{i,k-1}(m),$$

where $\sum_{i=0}^\infty|\mu_i|^p\le C\|a\|^p_{H^p(\mathbb{Z})}$ and $a_{i,k-1}$ satisfies:

- There exists a ball $B_{i,k-1}\subset\mathbb{Z}$ so that $\operatorname{supp} a_{i,k-1}\subseteq B_{i,k-1}$.
- $\|a_{i,k-1}\|_\infty\le1/(\#B_{i,k-1})^{1/p}$.
- $\sum_{m\in\mathbb{Z}}m^j a_{i,k-1}(m)=0$, $0\le j\le k-2$.

Therefore,

$$a(m)-a(m-1)=\sum_{i=0}^{\infty}\mu_i(a_{i,k-1}(m)-a_{i,k-1}(m-1))=\sum_{i=0}^{\infty}\mu_i c_{i,k-1}(m),$$

and one can easily prove that the sequences $c_{i,k-1}$ satisfy the required properties.

The proof ends by combining this result with the decomposition obtained in the previous lemma for a finite linear combination of translates of the sequence a with strictly positive coefficients. ∎

THEOREM 3.14. *Let* $0 < p \le 1$. *Then* $H^p(\mathbb{Z})$ *is continuously embedded in* $H^p_{\mathrm{at}}(\mathbb{Z})$.

Proof. This follows immediately from the previous theorem for $k = [1/p]$. ∎

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Departament de Matemàtica
Aplicada i Telemàtica
Universitat Politècnica de Catalunya
08034 Barcelona, Spain
E-mail: boza@mat.upc.es

Departament de Matemàtica
Aplicada i Anàlisi
Universitat de Barcelona
08071 Barcelona, Spain
E-mail: carro@cerber.mat.ub.es

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A constructive proof of the Beurling–Rudin theorem

by

RAYMOND MORTINI (Metz)

Abstract. A constructive proof of the Beurling–Rudin theorem on the characterization of the closed ideals in the disk algebra $A(\mathbb{D})$ is given.

Introduction. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, $\overline{\mathbb{D}}$ its closure and let $A(\mathbb{D})$ be the algebra of all functions continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$. Endowed with the supremum norm, $A(\mathbb{D})$ becomes a commutative, complex Banach algebra with unit element, the so-called disk algebra.

In 1957 Rudin [Ru] gave a complete characterization of the closed ideals in $A(\mathbb{D})$. Later, a similar but somewhat simpler and more functional analytic proof was given by Srinivasan and Wang [SrWa]. The proofs were based on Beurling's invariant subspace theorem for the shift operator on the Hilbert space H^2 of all square summable power series in $\mathbb{D}$, the Riesz theorem on the structure of analytic measures on the unit circle $\mathbb{T}$, the Hahn–Banach theorem and the Riesz representation theorem for bounded linear functionals on $C(\overline{\mathbb{D}})$.

In this paper we present an elementary and constructive proof of this theorem. For background material, the reader is referred to the books of J. Garnett [Ga] and K. Hoffman [Ho].

1. A Frostman type theorem for the sum of two inner functions. Let u be an inner function. By Frostman's well known result the inner function $(a-u)/(1-\overline{a}u)$ is a Blaschke product for all $a \in \mathbb{D}$ outside a possibly empty set E of logarithmic capacity zero, denoted by $\operatorname{cap} E = 0$ (see [Ga, p. 79]). Walter Rudin [Rud] extended this result by showing that for every analytic function f of bounded characteristic in $\mathbb{D}$ the inner factor of $f - a$ is a Blaschke product for all $a \in \mathbb{D} \setminus E$, where $\operatorname{cap} E = 0$. Here we have the following result of Donald Sarason (unpublished):

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