Contents of Volume 129, Number 1

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (1998): Vols. 127–131 (15 issues); $32 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by

drukarnia
herman & herman
02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

PRINTED IN POLAND

ISSN 0039-3223

STUDIA MATHEMATICA 129 (1) (1998)

The size of characters of compact Lie groups

by

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Abstract. Pointwise upper bounds for characters of compact, connected, simple Lie groups are obtained which enable one to prove that if μ is any central, continuous measure and n exceeds half the dimension of the Lie group, then $\mu^n \in L^1$. When μ is a continuous, orbital measure then μ^n is seen to belong to L^2. Lower bounds on the p-norms of characters are also obtained, and are used to show that, as in the abelian case, m-fold products of Sidon sets are not p-Sidon if $p < 2m/(m+1)$.

1. Introduction. The purpose of this paper is to obtain estimates on the size of characters of representations of compact, connected, simple Lie groups in order to study the asymptotic behaviour of the Fourier transform of central measures and to investigate Sidonicity problems.

In [13] Ragozin proved the striking fact that if G was such a group, and μ was a continuous, central measure on G, then $\mu^{\dim G} \in L^1(G)$. (The product here is convolution.) Consequently, $\widehat{\mu}(\lambda) \to 0$ as the degree of the representation λ tends to infinity, and also (see [15]) $\frac{\operatorname{Tr}\lambda(x)}{\deg\lambda} \to 0$ as $\deg\lambda \to \infty$ when x is not in the centre of G. In the first part of the paper we prove that if x does not belong to the centre of G and $r = \operatorname{rank} G$, then

$$\frac{|\operatorname{Tr}\lambda(x)|}{\deg\lambda} \le c(x)(\deg\lambda)^{-2/(\dim G - r)}.$$

From this we are able to show that if μ is any continuous, central measure on G then $\mu^n \in L^1(G)$ for all $n > \dim G/2$. We do this by obtaining estimates on the rate of decay of the Fourier transform which are sharp enough to prove that if $n > \dim G/2$ and μ belongs to a certain class of central measures (a class which includes all continuous, orbital measures) then $\mu^n \in L^2(G)$.

In the second part of the paper we obtain lower bounds on the p-norms of characters. Earlier results of this nature were found in [3] and [4]. The main difference between the earlier results and ours was that the earlier estimates involved an unknown constant which depended on the group G and/or p,

1991 *Mathematics Subject Classification*: Primary 43A80; Secondary 22E46, 43A65.
This research is partially supported by NSERC.

and this was not adequate for answering Sidonicity questions. We are able to show that if G is the product of compact, simply connected, simple Lie groups whose ranks tend to infinity, then the m-fold product of the FTR set of G (an m-fold product of Sidon sets) is not p-Sidon if $p < 2m/(m+1)$. It remains unknown if this set is $2m/(m+1)$-Sidon.

NOTATION. We will be using the following notation throughout the paper: Given a compact, connected, simple Lie group G of rank r we will let $Z(G)$ denote its centre, W its Weyl group, T a maximal torus of G, and t its Lie algebra. We denote by Φ the set of roots for (G, T), and by Φ^+ the positive roots relative to a fixed base $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. We let $\lambda_1, \ldots, \lambda_r$ be the fundamental dominant weights relative to Δ, and Λ^+ the set of all dominant weights. This set is in 1-1 correspondence with $\widehat{G}$; $\sigma_\lambda \in \widehat{G}$ is indexed by its highest weight $\lambda \in \Lambda^+$. Its degree will be denoted by d_λ. The weights of $\lambda \in \Lambda^+$ are given by

$$\Pi(\lambda) = \{\mu \in \Lambda : w(\mu) < \lambda \text{ for all } w \in W\}$$

where $\mu < \lambda$ means $\lambda - \mu$ is a non-negative integral sum of positive roots. We set $\varrho = \sum_{j=1}^r \lambda_j$. By the Fourier transform of a central measure μ on G we mean

$$\widehat{\mu}(\lambda) = \int_G \frac{\operatorname{Tr}\lambda(x)}{d_\lambda}\, d\mu$$

(rather than the appropriate scalar multiple of the $d_\lambda \times d_\lambda$ identity matrix).

2. Upper bounds for the trace function

2.1. *Root systems.* First we need a result about root systems which may be of independent interest. We will write ∂_α for the directional derivative in the direction of α in Φ; we think of α as both in the Euclidean space and its dual. We denote by q the function $q(x) = \prod_{\alpha\in\Phi^+} \alpha(x)$.

THEOREM 1. *If Φ is any root system then*

$$\prod_{\alpha\in\Phi^+} \partial_\alpha(q) \geq \prod_{\alpha\in\Phi^+} (\alpha, \alpha).$$

To prove this we need a lemma.

LEMMA 2. *If Φ is an irreducible root system with corresponding Weyl group W, then*

$$\prod_{\alpha\in\Phi^+} \partial_\alpha(q) = |W| \prod_{\alpha\in\Phi^+} (\varrho, \alpha).$$

Proof. It is shown in the proof of [16], (4.14.5), that if λ is any linear functional on the Euclidean space, then

$$\sum_{w\in W} \det w\,(w(\lambda))^m = \begin{cases} 0 & \text{if } m < |\Phi^+|, \\ m!k(\lambda)q & \text{if } m = |\Phi^+|, \end{cases}$$

where

$$k(\lambda) = |W| \frac{\prod_{\alpha\in\Phi^+}(\lambda, \alpha)}{\prod_{\alpha\in\Phi^+} \partial_\alpha(q)}.$$

Suppose now that λ belongs to the dual of a compact, connected, simple Lie group with root system Φ. From the Weyl character formula and the observation above it follows that there is a sequence of homogeneous polynomials g_m of degree m satisfying

$$\operatorname{Tr}\lambda(x) = \frac{\sum_{w\in W} \det w\,(w(\lambda+\varrho)(x))^{|\Phi^+|}/|\Phi^+|! + q(x)\sum_{m=1}^\infty g_m}{e^{-i\varrho(x)}\prod_{\alpha\in\Phi^+}(e^{i\alpha(x)} - 1)}.$$

Since $(e^{i\alpha(x)} - 1)/\alpha(x) \to 1$ as $x \to 0$, by taking limits we see that we have

$$d_\lambda = \operatorname{Tr}\lambda(0) = \lim_{x\to 0} \frac{\sum_{w\in W} \det w\,(w(\lambda+\varrho)(x))^{|\Phi^+|}}{|\Phi^+|!e^{-i\varrho(x)}q(x)} = k(\lambda+\varrho).$$

Thus

$$\prod_{\alpha\in\Phi^+} \partial_\alpha(q) = \frac{|W|\prod_{\alpha\in\Phi^+}(\lambda+\varrho, \alpha)}{d_\lambda} = |W| \prod_{\alpha\in\Phi^+} (\varrho, \alpha)$$

where the last equality is from the Weyl dimension formula. ∎

Proof of Theorem 1. If Φ is not irreducible then Φ decomposes as $\bigcup_{j=1}^k \Phi_j$, where the Φ_j are irreducible pairwise orthogonal root systems with positive roots Φ_j^+ satisfying $\bigcup_{j=1}^k \Phi_j^+ = \Phi^+$. Hence if $q_j = \prod_{\alpha\in\Phi_j^+} \alpha$, then

$$\prod_{\alpha\in\Phi^+} \partial_\alpha(q) = \prod_{j=1}^k \prod_{\alpha\in\Phi_j^+} \partial_\alpha(q_j),$$

and so an application of the lemma shows that it suffices to prove

$$|W| \prod_{\alpha\in\Phi^+} (\varrho, \alpha) \geq \prod_{\alpha\in\Phi^+} (\alpha, \alpha)$$

whenever Φ^+ is the set of positive roots of an irreducible root system with Weyl group W.

This can be proven by considering each of the classical and exceptional simple Lie groups separately. We will give the proof for type B_n, $n \geq 2$, (the proof is typical) and leave the others for the reader.

We first note that it is equivalent to prove

$$|W| \prod_{\alpha\in\Phi^+} \langle \varrho, \alpha\rangle \geq 2^{|\Phi^+|},$$

and that is what we will actually demonstrate.

For type B_n the Weyl group has cardinality $2^n n!$ and there are n^2 positive roots. These include

$$\Big\{\sum_{k=i}^{j}\alpha_k : 1\le i\le j\le n\Big\},$$

as well as

$$\Big\{\alpha_1+\ldots+\alpha_{j-1}+2\sum_{k=j}^{n}\alpha_k : j=2,3,\ldots,n\Big\}.$$

Thus

$$|W|\prod_{\alpha\in\Phi^+}\langle\varrho,\alpha\rangle\ge 2^n n! 1^n 2^{n-1}\ldots n\prod_{j=2}^{n}(2n-j+1)\equiv F(n).$$

Now $F(2)=48\ge 2^{2^2}$, so assume inductively that $F(n)\ge 2^{n^2}$. Clearly,

$$F(n+1)=\frac{F(n)2(n+1)(n+1)!(2n+1)2n}{n+1}.$$

Applying the induction assumption we obtain

$$F(n+1)\ge 2^{n^2}4n(2n+1)(n+1)!,$$

and one can now easily verify that $F(n+1)\ge 2^{(n+1)^2}$ completing the induction step. ■

2.2. *Pointwise upper bounds.* If $x\in T$ and $\alpha(x)\notin 2\pi\mathbb{Z}$ for any $\alpha\in\Phi^+$ (such x are called *regular*), then it is immediate from the Weyl character formula that

$$\frac{|\mathrm{Tr}\,\lambda(x)|}{d_\lambda}\le\frac{|W|}{d_\lambda|\prod_{\alpha\in\Phi^+}(e^{i\alpha(x)}-1)|}=\frac{c(x)}{d_\lambda}.$$

Hence if μ is compactly supported on the regular elements and their conjugates, an obvious consequence of the continuity of $\prod_{\alpha\in\Phi^+}(e^{i\alpha(x)}-1)$ is that $|\widehat{\mu}(\lambda)|\le O(d_\lambda^{-1})$. This means that the interesting pointwise upper bounds for the trace functions and asymptotic bounds for the Fourier transform are for the *singular* (non-regular) elements and the central measures not supported on the conjugates of the regular elements.

THEOREM 3. *If $x\notin Z(G)$ and $\lambda=\sum a_j\lambda_j\in\widehat{G}$ with $a_\lambda=\max a_j$, then*

$$\frac{|\mathrm{Tr}\,\lambda(x)|}{d_\lambda}\le\frac{c(x)}{a_\lambda+1}$$

(*where $c(x)$ does not depend on λ*).

It is clear from the Weyl dimension formula that $d_\lambda\le(a_\lambda+1)^{|\Phi^+|}$, so we immediately have the result mentioned in the introduction.

COROLLARY 4. *If $x\notin Z(G)$ then*

$$\frac{|\mathrm{Tr}\,\lambda(x)|}{d_\lambda}\le c(x)d_\lambda^{-1/|\Phi^+|}=c(x)d_\lambda^{-2/(\dim G-r)}.$$

Proof of Theorem 3. In the proof c will denote a positive constant which may change. Note that since the trace function is central, we may assume without loss of generality that $x\in T$. Let $\Phi(x)=\{\alpha\in\Phi:\alpha(x)\in 2\pi\mathbb{Z}\}$ and let $\Phi(x)^+=\Phi(x)\cap\Phi^+$. It is easy to check from the definition that $\Phi(x)$ is a subroot system. The fact that $\Phi(x)^+$ is a complete set of positive roots of this subroot system can be seen from the construction of the positive roots as those lying on the "positive" side of a hyperplane ([9], 10.1).

We will denote by q_x the function $q_x=\prod_{\alpha\in\Phi(x)^+}\alpha$. Notice that Theorem 1 says

$$\prod_{\alpha\in\Phi(x)^+}\partial_\alpha(q_x)\ge\prod_{\alpha\in\Phi(x)^+}(\alpha,\alpha)$$

and since there are only finitely many choices for $\Phi(x)^+$, the latter product dominates a strictly positive constant independent of x and λ (interpreting the empty product as 1).

Now

$$\lim_{z\to x}\frac{\prod_{\alpha\in\Phi(x)^+}(e^{i\alpha(z)}-1)}{\prod_{\alpha\in\Phi(x)^+}(\alpha(z)-\alpha(x))}=1,$$

thus the Weyl character formula implies that

$$\mathrm{Tr}\,\lambda(x)=\lim_{z\to x}\bigg(\frac{\sum_{w\in W}\det w\exp i(w(\varrho+\lambda),z)}{\prod_{\alpha\in\Phi(x)^+}(\alpha(z)-\alpha(x))}\bigg)\frac{e^{i\varrho(x)}}{\prod_{\alpha\in\Phi^+\setminus\Phi(x)^+}(e^{i\alpha(x)}-1)}.$$

Taking the directional derivatives we obtain

$$\mathrm{Tr}\,\lambda(x)=\frac{e^{i\varrho(x)}\sum_{w\in W}\det w\prod_{\alpha\in\Phi(x)^+}(w(\varrho+\lambda),\alpha)\exp i(w(\varrho+\lambda),x)}{\prod_{\alpha\in\Phi(x)^+}\partial_\alpha(q_x)\prod_{\alpha\in\Phi^+\setminus\Phi(x)^+}(e^{i\alpha(x)}-1)}$$

and hence

$$\frac{|\mathrm{Tr}\,\lambda(x)|}{d_\lambda}\le\frac{c}{|\prod_{\alpha\in\Phi^+\setminus\Phi(x)^+}(e^{i\alpha(x)}-1)|}\max_{w\in W}\bigg\{\frac{\prod_{\alpha\in\Phi(x)^+}|(w(\varrho+\lambda),\alpha)|}{\prod_{\alpha\in\Phi^+}(\varrho+\lambda,\alpha)}\bigg\}.$$

For any fixed $w\in W$, $\{w^{-1}(\alpha):\alpha\in\Phi(x)^+\}$ is a subset of Φ of cardinality equal to $|\Phi(x)^+|$, where at most one choice of β or $-\beta$ is made for each $\beta\in\Phi^+$. Since $(w(\varrho+\lambda),\alpha)=(\varrho+\lambda,w^{-1}(\alpha))$, this observation implies that

$$\frac{\prod_{\alpha\in\Phi(x)^+}|(w(\varrho+\lambda),\alpha)|}{\prod_{\alpha\in\Phi^+}(\varrho+\lambda,\alpha)}=\frac{1}{\prod_{\alpha\in\Phi^+\setminus\pm w^{-1}(\Phi(x)^+)}(\varrho+\lambda,\alpha)}.$$

Next we show that for any given $w\in W$ the $\mathbb{Z}$-span of $w^{-1}(\Phi(x)^+)$ does not contain Φ^+. Assume otherwise. Since $x\notin Z(G)$ there is some $\alpha\in\Phi^+$

with $\alpha(x) \notin 2\pi\mathbb{Z}$ (see [1]). Let $\beta = w^{-1}(\alpha)$. By our assumption there must exist integers n_i and roots $\beta_i \in \Phi(x)^+$ with $\beta = \sum n_i w^{-1}(\beta_i)$. But then $\alpha(x) = w(\beta)(x) = \sum n_i \beta_i(x) \in 2\pi\mathbb{Z}$, which gives a contradiction.

Let $\alpha_1, \ldots, \alpha_r$ denote the simple roots in Φ. It is an easy exercise to see that the $\mathbb{Z}$-span of the set

$$X_k \equiv \Big\{ \sum_{i=1}^{r} \varepsilon_i \alpha_i \in \Phi^+ : \varepsilon_k \neq 0 \Big\}$$

contains all of Φ, and consequently there must be some $\beta \in X_k$ with $\beta \notin \pm w^{-1}(\Phi(x)^+)$. But then

$$\prod_{\alpha \in \Phi^+ \setminus \pm w^{-1}(\Phi(x)^+)} (\varrho + \lambda, \alpha) \geq c\langle \varrho + \lambda, \beta \rangle \geq c(a_\lambda + 1).$$

Thus

$$\frac{|\operatorname{Tr} \lambda(x)|}{d_\lambda} \leq \frac{c}{|\prod_{\alpha \in \Phi^+ \setminus \Phi(x)^+} (e^{i\alpha(x)} - 1)|(a_\lambda + 1)} = \frac{c(x)}{a_\lambda + 1}. \ \blacksquare$$

Recall that a *central measure* is one which commutes with all other measures under convolution.

COROLLARY 5 [13]. *If μ is a central, continuous measure then $\widehat{\mu}(\lambda) \to 0$ as $d_\lambda \to \infty$.*

Proof. Since $\mu(Z(G)) = 0$, we have $\operatorname{Tr} \lambda(x)/d_\lambda \to 0$ as $d_\lambda \to \infty$ μ a.e. By the dominated convergence theorem $\widehat{\mu}(\lambda) \to 0$. $\blacksquare$

2.3. *Applications.* As a result of Theorem 3 we are able to improve upon Ragozin's work. But first we must obtain a preliminary result on the size of the Fourier transform of certain continuous measures.

LEMMA 6. *Let $\beta_1, \ldots, \beta_j \in \Phi^+$ and let K be a compact subset of*

$$\{x \in T : \Phi(x)^+ = \{\beta_1, \ldots, \beta_j\}\}.$$

There is a constant $c(K)$ such that

$$\frac{|\operatorname{Tr} \lambda(x)|}{d_\lambda} \leq \frac{c(K)}{a_\lambda + 1} \quad \textit{for all } x \in K.$$

Proof. The proof of Theorem 3 shows that if we let

$$D(x) = \prod_{\alpha \in \Phi^+ \setminus \{\beta_1, \ldots, \beta_j\}} (e^{i\alpha(x)} - 1),$$

then for $x \in K$,

$$\frac{|\operatorname{Tr} \lambda(x)|}{d_\lambda} \leq \frac{c}{(a_\lambda + 1)|D(x)|}.$$

Since D is continuous on the compact set K, and never vanishes, the result follows immediately. $\blacksquare$

NOTATION. Set $P_j = \{x \in T : |\Phi(x)^+| = j\}$ and set $P_j^G = \bigcup_{g \in G} g^{-1} P_j g$. Then P_0 is the set of regular elements and $P_{|\Phi^+|}^G = Z(G)$.

COROLLARY 7. *If μ is a central measure, compactly supported on P_j^G for some $j \neq |\Phi^+|$, then $|\widehat{\mu}(\lambda)| \leq O(1/(a_\lambda + 1))$. Moreover, if $n > \dim G/2$ then $\{\widehat{\mu}(\lambda)^n I_{d_\lambda}\} \in l^2$, and if in addition $n \in \mathbb{N}$ then $\mu^n \in L^2$.*

Proof. For each subset F of Φ^+ having cardinality j, let

$$B_F = \{x \in T : \Phi(x)^+ = F\} \cap \operatorname{supp} \mu.$$

If $\alpha \in F$, $x_n \in B_F$ and $x_n \to y$, then by continuity $\alpha(y) \in 2\pi\mathbb{Z}$, so that $\Phi(y)^+ \supseteq F$. But also $y \in \operatorname{supp} \mu$, so $y \in P_j$, and thus $y \in B_F$ proving that B_F is closed. By the lemma there is a constant $c(F)$ so that

$$\frac{|\operatorname{Tr} \lambda(x)|}{d_\lambda} \leq \frac{c(F)}{a_\lambda + 1} \quad \text{for all } x \in B_F.$$

Let $c = \max\{c(F) : F \subseteq \Phi^+, |F| = j\}$. Since μ is central the support of μ is contained in the set of conjugates of the union of the sets B_F, and hence

$$\frac{|\operatorname{Tr} \lambda(x)|}{d_\lambda} \leq \frac{c}{a_\lambda + 1} \quad \text{for all } x \in \operatorname{supp} \mu.$$

This clearly suffices to prove the first claim in the corollary.

To verify the second claim we use the first part of the corollary and the fact that $d_\lambda \leq (a_\lambda + 1)^{|\Phi^+|}$ to obtain

$$\|\widehat{\mu}^n\|_2^2 = \sum_{\sigma \in \widehat{G}} d_\sigma \operatorname{Tr} |\widehat{\mu}(\sigma)^n I_{d_\sigma}|^2 \leq c \sum_{\sigma \in \widehat{G}} (a_\sigma + 1)^{2|\Phi^+| - 2n}.$$

But there are at most $r(k+1)^{r-1}$ points in $\mathbb{N}^r$ with maximum coordinate k, thus

$$\sum_{\sigma \in \widehat{G}} (a_\sigma + 1)^{2|\Phi^+| - 2n} \leq \sum_{k=0}^{\infty} r(k+1)^{2|\Phi^+| - 2n + r - 1},$$

and the latter sum is finite provided $2|\Phi^+| - 2n + r < 0$, i.e. $n > |\Phi^+| + r/2 = \dim G/2$. $\blacksquare$

More generally, the same idea shows that if $n > \dim G/2$ and $\mu_1, \ldots, \mu_n$ are central measures, compactly supported on P_j^G for some $j \neq |\Phi^+|$, then $\mu_1 * \ldots * \mu_n \in L^2$. In particular, if μ and v are central and compactly supported off $Z(SU(2))$, then μ and v are compactly supported on P_0^G and thus $\mu * v \in L^2$. This was previously observed by Vrem [17]. Also, if $p \geq 2$, $1/p + 1/p' = 1$, and the integer n exceeds $\dim G/p'$, then the same type of argument again, coupled with the Hausdorff–Young inequality, implies that if μ is a central measure, compactly supported on P_j^G for some $j \neq |\Phi^+|$, then $\mu^n \in L^p$ (or $C(G)$ if $p' = 1$). Hence $\mu^{\dim G} \in L^p$ for all $p < \infty$.

Corollary 7 can be improved if we assume μ is supported on the conjugates of the regular elements. First we require a lower bound for d_λ.

PROPOSITION 8. *If* $\lambda = \sum_{j=1}^r a_j \lambda_j$ *then* $d_\lambda \geq c(G) \prod_{j=1}^r (a_j+1)^{|\Phi^+|/r}$.

Proof. In the appendix we show that for each of the compact, connected, simple Lie groups (except type A_r, r even, which we handle separately below) there is a way to partition the positive roots into classes Q_j, $j = 1, \ldots, r$, each of size $|\Phi^+|/r$, where if $\alpha \in Q_j$ then $\langle \alpha, \lambda_j \rangle \neq 0$.

This property ensures that

$$\prod_{\alpha \in Q_j} \langle \varrho + \lambda, \alpha \rangle \geq (a_j + 1)^{|\Phi^+|/r},$$

thus

$$d_\lambda = \frac{\prod_{j=1}^r \prod_{\alpha \in Q_j} \langle \varrho + \lambda, \alpha \rangle}{\prod_{\alpha \in \Phi^+} \langle \varrho, \alpha \rangle} \geq c(G) \prod_{j=1}^r (a_j + 1)^{|\Phi^+|/r}.$$

For type A_r, r even, we partition (see appendix) the positive roots except for

$$\Big\{ \sum_{k=j}^r \alpha_k : j \leq \frac{r}{2} \Big\}$$

into classes Q_j each of cardinality $r/2$, and again satisfying $\langle \alpha, \lambda_j \rangle \neq 0$ if $\alpha \in Q_j$. This partitioning gives the formula

$$d_\lambda = c(G) \prod_{j=1}^r \prod_{\alpha \in Q_j} \langle \varrho + \lambda, \alpha \rangle \prod_{j=1}^{r/2} \Big\langle \varrho + \lambda, \sum_{k=j}^r \alpha_k \Big\rangle.$$

By using the inequality $a + b \geq \sqrt{ab}$ for $a, b \geq 0$, we obtain

$$\Big\langle \varrho + \lambda, \sum_{k=j}^r \alpha_k \Big\rangle = \sum_{k=j}^r (a_k + 1) \geq \sqrt{(a_j+1)(a_{j+r/2}+1)}$$

for $j = 1, \ldots, r/2$. Hence

$$d_\lambda \geq c(G) \prod_{j=1}^r (a_j+1)^{r/2} \prod_{j=1}^{r/2} (a_j+1)^{1/2} (a_{j+r/2}+1)^{1/2}$$
$$= c(G) \prod_{j=1}^r (a_j+1)^{(r+1)/2},$$

and as $|\Phi^+|/r = (r+1)/2$ we are done. ∎

COROLLARY 9. *If μ is a central measure, compactly supported on the regular elements of G and their conjugates, then* $\{\widehat{\mu}(\lambda)^n I_{d_\lambda}\} \in l^2$ *whenever* $n > \dim G/(\dim G - r)$.

Proof. In the opening paragraph of Section 2.2 we observed that $|\widehat{\mu}(\lambda)| \leq O(d_\lambda^{-1})$. Thus

$$\|\widehat{\mu}^n\|_2^2 \leq \sum_{\lambda \in \widehat{G}} d_\lambda^{2-2n}.$$

Now, for any $t \leq 0$,

$$\sum_{\lambda \in \widehat{G}} d_\lambda^t \leq c(G) \sum_{\lambda = (a_1, \ldots, a_r)} \prod_{i=1}^r (a_i+1)^{t|\Phi^+|/r} \leq c(G) \Big(\sum_{k=0}^\infty (k+1)^{t|\Phi^+|/r} \Big)^r,$$

and this sum is clearly finite if $t < -r/|\Phi^+|$. Replacing t with $2 - 2n$ and solving for n gives the result. ∎

REMARK 1. (i) Because $\dim G < 2(\dim G - r)$ for all of our groups, this corollary implies that $\mu * \mu \in L^2$ for all central measures μ, compactly supported on the conjugates of the regular elements.

(ii) Ragozin observed that one application of his work was to show that a compact, connected, simple Lie group does not admit infinite central Sidon sets. In contrast, it is known that the dual of any compact, connected group contains an infinite central $(a, 1)$-Sidon set for any $a < 1$ ([6]; see our Section 3 for definitions). This implies that there are infinite subsets E of the dual, and central continuous measures μ satisfying $\widehat{\mu}(\lambda) \geq d_\lambda^{a-1}$ for all $\lambda \in E$ and for any $a < 1$. Hence we cannot hope for asymptotic estimates as we have in Corollaries 7 and 9 without some restriction on the class of measures to which they pertain.

EXAMPLE 10. An interesting class of singular, continuous, central measures are the orbital measures, μ_x, supported on the conjugacy class $C(x)$ containing $x \notin Z(G)$. These are defined by

$$\int_G f \, d\mu_x = \int_G f(gxg^{-1}) \, dm_G(g) \quad \text{for } f \in C(G).$$

The orbital measures are examples of measures supported on P_j^G. Ragozin has observed that for all $n < \dim G/\dim C(x)$, μ_x^n is singular to Haar measure on G. Since

$$\dim C(x) = 2(|\Phi^+| - |\Phi(x)^+|)$$

(see [12]), our results prove, in contrast, that when x is regular then $\{\mu_x^n\} \in l^2$ whenever $n > \dim G/\dim C(x)$. It would be interesting to know if this remains true for arbitrary $x \notin Z(G)$.

We are now ready for our improvement of Ragozin's result.

THEOREM 11. *If $\mu_1, \ldots, \mu_n$ are central, continuous measures and* $n > \dim G/2$ *then* $\mu_1 * \ldots * \mu_n \in L^1(G)$.

Proof. The main idea of the proof is to show that each central, continuous measure μ_i can be approximated, in measure norm, by a sequence of measures $v_{m,i}$ with $v_{m,i}^n \in L^2$ for each $i = 1, \ldots, n$. We then let $v_m = v_{m,1} * \ldots * v_{m,n}$ and note that Parseval's identity implies that $v_m \in L^2$. In particular, each v_m is in L^1, and as $v_m \to \mu_1 * \ldots * \mu_n$ in $M(G)$ this implies $\mu_1 * \ldots * \mu_n \in L^1(G)$.

So it only remains to see how to do the approximation for a given central, continuous measure μ. We continue to use the P_j and P_j^G notation. The sets P_j^G are disjoint; they are unions of conjugacy classes; and their union over all $j \neq |\Phi^+|$ is the complement of $Z(G)$.

Since μ is central and continuous, $\mu(Z(G)) = 0$, and thus if $\omega_j \equiv 1_{P_j^G}\mu$, then $\mu = \sum_{j=0}^{|\Phi^+|-1} \omega_j$. Now

$$P_j = \overline{P}_j \setminus \bigcup_{k=j+1}^{|\Phi^+|} \overline{P}_k.$$

Being a closed subset of a metric space, $\bigcup_{k=j+1}^{|\Phi^+|} \overline{P}_k$ is a G_δ, and thus P_j is an F_σ, say $P_j = \bigcup_{m=1}^{\infty} F_{m,j}$, where the sets $F_{m,j}$ are closed, nested subsets of P_j. Let $F_{m,j}^G = \bigcup g^{-1} F_{m,j} g$ and $\omega_{m,j} \equiv 1_{F_{m,j}^G}\omega_j = 1_{F_{m,j}^G}\mu$. Since $F_{m,j}^G$ is a union of conjugacy classes, $\omega_{m,j}$ is a central measure. Clearly, $\bigcup_m F_{m,j}^G = P_j^G$, so by continuity of measures $\omega_{m,j} \to \omega_j$ in measure norm. Since $F_{m,j}$ is compact, one can check that $F_{m,j}^G$ is compact, and thus by Corollary 7, $\omega_{m,j}^n \in L^2$.

Finally, we let $v_m = \sum_{j=1}^{|\Phi^+|-1} \omega_{m,j}$. Then $v_m \to \sum_{j=0}^{|\Phi^+|-1} \omega_j = \mu$ in $M(G)$ and we observe that $v_m^n \in L^2$, being a finite sum of measures having the same property. ■

A measure μ is called *L^p-improving* if there is some $p < 2$ with the property that $\mu * f \in L^2$ whenever $f \in L^p$.

COROLLARY 12. *If μ is any central measure, compactly supported on P_j^G for some $j \neq |\Phi^+|$, then μ is L^p-improving. Indeed, $\mu * L^p \subseteq L^2$ for $p > 2 - 4/(2 + \dim G)$. If, moreover, μ is supported on the regular elements then $\mu * L^p \subseteq L^2$ for $p > 1 + r/(2\dim G - r)$.*

Proof. Suppose μ is any central measure supported on P_j^G for some $j \neq |\Phi^+|$. From Corollary 7 it follows that the operator T_n defined on $L^1(G)$ by $\widehat{T_n(f)}(\sigma) = \widehat{\mu}(\sigma)^n \widehat{f}(\sigma)$ maps $L^1(G)$ to $L^2(G)$. Because also T_0 defined by $\widehat{T_0(f)}(\sigma) = \widehat{\mu}(\sigma)^0 \widehat{f}(\sigma)$ maps L^2 to L^2, an application of Stein's interpolation theorem (cf. [5]) gives $\mu * L^p \subseteq L^2$ for all $p > 2 - 4/(2 + \dim G)$.

The arguments are the same if μ is supported on the regular elements, but instead use the better result: $\{\widehat{\mu}(\lambda)^n I_{d_\lambda}\} \in l^2$ for $n > \dim G/(\dim G - r)$. ■

This property was discovered by Ricci and Travaglini [14] for orbital measures μ_x when x is a regular element. In fact, their sophisticated arguments yield the characterization: $\mu_x * L^p \subseteq L^2$ if and only if $p \geq 1 + r/(2\dim G - r)$.

3. Lower bounds for the trace function

3.1. *p-norm lower bounds*

THEOREM 13. *There is a constant c so that if G is any compact, connected, simple Lie group and $\lambda = \sum_{j=1}^r a_j\lambda_j \in \widehat{G}$, with $a_\lambda = \max a_j$, then*

$$\frac{\|\mathrm{Tr}\,\lambda\|_p}{d_\lambda} \geq \frac{1}{2}(cr^2 a_\lambda)^{-\dim G/p}$$

for all $1 \leq p < \infty$.

Proof. Fix $\lambda \in \widehat{G}$. Since $\mathrm{Tr}\,\lambda$ is a class function, the Weyl integration formula yields

$$\|\mathrm{Tr}\,\lambda\|_p^p = \frac{1}{|W|}\int_T |\mathrm{Tr}\,\lambda(x)|^p |D(x)|^2\,dx$$

where $|D(x)|^2 = \prod_{\alpha\in\Phi^+} |e^{i\alpha(x)} - 1|^2$.

We let $m(\mu)$ denote the multiplicity of μ in λ restricted to T. With this notation, for $x \in T$ we have

$$\mathrm{Tr}\,\lambda(x) = \sum_{\mu\in\Pi(\lambda)} m(\mu) e^{i\mu(x)}.$$

If $x = (x_j)_{j=1}^r$ and $\mu = \sum_{j=1}^r \mu_j\lambda_j$, then since $|\mu_j| \leq cr|a_\lambda|$ we have

$$\begin{aligned}|\mathrm{Tr}\,\lambda(x) - d_\lambda| &\leq \sum_{\mu\in\Pi(\lambda)} m(\mu) \sum_{j=1}^r |\mu_j x_j| \\ &\leq r\max_j\{|\mu_j x_j| : \mu \in \Pi(\lambda)\} \sum_{\mu\in\Pi(\lambda)} m(\mu) \\ &\leq cr^2 a_\lambda \max_j |x_j| d_\lambda.\end{aligned}$$

Let

$$B_\lambda = \left\{(y_j) \in T : \max|y_j| \leq \frac{1}{2ca_\lambda r^2}\right\}.$$

If $x \in B_\lambda$ then our calculations show that $|\mathrm{Tr}\,\lambda(x)| \geq d_\lambda/2$, while if $\alpha \in \Phi^+$ then $|\alpha(x)| \leq 3/4$ so that $|\sin\alpha(x)/2| \geq |\alpha(x)|/4$. Hence for $x \in B_\lambda$,

$$|D(x)| = \prod_{\alpha\in\Phi^+}\left|2\sin\frac{\alpha(x)}{2}\right| \geq \prod_{\alpha\in\Phi^+}\left|\frac{\alpha(x)}{2}\right|.$$

Combining these facts we see that

$$\|\operatorname{Tr}\lambda\|_p^p \ge \frac{1}{|W|}\int_{B_\lambda} |\operatorname{Tr}\lambda(x)|^p |D(x)|^2\,dx$$
$$\ge \left(\frac{1}{2}d_\lambda\right)^p \frac{1}{|W|}\int_{B_\lambda}\prod_{\alpha\in\Phi^+}\left|\frac{\alpha(x)}{2}\right|^2 dx$$
$$\ge \left(\frac{1}{2}d_\lambda\right)^p \frac{1}{|W|}\int\cdots\int_{-1/(2cr^2a_\lambda)}^{1/(2cr^2a_\lambda)}\prod_{\alpha\in\Phi^+}\left|\frac{\alpha(x)}{2}\right|^2 dx_1\ldots dx_r.$$

Observe that $\prod_{\alpha\in\Phi^+}|\alpha(x)|^2/4$ is a homogeneous polynomial of degree $2|\Phi^+|$, in r variables. If p is any homogeneous polynomial of degree, say d, in r variables then

$$\int_{-R}^{R}\cdots\int_{-R}^{R} p(x_1,\ldots,x_r)\,dx_1\ldots dx_r$$
$$= R^{r+d}\int_{-1}^{1}\cdots\int_{-1}^{1} p(x_1,\ldots,x_r)\,dx_1\ldots dx_r = R^{r+d}c(p)$$

where $c(p)$ is a constant which depends only on the polynomial p, and in particular, is independent of R. To evaluate this constant when $p = \prod_{\alpha\in\Phi^+}|\alpha(x)|^2/4$ it suffices to compute

$$\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{\alpha\in\Phi^+}\left|\frac{\alpha(x)}{2}\right|^2 dx_1\ldots dx_r.$$

For this we use the fact that $|\alpha(x)|/2 \ge |\sin\alpha(x)/2|$, so

$$c(p)\pi^{2|\Phi^+|+r} = \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{\alpha\in\Phi^+}\left|\frac{\alpha(x)}{2}\right|^2 dx_1\ldots dx_r$$
$$\ge \int_T |D(x)|^2\,dx = |W|\int_G dg = |W|,$$

with the penultimate equality being the Weyl integration formula applied again. Thus

$$\frac{1}{|W|}\int\cdots\int_{-1/(2cr^2a_\lambda)}^{1/(2cr^2a_\lambda)}\prod_{\alpha\in\Phi^+}\left|\frac{\alpha(x)}{2}\right|^2 dx_1\ldots dx_r \ge \frac{1}{(cr^2a_\lambda)^{2|\Phi^+|+r}}$$

(with a new choice of constant c), and since $2|\Phi^+|+r = \dim G$, the result is proved. ■

3.2. *Applications to Sidonicity.* Let G be any compact group with dual object $\widehat{G}$. We continue to write d_λ for the degree of $\lambda\in\widehat{G}$. Let $a\in\mathbb{R}$, $1\le p<\infty$.

DEFINITION 14. Let $t=(3p-2a)/(3p-2)$ and $r=2p/(3p-2)$. We call $E\subseteq\widehat{G}$ *local* (a,p)-Λ if there is a constant c satisfying

$$\|d_\lambda \operatorname{Tr} A\lambda\|_{2s} \le c\sqrt{s}\,(d_\lambda^t \operatorname{Tr}|A|^r)^{1/r}$$

for all $s\ge1$, $\lambda\in E$ and $d_\lambda\times d_\lambda$ matrices A. When the above inequality holds for $A=I_{d_\lambda}$ we call E *local central* (a,p)-Λ.

One can also define (central) (a,p)-Λ sets, however these are not of interest in this paper.

Obviously, it is easier to be local central (a,p)-Λ as a decreases or p increases. Two other obvious facts are:

PROPOSITION 15. (a) *$\widehat{G}$ is local central $(p-1,p)$-Λ.*

(b) *If $(b+1)/q\le(a+1)/p$ and E is local central (a,p)-Λ then E is local central (b,q)-Λ.*

Because of (a) we are only interested in the case $a>p-1$.

DEFINITION 16. The subset $E\subseteq\widehat{G}$ is called (*central*) (a,p)-*Sidon* if there is a constant c such that

$$\|f\|_\infty \ge c\Big(\sum d_\lambda^a \operatorname{Tr}|\widehat{f}(\lambda)|^p\Big)^{1/p}$$

whenever $f(x)=\sum_{\lambda\in E} d_\lambda \operatorname{Tr}\widehat{f}(\lambda)\lambda(x)$ (and f is central); E is called *local* (*central*) (a,p)-*Sidon* if the inequality above holds for all $f(x)=d_\lambda\operatorname{Tr}A\lambda(x)$ with $\lambda\in E$ and A a (central) $d_\lambda\times d_\lambda$ matrix.

$(1,1)$-Sidon sets are generally called *Sidon sets* and have been extensively studied (cf. [10] and the references cited therein). One reason for the interest in (a,p)-Λ sets is that local (a,p)-Sidon sets are local (a,p)-Λ (see [8]). Indeed, a set is local Sidon if and only if it is local central $(1,1)$-Λ (see [11]).

Using the fact that local Sidon sets are local $(1,1)$-Λ, Cartwright and McMullen [2] characterized Sidon sets in compact connected groups by means of an "independent" set called the FTR set:

DEFINITION 17. Let G be a compact, simply connected, simple Lie group of rank r, with the notation as in the first section. We define the *FTR set* of G by

$$\mathrm{FTR}(G)=\begin{cases}\{\lambda_1,\lambda_r\} & \text{if } G \text{ is of type } A_r,\\ \{\lambda_1\} & \text{if } G \text{ is of type } B_r,\ C_r \text{ or } D_r\ (r\ge5),\\ \{\lambda_1,\lambda_3,\lambda_4\} & \text{if } G \text{ is of type } D_4,\\ \emptyset & \text{otherwise.}\end{cases}$$

For $m\in\mathbb{N}$ we define the *m-fold FTR set* by

$$\mathrm{FTR}^m(G)=\{\sigma\in\widehat{G}:\sigma\in(\mathrm{FTR}(G))^m\}$$

(where $\mathrm{FTR}^0(G)$ is understood to be $\{1\}$ and the product denotes tensor product).

The FTR sets are essentially the only non-trivial examples of Sidon sets in products of compact Lie groups. In light of what is known about p-Sidon sets in abelian groups, it is reasonable to ask whether m-fold FTR sets are $(1,p)$-Sidon if and only if $p \geq 2m/(m+1)$. Some partial evidence to support this supposition is discussed in [8]. For example, it is shown that if $G = \prod G_n$ where rank G_n tends to infinity, then the m-fold FTR set of G is not central (a,p)-Sidon if $p < (a+1)m/(m+1)$. Here we prove the corresponding result for local (a,p)-Sidon sets.

We first need some preliminary results about m-fold FTR sets.

PROPOSITION 18. (a) *If $n > 2m$ and $m \geq 2$ then*

$$\mathrm{FTR}^m(A_n) = \mathrm{FTR}^{m-2}(A_n) \cup \Big\{\lambda = \sum a_j\lambda_j : \sum_{j=1}^{m} j a_j + \sum_{j=n-m+1}^{n} (n-j+1)a_j = m\Big\}.$$

(b) *If $n > m+1$, $m \geq 2$ and G is of type B_n, C_n or D_n $(n \geq 5)$ then*

$$\mathrm{FTR}^m(G) = \mathrm{FTR}^{m-2}(G) \cup \Big\{\lambda = \sum a_j\lambda_j : \sum_{j=1}^{m} j a_j = m\Big\}.$$

Proof. For convenience, when $\lambda = \sum a_j\lambda_j$ we will write

$$S(\lambda) \equiv \sum_{j=1}^{m} j a_j + \sum_{j=n-m+1}^{n} (n-j+1)a_j \quad \text{if } \lambda \in \widehat{A_n},$$

and

$$S(\lambda) \equiv \sum_{j=1}^{m} j a_j \quad \text{if } \lambda \in \widehat{B}_n, \widehat{C}_n \text{ or } \widehat{D}_n.$$

Also, if the representation μ is a subrepresentation of v we will write $\mu \leq v$.

(a) As $\mathrm{FTR}(A_n) = \{\lambda_1, \lambda_n\}$, $\mathrm{FTR}^2(A_n)$ consists of all irreducible subrepresentations of $\lambda_1 \otimes \lambda_1, \lambda_1 \otimes \lambda_n$ and $\lambda_n \otimes \lambda_n$. It is well known (and easy to check) that $\lambda_1 \otimes \lambda_1 = 2\lambda_1 \oplus \lambda_2, \lambda_1 \otimes \lambda_n = 1 \oplus (\lambda_1 + \lambda_n)$ and $\lambda_n \otimes \lambda_n = 2\lambda_n \oplus \lambda_{n-1}$, thus the result is true for $m = 2$.

We proceed by induction assuming the result is true for m and that $2(m+1) < n$. If $\lambda \in \mathrm{FTR}^{m+1}(A_n)$ then $\lambda \leq \lambda_1 \otimes \sigma$ or $\lambda \leq \lambda_n \otimes \sigma$ for some $\sigma \in \mathrm{FTR}^m(A_n)$, and by the induction hypothesis we may assume $\sigma = \sum a_j\lambda_j$ where $S(\sigma) = m - 2k$ for some non-negative integer k.

The components of $\lambda_1 \otimes \sigma$ are those $\sigma - \lambda_j + \lambda_{j+1} \in \Lambda^+$ for $0 \leq j \leq n$ (see [7]). The definition of σ ensures that if $a_j \neq 0$ then either $j \leq m < n/2 - 1$ or $j \geq n - m + 1 > n/2 + 2$, and from this it is clear that if $\sigma - \lambda_j + \lambda_{j+1} \in \Lambda^+$ then $S(\sigma - \lambda_j + \lambda_{j+1}) = S(\sigma) + 1$. The induction hypothesis shows that either $\lambda = \sigma - \lambda_j + \lambda_{j+1} \in \mathrm{FTR}^{m-1}$ or $\lambda \in \{\eta : S(\eta) = m+1\}$.

The argument is similar if λ is a subrepresentation of $\lambda_n \otimes \sigma$. Thus

$$\mathrm{FTR}^{m+1}(A_n) \subseteq \mathrm{FTR}^{m-1}(A_n) \cup \{\lambda : S(\lambda) = m+1\}.$$

As $1 \in \mathrm{FTR}^2(A_n)$ we obviously have $\mathrm{FTR}^{m-1}(A_n) \subseteq \mathrm{FTR}^{m+1}(A_n)$. Assume now that $\lambda = \sum a_j\lambda_j$ and $S(\lambda) = m+1$. If $a_k \neq 0$ for some $1 < k \leq m$ then consider

$$\sigma = \sum_{j \neq k, k-1} a_j\lambda_j + (a_{k-1}+1)\lambda_{k-1} + (a_k - 1)\lambda_k.$$

Since $S(\sigma) = m$, $\sigma \in \mathrm{FTR}^m(A_n)$ and $\lambda = \sigma - \lambda_{k-1} + \lambda_k \leq \lambda_1 \otimes \sigma \in \mathrm{FTR}^{m+1}(A_n)$. If $a_k \neq 0$ for some $n - m + 1 \leq k < n$, then one can similarly show that $\lambda \in \mathrm{FTR}^m(A_n) \otimes \lambda_n \subseteq \mathrm{FTR}^{m+1}(A_n)$. Otherwise $\lambda = a\lambda_1 + (m + 1 - a)\lambda_n$ where without loss of generality $a \neq 0$. Then $\lambda = ((a-1)\lambda_1 + (m + 1 - a)\lambda_n) - \lambda_0 + \lambda_1$ and so belongs to $\mathrm{FTR}^m(A_n) \otimes \lambda_1 \subseteq \mathrm{FTR}^{m+1}(A_n)$. Thus $\{\lambda : S(\lambda) = m+1\} \subseteq \mathrm{FTR}^{m+1}(A_n)$, which completes the proof of the induction step.

(b) If G is of type B_n, C_n or D_n $(n \geq 5)$ then $\mathrm{FTR}^m(G) = \{\lambda \leq \lambda_1^m\}$. We again proceed by induction on m (directly checking the result for $m = 2$) and will use the fact shown in [7] that the size of m ensures that if $\sigma \in \mathrm{FTR}^m(G)$, then $\sigma \otimes \lambda_1$ has components $\sigma \pm (\lambda_{j-1} + \lambda_j) \in \Lambda^+$ for $1 \leq j \leq m+1$.

So let $\lambda \in \mathrm{FTR}^{m+1}(G)$. If $\lambda = \sigma + \lambda_{j-1} - \lambda_j$ for some $\sigma \in \mathrm{FTR}^m(G)$, then $S(\lambda) = S(\sigma) - 1$, and therefore, by the induction hypothesis, $\lambda \in \mathrm{FTR}^{m-1}(G)$. Otherwise $\lambda = \sigma - (\lambda_{j-1} - \lambda_j)$, and then one can easily see that $S(\lambda) = S(\sigma) + 1$, so $\lambda \in \mathrm{FTR}^{m-1}(G) \cup \{\lambda : S(\lambda) = m+1\}$.

For the converse, notice that once again we clearly have $\mathrm{FTR}^{m-1}(G) \subseteq \mathrm{FTR}^{m+1}(G)$. Suppose $\lambda = \sum a_j\lambda_j$ with $S(\lambda) = m+1$ and $a_k \neq 0$ for some $k > 1$. If

$$\sigma = \sum_{j \neq k, k-1} a_j\lambda_j + (a_{k-1}+1)\lambda_{k-1} + (a_k - 1)\lambda_k$$

then $S(\sigma) = m$, so $\sigma \in \mathrm{FTR}^m(G)$, and as $\lambda = \sigma - (\lambda_{k-1} - \lambda_k)$, $\lambda \in \mathrm{FTR}^{m+1}(G)$. The remaining case to consider is $\lambda = (m+1)\lambda_1$ and since we can then write $\lambda = m\lambda_1 - (\lambda_0 - \lambda_1)$ we clearly have $\lambda \in \mathrm{FTR}^{m+1}(G)$, completing the induction step for part (b). ■

PROPOSITION 19. *If $n > 2m$ and G is of type A_n, B_n, C_n or D_n $(n \geq 5)$ and $\lambda \in \mathrm{FTR}^m(G) \setminus \mathrm{FTR}^{m-2}(G)$ then $d_\lambda \geq \binom{n+1}{m}$.*

Proof. This is proved in [7], 4.2, for G of type A_n. Similar arguments (but easier), using the previous proposition, work for the other types. ■

Note that by definition of the m-fold FTR set, $d_\lambda = O(n^m)$ if $\lambda \in \mathrm{FTR}^m(G)$. We are now ready to prove the main result of this section.

THEOREM 20. *Let $G = \prod G_n$ where G_n are compact, simply connected, simple Lie groups with* rank G_n *tending to infinity. The following are equivalent:*

(1) *$\mathrm{FTR}^m(G)$ is a local central (a,p)-Λ set;*

(2) *an infinite subset of* $\bigcup_n \mathrm{FTR}^m(G_n) \setminus \mathrm{FTR}^{m-2}(G_n)$ *is local central* (a,p)-Λ;

(3) $p \ge (a+1)m/(m+1)$.

Proof. (1)⇒(2) is obvious as $\mathrm{FTR}^m(G_n) \subseteq \mathrm{FTR}^m(G)$.

(2)⇒(3). Suppose an infinite subset of $\bigcup_n \mathrm{FTR}^m(G_n) \setminus \mathrm{FTR}^{m-2}(G_n)$ is local central (a,p)-Λ. Then there is a sequence $\{\lambda_k\} \subseteq \mathrm{FTR}^m(G_{j_k}) \setminus \mathrm{FTR}^{m-2}(G_{j_k})$ satisfying

$$\|\mathrm{Tr}\,\lambda_k\|_{2s} \le c\sqrt{s}\, d_{\lambda_k}^{2-(a+1)/p},$$

where $\operatorname{rank} G_{j_k} = r_k \to \infty$. Choosing $2s = (\dim G_{j_k})/\varepsilon$ (for $\varepsilon > 0$ small as explained later) and applying Theorem 13 we have

$$\frac{d_{\lambda_k}}{2(2\pi c r_k^2 a_{\lambda_k})^{\varepsilon}} \le \|\mathrm{Tr}\,\lambda_k\|_{2s} \le c\varepsilon^{-1/2}(\dim G_{j_k})^{1/2} d_{\lambda_k}^{2-(a+1)/p}.$$

As $r_k \to \infty$ we may assume without loss of generality that G_{j_k} is one of A_n, B_n, C_n or D_n ($n \ge 5$), where $n > 2m$, and thus by the proposition above $d_{\lambda_k} \ge c_m r_k^m$. Also, $a_{\lambda_k} \le d_{\lambda_k}$ and $\dim G_{j_k} = O(r_k^2)$, so we must have the inequality

$$r_k^{m(1-\varepsilon)-2\varepsilon} \le \frac{c}{\sqrt{\varepsilon}} r_k^{1+m(2-(a+1)/p)}$$

holding (for a new constant $c = c(m)$). If $p < (a+1)m/(m+1)$, then by fixing ε sufficiently small and letting $r_k \to \infty$ we have a contradiction.

(3)⇒(1). First notice that

$$\mathrm{FTR}^m(G) = \bigcup_{j=0}^{m} \Big(\mathrm{FTR}^j\Big(\prod_{\operatorname{rank} G_n > \max(8,2m)} G_n\Big) \times \mathrm{FTR}^{m-j}\Big(\prod_{\operatorname{rank} G_n \le \max(8,2m)} G_n\Big)\Big)$$

and that $\mathrm{FTR}^j(\prod_{\operatorname{rank} G_n \le \max(8,2m)} G_n)$ is a finite set since $\operatorname{rank} G_n \to \infty$. As $\sigma \times E$ is a local central (a,p)-Λ set if and only if E is, and the class of local central (a,p)-Λ sets is closed under finite unions, we may as well assume $G = \prod G_n$ where $\operatorname{rank} G_n > 2m$ and $G_n \ne D_4$ or one of the exceptional groups.

Such groups G have the property that when $\lambda \in \mathrm{FTR}^m(G) \setminus \mathrm{FTR}^{m-2}(G)$ then $\lambda = \beta_1 \times \ldots \times \beta_k$ where $\beta_i \in \mathrm{FTR}^{m_i}(G_{j_i}) \setminus \mathrm{FTR}^{m_i-2}(G_{j_i})$ and $\sum m_i = m$ (see [7]). Moreover, if we set $\lambda_{1i} = \lambda_1(G_{j_i})$, then $d_{\beta_i} \ge c_m d_{\lambda_{1i}}^{m_i}$.

As $\|\mathrm{Tr}\,\lambda\|_{2s}^{2s}$ is the number of irreducible subrepresentations of λ^s counted by squared multiplicities, it is clear that

$$\|\mathrm{Tr}\,\lambda\|_{2s} = \prod_{i=1}^{k} \|\mathrm{Tr}\,\beta_i\|_{2s} \le \prod_{i=1}^{k} \|\mathrm{Tr}\,\lambda_{1i}\|_{2sm_i}^{m_i}.$$

Since $\mathrm{FTR}(G)$ is a Sidon set, it is a local central $(1,1)$-Λ set, and thus there is a constant c such that

$$\|\mathrm{Tr}\,\lambda_{1i}\|_{2s} \le c\sqrt{2s} \quad \text{for all } s \ge 1 \text{ and all } \lambda_{1i}.$$

By factoring $\|\mathrm{Tr}\,\lambda_{1i}\|_{2sm_i}^{m_i}$ as $\|\mathrm{Tr}\,\lambda_{1i}\|_{2sm_i}^{m_i/m}\|\mathrm{Tr}\,\lambda_{1i}\|_{2sm_i}^{m_i(1-1/m)}$, and observing that the trace of a representation is bounded everywhere by its degree, it follows that

$$\|\mathrm{Tr}\,\lambda\|_{2s} \le \prod_{i=1}^{k} c\sqrt{2sm_i}^{\,m_i/m} d_{\lambda_{1i}}^{m_i(1-1/m)} \le c(m)\sqrt{s}\prod_{i=1}^{k} d_{\beta_i}^{1-1/m} \le c(m)\sqrt{s}\, d_{\lambda}^{2-(a+1)/p},$$

which is the desired result. ∎

Because local (a,p)-Sidon sets are local (a,p)-Λ the following are obvious corollaries of the theorem.

Corollary 21. (a) $\mathrm{FTR}^m(G)$ *is not local* (a,p)-*Sidon if* $p < m(a+1)/(m+1)$.

(b) $\mathrm{FTR}^{m+1}(G)$ *is not local* $(a, 2m/(m+1))$-*Sidon for* $a > 1-2/(m+1)^2$ *and* $m \ge 1$.

Previously it was known that $\mathrm{FTR}^2(G)$ was not local Sidon [2]. Here we see that it is not even local $(1/2+\varepsilon, 1)$-Sidon for any $\varepsilon > 0$.

4. Appendix. We will list a partitioning for each of the classical Lie groups which satisfies the requirements of Proposition 8, leaving the verification and a suitable partitioning for the exceptional groups to the reader.

1. A_r, r odd: For $j = 1, \ldots, (r+1)/2$ let $Q_j = \{\sum_{k=j}^{i} \alpha_k : i = j, \ldots, j+(r+1)/2-1\}$.

For $j = (r+1)/2+1, \ldots, r$ let $Q_j = \{\sum_{k=j}^{i} \alpha_k : i = j, \ldots, r\} \cup \{\sum_{k=i}^{j} \alpha_k : i = 1, \ldots, j-(r+1)/2\}$.

2. A_r, r even: Define $Q_1, \ldots, Q_{r-1}$ as for A_{r-1}. Let $Q_r = \{\sum_{k=i}^{r} \alpha_k : i = r/2+1, \ldots, r\}$.

3. B_r: Let $Q_j = \{\sum_{k=j}^{i} \alpha_k : i = j, \ldots, r\} \cup \{\sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{r} \alpha_k : i = 1, \ldots, j-1\}$.

4. C_r: The same partitioning as for B_r works if one replaces $\sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{r} \alpha_k$ in B_r with $\sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{r-1} \alpha_k + \alpha_r$, and replaces $\sum_{k=i}^{r-1} \alpha_k + 2\alpha_r$ in B_r with $2\sum_{k=i}^{r-1} \alpha_k + \alpha_r$.

5. D_r: For $j < r-1$ we let $Q_j = \{\sum_{k=j}^{i} \alpha_k : i = j, \ldots, r-1\} \cup \{\sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{r-2} \alpha_k + \alpha_{r-1} + \alpha_r : i = 1, \ldots, j-1\}$.

Let $Q_{r-1} = \{\alpha_{r-1}, \sum_{k=i}^{r} \alpha_k : i = 1, \ldots, r-2\}$.

Let $Q_r = \{\alpha_r, \sum_{k=i}^{r-2} \alpha_k + \alpha_r : i = 1, \ldots, r-2\}$.

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Received March 22, 1996
Revised version August 11, 1997 (3640)

STUDIA MATHEMATICA 129 (1) (1998)

Convex sets in Banach spaces and a problem of Rolewicz

by

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Abstract. Let $\mathcal{B}_X$ be the set of all closed, convex and bounded subsets of a Banach space X equipped with the Hausdorff metric. In the first part of this work we study the density character of $\mathcal{B}_X$ and investigate its connections with the geometry of the space, in particular with a property shared by the spaces of Shelah and Kunen. In the second part we are concerned with the problem of Rolewicz, namely the existence of support sets, for the case of spaces $C(K)$.

1. Introduction. In this paper we discuss some topics concerning the set $\mathcal{B}_X$ of all bounded, closed, convex and nonempty subsets of a real Banach space X. The Hausdorff distance between $C_1, C_2 \in \mathcal{B}_X$ is given by

$$d(C_1, C_2) = \inf\{\varepsilon > 0 : C_1 \subset C_2 + \varepsilon B_{\|\cdot\|}, C_2 \subset C_1 + \varepsilon B_{\|\cdot\|}\},$$

where $B_{\|\cdot\|}$ is the unit ball of X. It is well known that $(\mathcal{B}_X, d)$ is a complete metric space [11] and, hence, a Baire space.

The first part is devoted to the study of the density character of $\mathcal{B}_X$ and its interplay with different geometrical properties. These properties are property α, the (weak*) Mazur intersection property and the following cornerstone property, which we shall name the *Kunen–Shelah property*: among any uncountable family of elements of X, there is one that belongs to the closed convex hull of the rest. Shelah [23] (assuming the diamond principle for $\aleph_1$) and Kunen [18] (assuming the continuum hypothesis) constructed Banach spaces $\mathcal{S}$ and $\mathcal{K}$ respectively with the above property. Most of our work in Section 2 will tend to emphasize the effects of the Kunen–Shelah property on the topological properties of $\mathcal{B}_X$. For instance, we prove here that, in many cases, spaces enjoying this property satisfy $\operatorname{dens} X = \operatorname{dens} \mathcal{B}_X$ while, in general, $\operatorname{dens} \mathcal{B}_X = 2^{\operatorname{dens} X}$. Moreover, assuming $\mathfrak{c} < 2^{\omega_1}$ (where ω_1 is the first uncountable ordinal), an Asplund space X with $\operatorname{dens} X = \mathfrak{c}$

1991 *Mathematics Subject Classification*: Primary 46B20.
Supported in part by DGICYT grant PB 94-0243 and PB 93-0452