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Factorization of operators on C^*-algebras

by

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Abstract. Let $\mathcal{A}$ be a C^*-algebra. We prove that every absolutely summing operator from $\mathcal{A}$ into ℓ_2 factors through a Hilbert space operator that belongs to the 4-Schatten–von Neumann class. We also provide finite-dimensional examples that show that one cannot replace the 4-Schatten–von Neumann class by the p-Schatten–von Neumann class for any $p < 4$. As an application, we show that there exists a modulus of capacity $\varepsilon \to N(\varepsilon)$ so that if $\mathcal{A}$ is a C^*-algebra and $T \in \Pi_1(\mathcal{A}, \ell_2)$ with $\pi_1(T) \le 1$, then for every $\varepsilon > 0$, the ε-capacity of the image of the unit ball of $\mathcal{A}$ under T does not exceed $N(\varepsilon)$. This answers positively a question raised by Pełczyński.

1. Introduction. It is a well-known consequence of a classical result of Grothendieck that if X is a Banach space and X^{**} is isomorphic to a quotient of a $C(K)$-space then every absolutely summing operator from X into ℓ_2 factors through a Hilbert–Schmidt operator. The present paper is an attempt to get a generalization of this fact for the setting of arbitrary C^*-algebras. Different structures of operators defined on arbitrary C^*-algebras were considered by Pisier in [13] and [14]; for instance he proved that every (p,q)-summing operator on an arbitrary C^*-algebra admits a factorization similar to that of operators on $C(K)$-spaces, and every operator from any C^*-algebra into any Banach space of cotype 2 factors through Hilbert space. Using the notion of C^*-summing operators introduced by Pisier in [13], the author proved in [15] that absolutely summing operators from C^*-algebras into reflexive spaces are compact. The main result of this paper states that for the case of C^*-algebras and the range space being a Hilbert space, one can factor every absolutely summing operator through a Hilbert space operator that belongs to the 4-Schatten–von Neumann class (see definition below). The basic idea of the proof of this result is the factorization of C^*-summing operators used in [15] and some well-known coincidence of different classes of Hilbert space operators. This result allows us to prove a quantitative result on the compactness of absolutely summing operators from

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C^*-algebras into Hilbert spaces, answering a question raised by Pełczyński in [10] (Problem 3′) for the space of compact operators on a Hilbert space. A finite-dimensional approach shows that unlike the commutative case of $C(K)$-spaces, one cannot expect to factor every absolutely summing operator from general non-commutative C^*-algebras into Hilbert spaces through Hilbert–Schmidt operators. In fact, our examples show that the result stated above cannot be improved to the case of the p-Schatten–von Neumann class for any $p < 4$.

Our terminology and notation are standard. We refer to [2] and [20] for definitions from Banach space theory and [6] and [17] for basic properties from C^*-algebra and operator algebra theory.

2. Preliminaries. In this section we recall some definitions and facts which we use in the sequel. Throughout, the word "operator" will always mean a linear bounded operator and $\mathcal{L}(E,F)$ will stand for the space of all operators from E into F.

DEFINITION 1. Let E and F be Banach spaces and $1 \le p < \infty$. An operator $T \in \mathcal{L}(E,F)$ is said to be *absolutely p-summing* (or simply *p-summing*) if there exists a constant C such that for any finite sequence $(e_1, \ldots, e_n)$ of E, one has

$$\Big(\sum_{i=1}^{n} \|Te_i\|^p\Big)^{1/p} \le C \sup\Big\{\Big(\sum_{i=1}^{n} |\langle e_i, e^*\rangle|^p\Big)^{1/p} : e^* \in E^*,\ \|e^*\| \le 1\Big\}.$$

The least constant C for the inequality above to hold will be denoted by $\pi_p(T)$. It is well known that the class of all absolutely p-summing operators from E to F is a Banach space under the norm $\pi_p(\cdot)$. This Banach space will be denoted by $\Pi_p(E,F)$.

DEFINITION 2. Let $1 \le q \le p < \infty$. An operator $T \in \mathcal{L}(E,F)$ is said to be *(p,q)-summing* if there is a constant $K \ge 0$ for which

$$\Big(\sum_{k=1}^{n} \|Te_i\|^p\Big)^{1/p} \le K \sup\Big\{\Big(\sum_{i=1}^{n} |\langle e^*, e_i\rangle|^q\Big)^{1/q} : e^* \in E^*,\ \|e^*\| \le 1\Big\}$$

for every finite sequence $(e_1, \ldots, e_n)$ in E.

As above, the least constant K for which the inequality holds is the (p,q)-summing norm of T and is denoted by $\pi_{p,q}(T)$. The class of (p,q)-summing operators from E into F is a Banach space under the norm $\pi_{p,q}(\cdot)$. This class will be denoted by $\Pi_{p,q}(E,F)$.

Another class of operators relevant for our discussion is the Schatten–von Neumann class.

DEFINITION 3. For $1 \le p < \infty$ and for H_1 and H_2 Hilbert spaces, the pth *Schatten–von Neumann class* consists of all compact operators $U : H_1 \to H_2$ that have a representation of the form

$$(\dagger) \qquad U = \sum_{n=1}^{\infty} \alpha_n(\cdot, e_n) f_n,$$

where $(e_n)_n$ is an orthonormal sequence in H_1, $(f_n)_n$ is an orthonormal sequence in H_2, and $(\alpha_n)_n \in \ell_p$.

We will refer to (†) as an *orthonormal representation* of U. It is well known that one can always choose the sequence $(\alpha_n)_n$ in the representation (†) to satisfy $0 \le \alpha_{n+1} \le \alpha_n$ for all admissible indices. The pth *Schatten–von Neumann norm* is defined by

$$\sigma_p(U) = \Big(\sum_{n=1}^{\infty} |\alpha_n|^p\Big)^{1/p}$$

and the pth Schatten–von Neumann class is denoted by $S_p(H_1, H_2)$.

DEFINITION 4. Let E and F be Banach spaces, $1 \le p \le \infty$. We say that an operator $T \in \mathcal{L}(E,F)$ is *L_p-factorable* if there exist a measure space (Ω, Σ, μ) and operators $U_1 \in \mathcal{L}(E, L_p(\mu))$ and $U_2 \in \mathcal{L}(L_p(\mu), F^{**})$ such that $i_F \circ T = U_2 \circ U_1$ where $i_F : F \to F^{**}$ denotes the natural embedding.

The *L_p-factorable norm* of T is defined by $\gamma_p(T) := \inf\{\|U_1\| \cdot \|U_2\|\}$ where the infimum is taken over all possible factorizations as above.

For detailed discussion of p-summing operators, (q,p)-summing operators, p-Schatten–von Neumann operators and L_p-factorable operators, we refer to [3], [12] and [18].

We will now recall some basic facts on C^*-algebras and von Neumann algebras. Let $\mathcal{A}$ be a C^*-algebra; we denote by $\mathcal{A}_h$ the set of Hermitian (self-adjoint) elements of $\mathcal{A}$. For $x \in \mathcal{A}$ and $f \in \mathcal{A}^*$, we denote by xf (resp. fx) the element of $\mathcal{A}^*$ defined by $xf(y) = f(yx)$ (resp. $fx(y) = f(xy)$) for every $y \in \mathcal{A}$.

DEFINITION 5. A von Neumann algebra is said to be *σ-finite* if it admits at most countably many mutually orthogonal distinct projections.

We refer to [6] and [17] for some characterizations and examples of σ-finite von Neumann algebras.

3. Main theorem

THEOREM 1. *Let $\mathcal{A}$ be a C^*-algebra and $T \in \Pi_1(\mathcal{A}, \ell_2)$. Then for every $\varepsilon > 0$, there exist a Hilbert space H and operators $J : A \to H$ and $K : H \to \ell_2$ such that:*

(1) $T = K \circ J$;
(2) $\|J\| \le 1$;
(3) $K \in S_4(H, \ell_2)$ *with* $\sigma_4(K) \le 2(1+\varepsilon)\pi_1(T)$.

To prove this theorem, we will first consider the following particular case:

PROPOSITION 1. *Let $\mathcal{M}$ be a σ-finite von Neumann algebra and assume that $T \in \Pi_1(\mathcal{M}, \ell_2)$ is weak* to weakly continuous operator. For every $\varepsilon > 0$, there exist a Hilbert space H and operators $J : \mathcal{M} \to H$ and $K : H \to \ell_2$ such that*:

(1) $T = K \circ J$;
(2) $\|J\| \le 1$;
(3) $K \in S_4(H, \ell_2)$ *with* $\sigma_4(K) \le 2(1+\varepsilon)\pi_1(T)$.

Proof. The proof is based on the factorization technique used in [15]. We will repeat the argument for completeness.

Let $T \in \Pi_1(\mathcal{M}, \ell_2)$ and assume that T is weak* to weakly continuous. Fix $\delta > 0$ such that $(1+\delta)^{1/2} \le (1+\varepsilon)$.

By [15] (Proposition 1.1) and [13] (Lemma 4.1), there exists a normal positive functional g on $\mathcal{M}$ such that $\|g\| \le 1$ and

$$\|Tx\| \le \pi_1(T) g(|x|) \qquad \text{for every } x \in \mathcal{M}_h.$$

Since the von Neumann algebra $\mathcal{M}$ is σ-finite, there exists a faithful normal functional f_0 in $\mathcal{M}_*$ where $\mathcal{M}_*$ denotes the predual of $\mathcal{M}$ (see [17], Proposition II-3.19). We can choose f_0 so that $\|f_0\| \le \delta$. Let $f = (g + f_0)/(1+\delta)$; clearly, $\|f\| \le 1$ and

$$\|Tx\| \le (1+\delta)\pi_1(T) f(|x|) \qquad \text{for every } x \in \mathcal{M}_h.$$

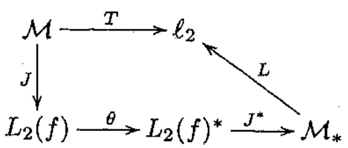
From Lemma 2 of [15], we deduce that

$$\|Tx\| \le 2(1+\delta)\pi_1(T)\|xf + fx\|_{\mathcal{M}_*} \qquad \text{for every } x \in \mathcal{M}.$$

As in [15], we equip $\mathcal{M}$ with the scalar product

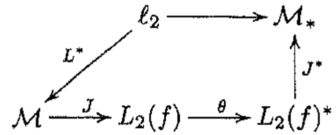
$$\langle x, y\rangle = f\Big(\frac{xy^* + y^*x}{2}\Big).$$

Since f is faithful, $\mathcal{M}$ with $\langle\cdot,\cdot\rangle$ is pre-Hilbertian. We denote the completion of this space by $L_2(\mathcal{M}, f)$ (or simply $L_2(f)$). From [15], we have the following factorization:



where $\theta(Jx) = \langle\cdot, J(x^*)\rangle$ for every $x \in \mathcal{M}$, and $L((xf + fx)/2) = Tx$ for every $x \in \mathcal{M}$, and J is the inclusion map (one can easily check as in [15] that $J^* \circ \theta \circ J(x) = (xf + fx)/2$).

Set $H := L_2(f)$ and $K := L \circ J^* \circ \theta$. Clearly, (1) and (2) are satisfied. To prove (3), let us consider the adjoint maps:



The proposition will be deduced from the following lemma:

LEMMA 1. *For every $p \ge 1$, $K^* \in \Pi_{2p,p}(\ell_2, H^*)$ with $\pi_{2p,p}(K^*) \le \pi_p(T)^{1/2}\|L\|^{1/2}$.*

To see the lemma, let $(z_n)_n$ be a sequence in ℓ_2 such that

$$\sup\Big\{\Big(\sum_{n=1}^{\infty} |\langle z_n, z\rangle|^p\Big)^{1/p} : z \in \ell_2,\ \|z\| \le 1\Big\} = C < \infty.$$

Then

$$\sup\Big\{\Big(\sum_{n=1}^{\infty} |\langle L^*(z_n), \xi\rangle|^p\Big)^{1/p} : \xi \in \mathcal{M}^*,\ \|\xi\| \le 1\Big\} \le \|L\|C.$$

Similarly,

$$\sup\Big\{\Big(\sum_{n=1}^{\infty} |\langle (L^*(z_n))^*, \xi\rangle|^p\Big)^{1/p} : \xi \in \mathcal{M}^*,\ \|\xi\| \le 1\Big\} \le \|L\|C$$

where $(L^*(z_n))^*$ is the adjoint of the operator $L^*(z_n)$ in $\mathcal{M}$ for every $n \in \mathbb{N}$. Since $((L^*(z_n))^*)_n$ is a sequence in $\mathcal{M}$, one can apply T. The fact that T is p-summing (because every 1-summing operator is p-summing for every $p > 1$) implies that

$$\begin{aligned}\Big(\sum_{n=1}^{\infty} \|T(L^*(z_n)^*)\|^p\Big)^{1/p} &\le \pi_p(T) \sup\Big\{\Big(\sum_{n=1}^{\infty} |\langle L^*(z_n)^*, \xi\rangle|^p\Big)^{1/p} : \|\xi\| \le 1\Big\}\\ &\le \pi_p(T)\|L\|C.\end{aligned}$$

But since $(z_n)_n$ is bounded (in fact, it is bounded by C) we get

$$\Big(\sum_{n=1}^{\infty} |\langle T((L^*(z_n))^*), z_n\rangle|^p\Big)^{1/p} \le \pi_p(T)\|L\|C^2.$$

Now for each $n \in \mathbb{N}$,

$$\begin{aligned}\langle T((L^*(z_n))^*), z_n\rangle &= \langle L \circ J^* \circ \theta \circ J(L^*(z_n)^*), z_n\rangle\\ &= \langle \theta \circ J(L^*(z_n)^*), J \circ L^*(z_n)\rangle\end{aligned}$$

$$= \langle J(L^*(z_n)), J(L^*(z_n))\rangle = \|J(L^*(z_n))\|^2_{L_2(f)}.$$

So

$$\Big(\sum_{n=1}^{\infty} \|J\circ L^*(z_n)\|^{2p}\Big)^{1/p} \le \pi_p(T)\|L\|C^2.$$

Hence

$$\Big(\sum_{n=1}^{\infty} \|K^*(z_n)\|^{2p}\Big)^{1/(2p)} \le \pi_p(T)^{1/2}\|L\|^{1/2}C,$$

which shows that $K^* \in \Pi_{2p,p}(\ell_2, H^*)$ with $\pi_{2p,p}(K^*) \le \pi_p(T)^{1/2}\|L\|^{1/2}$. The lemma is proved.

To complete the proof of the proposition, we apply the above lemma for $p = 2$; we get $\pi_{4,2}(K^*) \le \pi_2(T)^{1/2}\|L\|^{1/2}$. We also note from the proof of Theorem 1 of [15] that the set $\{xf + fx : x \in \mathcal{M}\}$ is norm dense in $\mathcal{M}_*$, so from the estimate

$$\Big\|L\Big(\frac{xf+fx}{2}\Big)\Big\| = \|Tx\| \le 2(1+\delta)\pi_1(T)\|xf+fx\|_{\mathcal{M}_*} \quad \text{for every } x \in \mathcal{M},$$

we get

$$\|L(xf+fx)\| \le 4(1+\delta)\pi_1(T)\|xf+fx\|_{\mathcal{M}_*} \quad \text{for every } x \in \mathcal{M}.$$

We conclude that $\|L\| \le 4(1+\delta)\pi_1(T)$ and therefore

$$\pi_{4,2}(K^*) \le \pi_2(T)^{1/2}2(1+\delta)^{1/2}\pi_1(T)^{1/2} \le 2(1+\varepsilon)\pi_1(T).$$

From a result of Mityagin (which appeared for the first time in a paper of Kwapień [8]; see also [3], Theorem 10.3, or [18], Proposition 11.8), the space $\Pi_{4,2}(\ell_2, H^*)$ is isometrically isomorphic to $S_4(\ell_2, H^*)$ so

$$\sigma_4(K^*) \le 2(1+\varepsilon)\pi_1(T)$$

and from Proposition 4.5 of [3] (p. 80), $K \in S_4(H, \ell_2)$ with

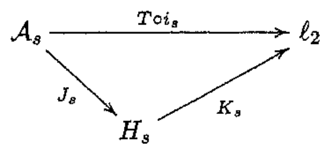
$$\sigma_4(K) = \sigma_4(K^*) \le 2(1+\varepsilon)\pi_1(T).$$

The proof of the proposition is complete. ■

Proof of Theorem 1. Assume first that $\mathcal{A}$ is separable and $T \in \Pi_1(\mathcal{A}, \ell_2)$. The space $\mathcal{A}^{**}$ is a von Neumann algebra and $T^{**} \in \Pi_1(\mathcal{A}^{**}, \ell_2)$. Let $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^{**}$ be the natural embedding and choose a countable dense subset $(a_n)_n$ of $\mathcal{A}$. If $\mathcal{M}$ is the von Neumann algebra generated by $\{i_{\mathcal{A}}(a_n) : n \ge 1\}$, then $\mathcal{M}$ is σ-finite. Also, if we denote by I the inclusion of $\mathcal{M}$ into $\mathcal{A}^{**}$, then I is weak* to weak* continuous. From Proposition 1, the operator $T^{**}\circ I$ factors through a Hilbert space operator K that belongs to the class S_4 and so does $T = T^{**}\circ I \circ i_{\mathcal{A}}$. One can easily verify that this factorization satisfies the conclusion of the theorem.

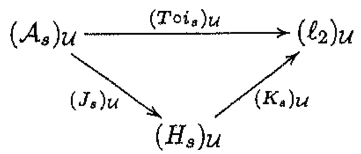
For the general case, we will use the ultraproduct technique. Let $(\mathcal{A}_s)_{s\in S}$ be the collection of all separable C^*-subalgebras of $\mathcal{A}$. As a particular case of Theorem 3.3 of [16] (which is the C^*-version of Proposition 6.2 of [5]), there exists a subset Λ of S and an ultrafilter $\mathcal{U}$ on Λ such that $\mathcal{A}$ is (completely) isometric to a subspace of $(\mathcal{A}_s)_{\mathcal{U}}$. Inspecting the proof in [16], one notices that in our case $\Lambda = S$.

Let $T : \mathcal{A} \to \ell_2$ be a 1-summing operator and $i_s : \mathcal{A}_s \to \mathcal{A}$ be the inclusion map. It is clear that $T\circ i_s \in \Pi_1(\mathcal{A}_s, \ell_2)$ with $\pi_1(T\circ i_s) \le \pi_1(T)$. From the separable case above, there exists a Hilbert space H_s such that the following diagram commutes:



with $\|J_s\| \le 1$ and $\sigma_4(K_s) \le 2(1+\varepsilon)\pi_1(T)$.

From this, one can verify that the following diagram commutes:



It is clear that $\|(J_s)_{\mathcal{U}}\| \le 1$ and since S_4 is a maximal ideal operator, we get $(K_s)_{\mathcal{U}} \in S_4((H_s)_{\mathcal{U}}, (\ell_2)_{\mathcal{U}})$ with $\sigma_4((K_s)_{\mathcal{U}}) \le \lim_{s,\mathcal{U}} \sigma_4(K_s) \le 2(1+\varepsilon)\pi_1(T)$ (see [5], Theorem 8.1).

Let $Q : (\ell_2)_{\mathcal{U}} \to \ell_2$ be defined by $Q((y_s)_s) = \text{weak-lim}_{s,\mathcal{U}}\, y_s$ and $I : \mathcal{A} \to (\mathcal{A}_s)_{\mathcal{U}}$ be the isometric embedding. We claim that $Q\circ (T\circ i_s)_{\mathcal{U}} \circ I = T$.

To see this, notice that for every $x \in \mathcal{A}$, $I(x)_s = 0$ if $x \notin \mathcal{A}_s$ and $I(x)_s = x$ if $x \in \mathcal{A}_s$. So $(T\circ i_s)_{\mathcal{U}}(Ix) = (y_s)_{s\in S}$ where $y_s = 0$ if $x \notin \mathcal{A}_s$ and $y_s = Tx$ if $x \in \mathcal{A}_s$ and by the definition of Q the claim follows.

We get the conclusion of the theorem by setting $J = (J_s)_{\mathcal{U}} \circ I$, $K = Q\circ (K_s)_{\mathcal{U}}$ and $H = (H_s)_{\mathcal{U}}$. ■

For the next simple extension of Theorem 1, we refer to [19] for definitions and examples of JB^*-triples and JBW^*-triples.

COROLLARY 1. *If $\mathcal{A}$ is a JB^*-triple then every absolutely summing operator from $\mathcal{A}$ into ℓ_2 factors through an operator that belongs to the 4-Schatten–von Neumann class.*

Proof. Let $T : \mathcal{A} \to \ell_2$ be an absolutely summing operator. The space $\mathcal{A}^{**}$ is a JBW^*-triple. But every JBW^*-triple is (as a Banach space) isometric to a complemented subspace of a von Neumann algebra (see [1]).

From Theorem 1, T^{**} (and consequently T) factors through an operator that belongs to the class S_4. ■

REMARK 1. We remark that Lemma 1 is valid for any weak* to weakly continuous absolutely summing operator from a σ-finite von Neuman algebra into a general Banach space; in particular, the adjoint of any such operator belongs to the class ideal $\Pi_{2p,p}$ for every $p \geq 1$.

The following finite-dimensional examples show that one cannot improve Theorem 1 to the case of the p-Schatten–von Neumann class for $p < 4$. The type of operators considered below was suggested to the author by Pełczyński.

For $n \geq 1$, $B(\ell_2^n)$ (resp. $HS(\ell_2^n)$) denotes the space of $n \times n$ matrices with the usual operator norm (resp. the Hilbert–Schmidt norm).

Let $I_n : B(\ell_2^n) \to HS(\ell_2^n)$ be the identity operator and set $\alpha_n = \pi_1(I_n)$.

THEOREM 2. *For every $n \geq 1$, let $T_n = I_n/\alpha_n$. There exists an absolute constant $\beta > 0$ (independent of n) such that if H is a Hilbert space, and $J \in \mathcal{L}(B(\ell_2^n), H)$ and $K \in \mathcal{L}(H, HS(\ell_2^n))$ satisfy*:

(i) $\|J\| \leq 1$,

(ii) $T_n = K \circ J$,

then for every $p \geq 2$, $\sigma_p(K) \geq \beta n^{(4-p)/(2p)}$.

For the proof of this theorem, we will recall a few facts about the operator I_n. The following proposition can be found in [4] and [11].

PROPOSITION 2. (1) *There exists a universal constant $c > 0$ such that $\alpha_n = \pi_1(I_n) \leq cn$ for every $n \geq 1$.*

(2) *There exists a universal constant $c' > 0$ such that $\gamma_1(I_n) \geq c' n^{3/2}$ for every $n \geq 1$.*

Proof (of Theorem 2). Let H be a Hilbert space and J and K be operators as in the statement. Since $HS(\ell_2^n)$ is a finite-dimensional Hilbert space, $K : H \to HS(\ell_2^n)$ is a Hilbert–Schmidt operator. Similarly, the adjoint $K^* : HS(\ell_2^n) \to H$ is also a Hilbert–Schmidt operator. One can choose, by the Pietsch Factorization Theorem, a probability space $(\Omega, \Sigma, \lambda)$ such that

$$\begin{array}{ccc} HS(\ell_2^n) & \xrightarrow{K^*} & H \\ {\scriptstyle V}\downarrow & & \uparrow{\scriptstyle U} \\ L_\infty(\lambda) & \xrightarrow{i_2} & L_2(\lambda) \end{array}$$

with $\|V\| = 1$ and $\|U\| = \pi_2(K^*) = \pi_2(K)$. Taking the adjoints,

$$\begin{array}{ccc} H & \xrightarrow{K} & HS(\ell_2^n) \\ {\scriptstyle U^*}\downarrow & & \uparrow{\scriptstyle V^*} \\ L_2(\lambda) & \xrightarrow{i_2^*} & L_1(\lambda) \end{array}$$

Hence the operator T_n factors through $L_1(\lambda)$ as follows:

$$\begin{array}{ccc} B(\ell_2^n) & \xrightarrow{T_n} & HS(\ell_2^n) \\ & {\scriptstyle U_1}\searrow \quad \nearrow{\scriptstyle V^*} & \\ & L_1(\lambda) & \end{array}$$

where $U_1 = i_2^* \circ U^* \circ J$. From the definition of $\gamma_1(T_n)$ (see Definition 4 above), we get the following estimate:

$$\gamma_1(T_n) \leq \|U_1\| \cdot \|V^*\| \leq \|i_2^*\| \cdot \|U^*\| \cdot \|J\| \cdot \|V^*\| \leq \|U^*\| = \pi_2(K).$$

From the above proposition, $c'n^{1/2}/c \leq c'n^{3/2}/\alpha_n \leq \pi_2(K)$.

If we set $\beta := c'/c$, we get $\sigma_2(K) = \pi_2(K) \geq \beta n^{1/2}$ and the theorem is proved for the case $p = 2$.

For $p > 2$, note that $B(\ell_2^n)$ and $HS(\ell_2^n)$ are of dimension n^2 so we can assume without loss of generality that $\dim(H) = n^2$. Let $(s_i(K))_{1 \leq i \leq n^2}$ be the singular numbers of K. It is well known that for every $q > 0$, $\sigma_q(K) = (\sum_{i=1}^{n^2} s_i(K)^q)^{1/q}$. Using Hölder's inequality, we get, for every $p > 2$,

$$\begin{aligned} \sigma_2(K) = \Big(\sum_{i=1}^{n^2} s_i(K)^2\Big)^{1/2} &\leq \Big(\sum_{i=1}^{n^2} s_i(K)^p\Big)^{1/p} \Big(\sum_{i=1}^{n^2} 1\Big)^{(1-2/p)/2} \\ &= \sigma_p(K) n^{1-2/p}. \end{aligned}$$

Hence $\beta n^{1/2} \leq \sigma_2(K) \leq \sigma_p(K) n^{1-2/p}$, which implies that $\sigma_p(K) \geq \beta n^{-1/2+2/p} = \beta n^{(4-p)/(2p)}$. The proof of the theorem is complete. ■

The operator T_n satisfies $\pi_1(T_n) = 1$ but any factorization through any Hilbert space operator has large p-Schatten–von Neumann norm for $p < 4$. This shows that the class S_4 in the statement of Theorem 1 cannot be improved.

The results above lead us to the question of characterizing operators from a C^*-algebra into ℓ_2 that can be factored through Hilbert–Schmidt operators.

THEOREM 3. *Let $\mathcal{A}$ be a C^*-algebra. An operator $T : \mathcal{A} \to \ell_2$ factors through a Hilbert–Schmidt operator if and only if it is L_1-factorable.*

Proof. If T factors through a Hilbert–Schmidt operator then it is L_1-factorable since Hilbert–Schmidt operators are L_1-factorable.

Conversely, assume that T is L_1-factorable i.e. there exist a measure space (Ω,Σ,λ) and operators $U_1:\mathcal{A}\to L_1(\Omega,\Sigma,\lambda)$ and $U_2:L_1(\Omega,\Sigma,\lambda)\to \ell_2$ such that $T=U_2\circ U_1$. From Grothendieck's theorem U_2 is 1-summing. Since $L_1(\Omega,\Sigma,\lambda)$ is of cotype 2, U_1 factors through a Hilbert space (see [13]), which shows that T factors through a Hilbert–Schmidt operator. ■

4. Measure of compactness. In this section, we will provide an application of the main theorem to measure compactness of any absolutely summing operator from C^*-algebras into Hilbert spaces.

Let L be a normed linear space with norm $\|\cdot\|$ and A be a totally bounded set in L. For any given $\varepsilon>0$, we set $N_\varepsilon(A):=$ the infimum of integers m such that there exist subsets $E_1,\ldots,E_m$ of L whose diameters do not exceed 2ε and whose union contains A, i.e.,

$$\bigcup_{k=1}^{n} E_k\supseteq A \quad\text{and}\quad \operatorname{diam}(E_k)\le 2\varepsilon.$$

DEFINITION 6. $H_\varepsilon(A):=\log_2 N_\varepsilon(A)$ is called the *ε-capacity* of the set A.

The number $H_\varepsilon(A)$ (along with other related notions) has been extensively studied in the literature (see for instance [7]).

Our main result in this section answers positively a question raised by Pełczyński and can be viewed as a quantitative version of Theorem 1 of [15].

THEOREM 4. *There exists an absolute constant C such that if $\mathcal{A}$ is a C^*-algebra and $T\in\Pi_1(\mathcal{A},\ell_2)$ with $\pi_1(T)\le 1$, then for every $\varepsilon>0$,*

$$H_\varepsilon(T(B_{\mathcal{A}}))\le C/\varepsilon^4.$$

We will show that Theorem 4 is a consequence of the following result.

THEOREM 5. *Let H be a separable Hilbert space and $S\in S_p(H,\ell_2)$. Then for every $\varepsilon>0$,*

$$H_\varepsilon(S(B_H))\le\frac{\sigma_p(S)^p\varrho(p)}{\varepsilon^p}$$

where $\varrho(p)=(8^p/p+\int_0^{8^{-p}}\ln(1/t)\,dt+1)^p$.

The proof is based on a notion of entropy of operators (see [12], p. 168).

DEFINITION 7. Let E and F be Banach spaces and $S\in\mathcal{L}(E,F)$. The nth (*outer*) *entropy number* $e_n(S)$ of the operator S is the minimum of $\delta>0$ such that there exists a finite sequence $y_1,\ldots,y_q\in F$ with $q\le 2^{n-1}$ and $S(B_E)\subseteq\bigcup_{i=1}^q\{y_i+\delta B_F\}$.

This notion was formally introduced by Pietsch primarily motivated by some earlier work of Mityagin and Pełczyński on ε-capacity of operators (see [9]). It is clear that $e_{n+1}(S)\le e_n(S)$ for every operator S and every $n\in\mathbb{N}$.

For diagonal Hilbert space operators, the following proposition was proved by Pietsch.

PROPOSITION 3 ([12], p. 174). *Let $S\in\mathcal{L}(\ell_2)$ such that $S((\xi_n)_n)=(\alpha_n\xi_n)_{n\ge1}$ and $(\alpha_n)_n\in c_0$. Then*

$$\Big(\sum_{n=1}^\infty e_n(S)^p\Big)^{1/p}\le K_p\Big(\sum_{n=1}^\infty|\alpha_n|^p\Big)^{1/p}.$$

Proof of Theorem 5. Let $1<p<\infty$ and $S\in S_p(H,\ell_2)$. The operator S admits an orthonormal representation

$$S=\sum_{n=1}^\infty\alpha_n(\cdot,h_n)f_n, \tag{†}$$

where (h_n) and (f_n) are orthonormal sequences in H and ℓ_2 respectively, and $(\alpha_n)_n\in\ell_p$. We can choose this representation so that $0\le\alpha_{n+1}\le\alpha_n$ for all admissible indices. Let $(e_n)_n$ be the unit vector basis of ℓ_2. Let $Z=\overline{\operatorname{span}}\{h_n:n\in\mathbb{N}\}$ in H. Since $S(B_Z)=S(B_H)$, we can assume without loss of generality that $H=Z$.

Let $I:\ell_2\to H$ be defined by $Ie_n=h_n$ for every $n\in\mathbb{N}$ and $J:\ell_2\to\ell_2$ so that $J(f_n)=e_n$ for every $n\in\mathbb{N}$. Let $\widetilde{S}:=J\circ S\circ I$. Clearly, $\widetilde{S}\in S_p(\ell_2,\ell_2)$, and I and J are isometries.

For every $x\in\ell_2$, we have

$$\widetilde{S}x=\sum_{n=1}^\infty\alpha_n(Ix,h_n)J(f_n)=\sum_{n=1}^\infty\alpha_n(x,I^*h_n)e_n=\sum_{n=1}^\infty\alpha_n(x,e_n)e_n.$$

So for every $x=(x_n)_n\in\ell_2$, $Sx=(\alpha_nx_n)_{n\ge1}$. Hence $\widetilde{S}$ satisfies the assumption of the above proposition and therefore

$$\Big(\sum_{n=1}^\infty(e_n(\widetilde{S}))^p\Big)^{1/p}\le K_p\Big(\sum_{n=1}^\infty|\alpha_n|^p\Big)^{1/p}\le K_p\sigma_p(S).$$

For $\varepsilon>0$, define $k(\varepsilon):=\max\{k:e_k(\widetilde{S})\ge\varepsilon\}$. We have

$$(K_p\sigma_p(S))^p\ge\sum_{n=1}^\infty(e_n(\widetilde{S}))^p\ge\sum_{n=1}^{k(\varepsilon)}(e_n(\widetilde{S}))^p\ge\varepsilon^pk(\varepsilon)$$

so

$$k(\varepsilon)\le\Big(\frac{K_p\sigma_p(S)}{\varepsilon}\Big)^p.$$

From the definition of $k(\varepsilon)$ we get $e_{k(\varepsilon)+1}(\widetilde{S}) \le \varepsilon$, and the definition of the nth entropy of $\widetilde{S}$ implies that there exist $\delta \le \varepsilon$ and $\{y_1, \ldots, y_q\} \subseteq \ell_2$, with $q \le 2^{k(\varepsilon)}$, so that $\widetilde{S}(B_{\ell_2}) \subset \{y_1, \ldots, y_q\} + \delta B_{\ell_2}$, i.e. the set $\widetilde{S}(B_{\ell_2})$ can be covered by $2^{k(\varepsilon)}$ balls of radius $\delta \le \varepsilon$, so $N_\varepsilon(\widetilde{S}(B_{\ell_2})) \le 2^{k(\varepsilon)}$ and

$$H_\varepsilon(\widetilde{S}(B_{\ell_2})) \le k(\varepsilon) \le \frac{(\sigma_p(S)K_p)^p}{\varepsilon^p}.$$

Now since J is an isometry, $H_\varepsilon(\widetilde{S}(B_{\ell_2})) = H_\varepsilon(S \circ I(B_{\ell_2}))$; also, by the definition of I, $I(B_{\ell_2}) = B_H$ so

$$H_\varepsilon(S(B_H)) = H_\varepsilon(\widetilde{S}(B_{\ell_2})) \le \frac{\sigma_p(S)^p K_p^p}{\varepsilon^p}$$

and setting $\varrho(p) = K_p^p$, the theorem is proved.

The estimate on K_p can be found in Pietsch's book [12] (p. 174). ∎

Proof of Theorem 4. If $\mathcal{A}$ is a C^*-algebra and $T \in \Pi_1(\mathcal{A}, \ell_2)$ with $\pi_1(T) \le 1$, then one can deduce from Theorems 1 and 5 that $H_\varepsilon(T(B_{\mathcal{A}})) \le 3^4\varrho(4)/\varepsilon^4$. In fact, one can choose H, J and K (as in Theorem 1) so that $\sigma_4(K) \le 3$, so from Theorem 5, $H_\varepsilon(K(B_H)) \le 3^4\varrho(4)/\varepsilon^4$ and since $\|J\| \le 1$, $H_\varepsilon(T(B_{\mathcal{A}})) \le 3^4\varrho(4)/\varepsilon^4$. Hence if we set $C = 3^4\varrho(4)$, the proof of the theorem is complete. ∎

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