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Functions with derivatives in spaces of Morrey type and elliptic equations in unbounded domains

by

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Abstract. We introduce a sort of "local" Morrey spaces and show an existence and uniqueness theorem for the Dirichlet problem in unbounded domains for linear second order elliptic partial differential equations with principal coefficients "close" to functions having derivatives in such spaces.

Introduction. In [TTV] the space $M^{p,\lambda}(\Omega) = M^{p,\lambda}(\Omega,1)$ was introduced, for an open subset Ω of $\mathbb{R}^n$, with $p \in [1,\infty]$ and $\lambda \in [0,n[$, where $M^{p,\lambda}(\Omega,t)$, $t \in \mathbb{R}_+$, is defined as the space of functions $g \in L^p_{\mathrm{loc}}(\overline{\Omega})$ such that

$$(1)\qquad \|g\|_{M^{p,\lambda}(\Omega,t)} = \sup_{\substack{\tau\in]0,t]\\ x\in\Omega}} \Big(\tau^{-\lambda} \int_{\Omega\cap B(x,\tau)} |g|^p\Big)^{1/p} < \infty.$$

We observe that, when Ω is a bounded set, the space $M^{p,\lambda}(\Omega)$ reduces to the classical Morrey space, denoted by $L^M_{p,\lambda}(\Omega)$ (see [KJF]) or $L^{p,\lambda}(\Omega)$ (see [CF], [CFR]).

Subsequently in [V] the subspace $W^1M^{p,\lambda}(\Omega)$ of the functions $g \in W^{1,1}_{\mathrm{loc}}(\overline{\Omega})$ such that $g, g_x \in M^{p,\lambda}(\Omega)$, equipped with the norm

$$(2)\qquad \|g\|_{W^1M^{p,\lambda}(\Omega)} = \|g\|_{M^{p,\lambda}(\Omega)} + \|g_x\|_{M^{p,\lambda}(\Omega)},$$

has been defined, where

$$g_{x_i} = \frac{\partial g}{\partial x_i}, \quad g_x = \Big(\sum_{i=1}^n g_{x_i}^2\Big)^{1/2}.$$

It turns out (see Theorem 4.1 and Corollary 4.2 of [V]) that, if Ω is sufficiently regular, then there exists a bounded linear extension operator

$$p : W^1M^{p,\lambda}(\Omega) \to W^1M^{p,\lambda}(\mathbb{R}^n).$$

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Here we define more general spaces, denoted by $W^1M^{p,\lambda}_{\rm loc}(\overline{\Omega})$, whose functions are required to be in $W^1M^{p,\lambda}(\Omega)$ only when multiplied by functions of a suitable sequence in $C^\infty_0(\mathbb{R}^n)$, as specified in Section 1.

Actually, the extension operator p can be defined on $L^1_{\rm loc}(\overline{\Omega})$: in Section 2 we show that

$$p(W^1M^{p,\lambda}_{\rm loc}(\overline{\Omega})) \subset W^1M^{p,\lambda}_{\rm loc}(\mathbb{R}^n),$$

and that, if $g \in L^1_{\rm loc}(\overline{\Omega})$ satisfies certain bounds in Ω, then the same bounds hold for $p(g)$ in a suitable neighbourhood of Ω.

Then, for $n \geq 3$, we consider the Dirichlet problem

$$u \in W^2(\Omega) \cap W^1_0(\Omega), \qquad Lu = f, \tag{3}$$

for the linear operator $L : W^2(\Omega) \to L^2(\Omega)$ defined as

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + au, \tag{4}$$

where $a_{ij} = a_{ji} \in L^\infty(\Omega)$, a_i belongs to $M^{s,n-s}_0(\Omega)$, the closure of $C^\infty_0(\Omega)$ in $M^{s,n-s}(\Omega)$, for some $s \in]2,n]$, and a belongs to $\widetilde{M}^p(\Omega)$, the closure of $L^\infty(\Omega)$ in $M^{p,0}(\Omega)$ for a suitable $p \in \mathbb{R}_+$.

We recall that, when Ω is bounded, problem (3) has been dealt with by F. Chiarenza–M. Franciosi in [CF] under the following assumption, which is a generalization of the classical one due to C. Miranda [M]:

$$(a_{ij})_x \in VL^{s,n-s}(\Omega) \quad \text{for some } s \in]2,n], \quad i,j = 1,\ldots,n,$$

where $VL^{s,n-s}(\Omega)$ is the subspace of $L^{s,n-s}(\Omega)$ of functions g such that $\|g\|_{M^{p,\lambda}(\Omega,t)} \to 0$ as $t \to 0$.

Successively the problem has been studied when Ω is unbounded: 1) in [CLM$_1$] with

$$(a_{ij})_x \in M^{s,n-s}_0(\Omega) \quad \text{for some } s \in]2,n], \quad i,j = 1,\ldots,n;$$

2) in [CLM$_2$] with

$$a_{ij} \in W^1\widetilde{M}^{s,n-s}_{\rm loc}(\overline{\Omega}) \quad \text{for some } s \in]2,n],$$
$$\lim_{|x|\to\infty} a_{ij}(x) = a^0_{ij}, \; i,j = 1,\ldots,n,$$

where $W^1\widetilde{M}^{s,n-s}_{\rm loc}(\overline{\Omega})$ is a suitable subspace of $W^1M^{s,n-s}_{\rm loc}(\overline{\Omega})$ which will be specified in the next section.

In this paper we are concerned with problem (3) when Ω is an unbounded open set, and with more general assumptions on the coefficients a_{ij} of the operator L in comparison to [CLM$_1$] and [CLM$_2$].

We suppose that the coefficients a_{ij} are sufficiently close to uniformly elliptic coefficients e_{ij}, belonging to $W^1\widetilde{M}^{s,n-s}_{\rm loc}(\overline{\Omega})$ for some $s \in]2,n]$, and that both a_{ij} and e_{ij} are convergent at infinity.

We use the extension operator in order to get an approximation of the functions a_{ij} by means of regular coefficients a^k_{ij}, which we take to define, according to (4), a sequence of operators L_k, $k \in \mathbb{N}$.

We obtain an a-priori bound for the operators L and L_k. By using such bounds and a classical method due to M. Chicco (see [C$_1$], [C$_2$]) and applied in other papers (see [TT$_2$], [TT$_3$]), and assuming that

$$\operatorname*{ess\,inf}_{\Omega} a > 0$$

we find a solution u of (3) as limit of solutions u_k of related Dirichlet problems for L_k.

By Fredholm theory and compactness results for the multiplication operator

$$u \in W^1(\Omega) \to gu \in L^2(\Omega),$$

when g belongs to $M^{s,n-s}(\Omega)$ for some $s \in]2,n]$, we finally prove an existence and uniqueness theorem for (3).

1. The spaces $M^{p,\lambda}_{\rm loc}(\Omega)$ and $W^1M^{p,\lambda}_{\rm loc}(\Omega)$.

Let Ω be an open subset in $\mathbb{R}^n$. We denote by $\Sigma(\Omega)$ the σ-algebra of Lebesgue measurable subsets of Ω and, for every $E \in \Sigma(\mathbb{R}^n)$, by $|E|$ the Lebesgue measure of E and by χ_E the characteristic function of E. For every $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, we let $B(x,t)$ be the open sphere centred at x with radius t, and for $E \in \Sigma(\mathbb{R}^n)$ we put $E(x,t) = E \cap B(x,t)$, also setting $E(x) = E(x,1)$.

If $E \in \Sigma(\Omega)$, $p \in [1,\infty[$ and $g \in L^p(E)$, we put

$$|g|_{p,E} = \|g\|_{L^p(E)}.$$

We denote by $L^p_{\rm loc}(\overline{\Omega})$ the class of measurable functions g on Ω such that $\zeta g \in L^p(\Omega)$ for every $\zeta \in C^\infty_0(\mathbb{R}^n)$, and by $W^{1,p}_{\rm loc}(\overline{\Omega})$ the subspace of functions $g \in L^p_{\rm loc}(\overline{\Omega})$ such that $g_{x_i} \in L^p_{\rm loc}(\overline{\Omega})$, $i = 1,\ldots,n$.

For each $p \in [1,\infty[$ and $\lambda \in [0,n[$, with the above notations, we recall that $M^{p,\lambda}(\Omega,t)$ is the space of functions $g \in L^p_{\rm loc}(\overline{\Omega})$ such that

$$\|g\|_{M^{p,\lambda}(\Omega,t)} = \sup_{\substack{\tau\in]0,t] \\ x\in\Omega}} \tau^{-\lambda/p} |g|_{p,\Omega(x,\tau)} < \infty, \tag{1.1}$$

and

$$M^{p,\lambda}(\Omega) = M^{p,\lambda}(\Omega,1).$$

We also set $M^{p,0}(\Omega) = M^p(\Omega)$.

We recall that $g \in \widetilde{M}^{p,\lambda}(\Omega)$ if and only if there exists a function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$, convergent to zero as $\delta \to 0$, such that

$$\|\chi_E g\|_{M^{p,\lambda}(\Omega)} \leq \sigma(\delta) \tag{1.2}$$

for each $E \in \Sigma(\Omega)$ such that $\sup_{x\in\Omega}|E(x)| \le \delta$. Such a function will be called a modulus of continuity of g and denoted by $\sigma[g,\Omega]$.

For $r \in \mathbb{R}_+$, we set $B_r = B(0,r)$ and denote by ζ_r a function of class $C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\zeta_r \subset B_{2r}, \quad 0 \le \zeta_r \le 1, \quad \zeta_{r|B_r} = 1, \quad (\zeta_r)_x \le 2/r.$$

We also recall that

$$M_0^{p,\lambda}(\Omega) = \{g \in \widetilde{M}^{p,\lambda}(\Omega) : \lim_{r\to\infty}\|(1-\zeta_r)g\|_{M^{p,\lambda}(\Omega)} = 0\}, \tag{1.3}$$

also setting $M_0^{p,0}(\Omega) = M_0^p(\Omega)$.

We shall employ results about the multiplication operator

$$u \in W^1(\Omega) \to gu \in L^2(\Omega) \tag{1.4}$$

with g belonging to some of the spaces defined above, which have been proved in [TTV], [TT$_1$], [GTT].

LEMMA 1.1. *Let $n > 2$ and Ω be an open subset of $\mathbb{R}^n$ with the cone property.*

If $g \in M^{s,n-s}(\Omega)$ with $s \in]2,n]$, then $gu \in L^2(\Omega)$ for any $u \in W^1(\Omega)$, and the operator (1.4) *is bounded.*

If $g \in \widetilde{M}^{s,n-s}(\Omega)$, then there exists $c(\varepsilon) \in \mathbb{R}_+$ such that

$$|gu|_{2,\Omega} \le \varepsilon|u_x|_{2,\Omega} + c(\varepsilon)|u|_{2,\Omega} \quad \forall u \in W^1(\Omega).$$

If $g \in M_0^{s,n-s}(\Omega)$, then there exist $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega(\varepsilon)$ of Ω such that

$$|gu|_{2,\Omega} \le \varepsilon\|u\|_{W^1(\Omega)} + c(\varepsilon)|u|_{2,\Omega(\varepsilon)} \quad \forall u \in W^1(\Omega),$$

and the operator (1.4) *is compact.*

LEMMA 1.2. *Let Ω be an open subset of $\mathbb{R}^n$ with the cone property. Let $k \in \mathbb{N}$ and $p \in [2,\infty[$ be such that*

$$p = 2 \quad \text{if } n < 2k, \qquad p > 2 \quad \text{if } n = 2k, \qquad p = n/k \quad \text{if } n > 2k.$$

If $g \in M^p(\Omega)$, then for any $u \in W^k(\Omega)$ we have $gu \in L^2(\Omega)$, and the operator

$$u \in W^k(\Omega) \to gu \in L^2(\Omega) \tag{1.5}$$

is bounded.

If $g \in \widetilde{M}^p(\Omega)$, then there exists $c(\varepsilon) \in \mathbb{R}_+$ such that

$$|gu|_{2,\Omega} \le \varepsilon \sum_{|\alpha|=k}|D^\alpha u|_{2,\Omega} + c(\varepsilon)|u|_{2,\Omega} \quad \forall u \in W^k(\Omega).$$

If $g \in M_0^p(\Omega)$, then there exist $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega(\varepsilon)$ of Ω such that

$$|gu|_{2,\Omega} \le \varepsilon\|u\|_{W^k(\Omega)} + c(\varepsilon)|u|_{2,\Omega(\varepsilon)} \quad \forall u \in W^k(\Omega);$$

in this case the operator (1.5) *is compact.*

REMARK 1.3. We recall (see [TTV]) that $g \in M_0^{p,\lambda}(\Omega)$ if and only if $g \in M^{p,\lambda}(\Omega)$ and

$$\lim_{t\to 0}\|g\|_{M^{p,\lambda}(\Omega,t)} = 0, \qquad \lim_{|x|\to\infty}|g|_{p,\Omega(x)} = 0.$$

We also remark that, if Ω is bounded, then $M_0^{p,\lambda} = \widetilde{M}^{p,\lambda}(\Omega)$.

Now, we introduce (see [CLM$_2$]) a more general class of spaces by setting

$$M_{\mathrm{loc}}^{p,\lambda}(\overline{\Omega}) = \{g \in L^1_{\mathrm{loc}}(\overline{\Omega}) : \zeta_r g \in M^{p,\lambda}(\Omega)\ \forall r \in \mathbb{R}_+\}, \tag{1.6}$$

$$\widetilde{M}_{\mathrm{loc}}^{p,\lambda}(\overline{\Omega}) = \{g \in L^1_{\mathrm{loc}}(\overline{\Omega}) : \zeta_r g \in \widetilde{M}^{p,\lambda}(\Omega)\ \forall r \in \mathbb{R}_+\}. \tag{1.7}$$

The spaces

$$W^1M^{p,\lambda}(\Omega) = \{g \in W^{1,1}_{\mathrm{loc}}(\overline{\Omega}) : g, g_x \in M^{p,\lambda}(\Omega)\}, \tag{1.8}$$

defined in [V], can in turn be generalized by letting

$$W^1M_{\mathrm{loc}}^{p,\lambda}(\overline{\Omega}) = \{g \in W^{1,1}_{\mathrm{loc}}(\overline{\Omega}) : g, g_x \in M_{\mathrm{loc}}^{p,\lambda}(\overline{\Omega})\}, \tag{1.9}$$

$$W^1\widetilde{M}_{\mathrm{loc}}^{p,\lambda}(\overline{\Omega}) = \{g \in W^{1,1}_{\mathrm{loc}}(\overline{\Omega}) : g, g_x \in \widetilde{M}_{\mathrm{loc}}^{p,\lambda}(\overline{\Omega})\}. \tag{1.10}$$

2. The extension operator $p \in B(W^1M^{p,\lambda}(\Omega), W^1M^{p,\lambda}(\mathbb{R}^n))$. We shall consider an open subset Ω of $\mathbb{R}^n$ with the following regularity property (see [AD]):

(Φ) there exist a locally finite open covering $\{U_h\}_{h\in\mathbb{N}}$ of $\partial\Omega$ and homeomorphisms $\Phi_h = (\Phi_{h,1},\ldots,\Phi_{h,n}) : \overline{U}_h \to \overline{B}_1$, $h \in \mathbb{N}$, with constants $\delta, M \in \mathbb{R}_+$, $N \in \mathbb{N}$, such that

$$\bigcup_{h\in\mathbb{N}}\Phi_h^{-1}(B(0,1/4)) \supset \Omega_\delta^- = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \delta\}; \tag{2.1}$$

(2.2) every $x \in \mathbb{R}^n$ belongs to at most N of the U_h's;

(2.3) for every $h \in \mathbb{N}$,

$$\Phi_h(U_h \cap \Omega) = B_1^+ = \{x \in B_1 : x_n > 0\},$$
$$\Phi_h(U_h \cap \partial\Omega) = \{x \in B_1 : x_n = 0\};$$

(2.4) for every $h \in \mathbb{N}$, $(\Phi_{h,k})_x \in L^\infty(U_h)$, $(\Phi_{h,k}^{-1})_{x'} \in L^\infty(B_1)$, $k = 1,\ldots,n$, and

$$\|\Phi_{h,k}\|_{L^\infty(U_h)}, \|\Phi_{h,k}^{-1}\|_{L^\infty(B_1)}, \|(\Phi_{h,k})_{x_i}\|_{L^\infty(U_h)}, \|(\Phi_{h,k}^{-1})_{x'_i}\|_{L^\infty(B_1)} \le M,$$
$$\text{for } i = 1,\ldots,n.$$

In [V] (Theorem 3.3) the following extension result is proved, where

$$\gamma : (x_1, \ldots, x_n) \in \mathbb{R}^n \to (x_1, \ldots, -x_n) \in \mathbb{R}^n,$$

and $p \in [1, \infty[$, $\lambda \in [0, n]$.

THEOREM 2.1. *Let Ω be an open subset of $\mathbb{R}^n$ with property (Φ). Let $E \in \Sigma(\mathbb{R}^n)$, $E_0 = E \cap \Omega$, and $E_h = [(E \cap U_h) \cup \Phi_h^{-1}(\gamma(\Phi_h(E \cap U_h)))] \cap \Omega$, $h \in \mathbb{N}$, where $\{U_h\}_{h\in\mathbb{N}}$ and $\Phi_h : \overline{U}_h \longrightarrow \overline{B}_1$, $h \in \mathbb{N}$, are respectively the open covering of $\partial\Omega$ and the homeomorphisms of (Φ), satisfying (2.1)–(2.4). Then there exists a linear extension operator*

$$p : L^1_{\mathrm{loc}}(\overline{\Omega}) \to L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

such that $p(W^{1,1}_{\mathrm{loc}}(\overline{\Omega})) \subset W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, and

(A) *if*

$$\chi_{E_0} g, \chi_{E_h} g \in M^{p,\lambda}(\Omega), \quad h \in \mathbb{N},$$

then

$$\|\chi_E p(g)\|_{M^{p,\lambda}(\mathbb{R}^n)} \le c \sup_{h\in\mathbb{N}_0} \|\chi_{E_h} g\|_{M^{p,\lambda}(\Omega)}, \tag{2.5}$$

with $c \in \mathbb{R}_+$ depending only on n and on the constants implied in (Φ);

(B) *if also*

$$\chi_{E_0} g_x, \chi_{E_h} g_x \in M^{p,\lambda}(\Omega), \quad h \in \mathbb{N},$$

then

$$\|\chi_E (p(g))_x\|_{M^{p,\lambda}(\mathbb{R}^n)} \le c\big(\sup_{h\in\mathbb{N}_0} \|\chi_{E_h} g\|_{M^{p,\lambda}(\Omega)} + \sup_{h\in\mathbb{N}_0} \|\chi_{E_h} g_x\|_{M^{p,\lambda}(\Omega)}\big), \tag{2.6}$$

with $c \in \mathbb{R}_+$ depending on n and on the constants implied by (Φ).

The extension operator p maps $W^1M^{p,\lambda}(\Omega)$ into $W^1M^{p,\lambda}(\mathbb{R}^n)$, as a consequence of the following

COROLLARY 2.2. *Let Ω be an open subset of $\mathbb{R}^n$ with property (Φ). Then*

(A) $$g \in M^{p,\lambda}(\Omega) \Rightarrow p(g) \in M^{p,\lambda}(\mathbb{R}^n),$$

and we have

$$\|p(g)\|_{M^{p,\lambda}(\mathbb{R}^n)} \le c\|g\|_{M^{p,\lambda}(\Omega)}. \tag{2.7}$$

Moreover,

(B) $$g \in W^1M^{p,\lambda}(\Omega) \Rightarrow (p(g))_x \in M^{p,\lambda}(\mathbb{R}^n),$$

and we have

$$\|(p(g))_x\|_{M^{p,\lambda}(\mathbb{R}^n)} \le c\|g\|_{W^1M^{p,\lambda}(\Omega)}, \tag{2.8}$$

whence, by (A), *$p(g) \in W^1M^{p,\lambda}(\mathbb{R}^n)$ and*

$$\|p(g)\|_{W^1M^{p,\lambda}(\mathbb{R}^n)} \le 2c\|g\|_{W^1M^{p,\lambda}(\Omega)}, \tag{2.9}$$

with $c \in \mathbb{R}_+$ depending on n and on the constants implied by (Φ).

COROLLARY 2.3. *Let Ω be an open subset of $\mathbb{R}^n$ with property (Φ). Then*

(A) $$g \in L^\infty(\Omega) \Rightarrow p(g) \in L^\infty(\mathbb{R}^n),$$

and we have

$$\|p(g)\|_{L^\infty(\mathbb{R}^n)} \le c\|g\|_{L^\infty(\Omega)}. \tag{2.10}$$

Moreover,

(B) $$g \in W^{1,\infty}(\Omega) \Rightarrow p(g) \in W^{1,\infty}(\mathbb{R}^n),$$

and we have

$$\|(p(g))_x\|_{L^\infty(\mathbb{R}^n)} \le c\|g\|_{W^{1,\infty}(\Omega)}, \tag{2.11}$$

whence, by (A), *$p(g) \in W^{1,\infty}(\mathbb{R}^n)$ and*

$$\|p(g)\|_{W^{1,\infty}(\mathbb{R}^n)} \le 2c\|g\|_{W^{1,\infty}(\Omega)}, \tag{2.12}$$

with $c \in \mathbb{R}_+$ depending on n and on the constants implied in (Φ).

Proof. (A) If $g \in L^\infty(\Omega)$, then $g \in M^{p,\lambda}(\Omega)$ for every $p \in [1, \infty[$ and $\lambda \in [0, n]$. By using Corollary 2.2(A) with $p = 1$, $\lambda = n$, for every $x \in \mathbb{R}^n$ and $t \in]0, 1]$ we have

$$\frac{\omega_n}{|B(x,t)|} \int_{B(x,t)} |p(g)| \le c\|g\|_{M^{1,n}(\Omega,t)} \le c\omega_n \|g\|_{L^\infty(\Omega)},$$

with $\omega_n = |B_1|$, whence (2.10) follows as $t \to 0$ by the Lebesgue Theorem.

(B) The proof of (2.11) is similar, with $(p(g))_x$ instead of $p(g)$, applying (B) of Corollary 2.2 instead of (A). ■

REMARK 2.4. By construction of the extension p (see the proof of Theorem 3.3 of [V]), there exists $\eta \in \mathbb{R}_+$ such that certain bounds, if satisfied by $g \in L^1_{\mathrm{loc}}(\overline{\Omega})$ on Ω, hold for the extended function $p(g)$ on

$$\Omega(\eta) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, \Omega) < \eta\}. \tag{2.13}$$

Indeed, if $x \in \Omega(\eta) \setminus \overline{\Omega}$, then there exist $x_1 \in U_{k_1} \cap \Omega, \ldots, x_h \in U_{k_h} \cap \Omega$, with $h \in \{1, \ldots, N\}$, such that for every $g \in L^1_{\mathrm{loc}}(\Omega)$,

$$p(g)(x) = \theta_{k_1}(x)g(x_1) + \ldots + \theta_{k_h}(x)g(x_h),$$

where $\theta_{k_1}, \ldots, \theta_{k_h} \in C_0^\infty(\mathbb{R}^n)$ and $\theta_{k_1}(x) + \ldots + \theta_{k_h}(x) = 1$.

Moreover, since the U_j's are uniformly bounded, there exists $\widetilde{r} \in \mathbb{R}_+$ such that, for $r \ge \widetilde{r}$,

$$|x| \ge r \Rightarrow |x_k| \ge r - \widetilde{r}, \quad k = 1, \ldots, h,$$

for every $x \in \Omega(\eta) \setminus \overline{\Omega}$.

Reciprocally, every $y \in \Omega$ occurs, with its image $g(y)$, in at most N of the values $p(g)(x), x \in \Omega(\eta) \setminus \overline{\Omega}$.

We also remark that, if $\mathrm{supp}\, g$ is bounded, then $\mathrm{supp}\, p(g)$ is compact.

COROLLARY 2.5. *Let Ω be an open subset of $\mathbb{R}^n$ with property (Φ). Let $\mu \in \mathbb{R}$. If*

$$g \geq \mu \quad (\text{resp. } g \leq \mu) \quad \text{in } \Omega,$$

then

$$p(g) \geq \mu \quad (\text{resp. } p(g) \leq \mu) \quad \text{in } \Omega(\eta),$$

with $\Omega(\eta)$ given by Remark 2.4.

Proof. Let $x \in \Omega(\eta) \setminus \overline{\Omega}$. Then

$$p(g)(x) = \theta_{k_1}(x)g(x_1) + \ldots + \theta_{k_h}(x)g(x_h) \geq (\theta_{k_1}(x) + \ldots + \theta_{k_h}(x))\mu = \mu,$$

if $g \geq \mu$ in Ω; and analogously $p(g)(x) \leq \mu$ if $g \leq \mu$ in Ω. ■

COROLLARY 2.6. *Let Ω be an open subset of $\mathbb{R}^n$ with property (Φ). If*

$$g_\infty = \lim_{\substack{|x|\to\infty \\ x\in\Omega}} g(x)$$

then

$$g_\infty = \lim_{\substack{|x|\to\infty \\ x\in\Omega(\eta)}} p(g)(x).$$

Proof. We prove the result when $g_\infty \in \mathbb{R}$, the other cases being similar. We fix $\varepsilon \in \mathbb{R}_+$. Then there exists $r_\varepsilon > \widetilde{r}$ such that

$$g_\infty - \varepsilon \leq g(x) \leq g_\infty + \varepsilon$$

for every $x \in \Omega$ such that $|x| \geq r_\varepsilon - \widetilde{r}$. By reasoning as in Corollary 2.5, and observing that

$$x \in \Omega(\eta) \setminus \overline{\Omega} \text{ and } |x| \geq r_\varepsilon + \widetilde{r} \Rightarrow x_k \in \Omega \text{ and } |x_k| \geq r_\varepsilon, \quad k = 1, \ldots, h,$$

we obtain

$$g_\infty - \varepsilon \leq p(g)(x) \leq g_\infty + \varepsilon$$

for each $x \in \Omega(\eta)$ such that $|x| \geq r_\varepsilon$. ■

COROLLARY 2.7. *Let Ω be an open subset of $\mathbb{R}^n$ with property (Φ).* Then

(A) $$g \in M^{p,\lambda}_{\text{loc}}(\overline{\Omega}) \Rightarrow p(g) \in M^{p,\lambda}_{\text{loc}}(\mathbb{R}^n);$$

(B) $$g \in W^1 M^{p,\lambda}_{\text{loc}}(\overline{\Omega}) \Rightarrow p(g) \in W^1 M^{p,\lambda}_{\text{loc}}(\mathbb{R}^n).$$

Proof. (A) Let $g \in M^{p,\lambda}_{\text{loc}}(\overline{\Omega})$. Let $r \in \mathbb{R}_+$, $F \in \Sigma(\mathbb{R}^n)$ and set $E = B_{2r} \cap F$ in Theorem 2.1.

Since the E_j's, $j \in \mathbb{N}_0$, are bounded (because of (2.4) of property (Φ) and of the definition of γ), for every $h \in \mathbb{N}_0$ there exists $r_h \in \mathbb{R}_+$ such that $\chi_{E_h} \leq \zeta_{r_h}$, and therefore

$$\chi_{E_h} g \in M^{p,\lambda}(\Omega), \quad h \in \mathbb{N}_0.$$

Hence we can apply Theorem 2.1(A) to get

$$\|\chi_E p(g)\|_{M^{p,\lambda}(\mathbb{R}^n)} \leq c \sup_{h\in\mathbb{N}_0} \|\chi_{E_h} g\|_{M^{p,\lambda}(\Omega)}. \tag{2.14}$$

Since, by property (Φ), B_{2r} intersects a finite number of U_h (see [V]), there exists $N_r \in \mathbb{N}$ such that

$$\|\chi_E p(g)\|_{M^{p,\lambda}(\mathbb{R}^n)} \leq c \max_{h=0,1,\ldots N_r} \|\chi_{E_h} g\|_{M^{p,\lambda}(\Omega)}. \tag{2.15}$$

We observe that

$$E_0 = (B_{2r} \cap F) \cap \Omega = B_{2r} \cap (F \cap \Omega) = B_{2r} \cap F_0,$$

and that, since U_h are bounded, there exists $\widetilde{r} \in \mathbb{R}_+$ such that $B_{2r} \cup (\bigcup_{h=1}^{N_r} U_h) \subset B_{\widetilde{r}}$ and

$$\begin{aligned} E_h &= [(E \cap U_h) \cup \Phi_h^{-1}(\gamma(\Phi_h(E \cap U_h)))] \cap \Omega \\ &\subset \Big[B_{2r} \cup \bigcup_{h=1}^{N_r} U_h\Big] \cap [(F \cap U_h) \cup \Phi_h^{-1}(\gamma(\Phi_h(F \cap U_h)))] \cap \Omega \\ &\subset B_{\widetilde{r}} \cap F_h. \end{aligned}$$

Therefore, from (2.15) we deduce that

$$\begin{aligned} \|\chi_F \zeta_r p(g)\|_{M^{p,\lambda}(\mathbb{R}^n)} &\leq \|\chi_F \chi_{B_{2r}} p(g)\|_{M^{p,\lambda}(\mathbb{R}^n)} = \|\chi_E p(g)\|_{M^{p,\lambda}(\mathbb{R}^n)} \\ &\leq c \max_{h=0,1,\ldots,N_r} \|\chi_{E_h} g\|_{M^{p,\lambda}(\Omega)} \\ &\leq c \max_{h=0,1,\ldots,N_r} \|\chi_{B_{\widetilde{r}}} \chi_{F_h} g\|_{M^{p,\lambda}(\Omega)} \\ &\leq c \max_{h=0,1,\ldots,N_r} \|\zeta_{\widetilde{r}} \chi_{F_h} g\|_{M^{p,\lambda}(\Omega)} \end{aligned} \tag{2.16}$$

for any $F \in \Sigma(\mathbb{R}^n)$. Applying (2.16) with $F = \mathbb{R}^n$, we obtain $\zeta_r p(g) \in M^{p,\lambda}(\mathbb{R}^n)$.

(B) If we assume that also $g_x \in M^{p,\lambda}_{\text{loc}}(\overline{\Omega})$, then by the same argument, but using (B) instead of (A) of Theorem 2.1, we get the inequality

$$\begin{aligned} &\|\chi_F \zeta_r (p(g))_x\|_{M^{p,\lambda}(\mathbb{R}^n)} \\ &\quad \leq c\big(\max_{h=0,1,\ldots,N_r} \|\zeta_{\widetilde{r}} \chi_{F_h} g\|_{M^{p,\lambda}(\Omega)} + \max_{h=0,1,\ldots,N_r} \|\zeta_{\widetilde{r}} \chi_{F_h} g_x\|_{M^{p,\lambda}(\Omega)}\big), \end{aligned} \tag{2.17}$$

which, applied with $F = \mathbb{R}^n$, yields $\zeta_r (p(g))_x \in M^{p,\lambda}(\mathbb{R}^n)$. ■

COROLLARY 2.8. *Let Ω be an open subset of $\mathbb{R}^n$ with property (Φ). Then*

(A) $$g \in \widetilde{M}^{p,\lambda}_{\text{loc}}(\overline{\Omega}) \Rightarrow p(g) \in \widetilde{M}^{p,\lambda}_{\text{loc}}(\mathbb{R}^n);$$

(B) $$g \in W^1 \widetilde{M}^{p,\lambda}_{\text{loc}}(\overline{\Omega}) \Rightarrow p(g) \in W^1 \widetilde{M}^{p,\lambda}_{\text{loc}}(\mathbb{R}^n).$$

Proof. (A) Suppose $g \in \widetilde{M}^{p,\lambda}_{\text{loc}}(\overline{\Omega})$. From Corollary 2.7(A), we deduce that $p(g) \in M^{p,\lambda}_{\text{loc}}(\mathbb{R}^n)$. Let $\delta \in \mathbb{R}_+$. Since the U_j are uniformly bounded, by reasoning as in [V], Corollary 4.3, there exists $\xi = \xi(n, M) \in \mathbb{R}_+$ such that

$$\sup_{x\in\Omega} |F_h(x)| \le \xi\delta, \qquad h \in \mathbb{N}_0,$$

for any $F \in \Sigma(\mathbb{R}^n)$ with $\sup_{x\in\Omega} |F(x)| \le \delta$, and a fortiori if $\sup_{x\in\mathbb{R}^n} |F(x)| \le \delta$.

Therefore, by (2.16) we get

$$\sigma[\zeta_r p(g), \mathbb{R}^n](\delta) \le c\sigma[\zeta_{\bar r} g, \Omega](\xi\delta) \to 0 \quad \text{as } \delta \to 0, \tag{2.18}$$

whence $\zeta_r p(g) \in \widetilde{M}^{p,\lambda}(\mathbb{R}^n)$.

(B) If we also assume that $g_x \in \widetilde{M}^{p,\lambda}_{\mathrm{loc}}(\bar\Omega)$, then by Corollary 2.7(B), $p(g) \in W^1 M^{p,\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$. By proceeding as in case (A), from (2.17) we obtain

$$\begin{aligned} &\sigma[\zeta_r (p(g))_x, \mathbb{R}^n](\delta) \\ &\qquad \le c(\sigma[\zeta_{\bar r} g, \Omega](\xi\delta) + \sigma[\zeta_{\bar r} g_x, \Omega](\xi\delta)) \to 0 \quad \text{as } \delta \to 0, \end{aligned} \tag{2.19}$$

whence $\zeta_r (p(g))_x \in \widetilde{M}^{p,\lambda}(\mathbb{R}^n)$. ∎

3. A-priori bounds. Denote by $E(\nu, \Omega)$, for any $\nu \in \mathbb{R}_+$, the class of $n \times n$ real matrix-valued functions $((e_{ij}))$ such that

$$e_{ij} = e_{ji} \in L^\infty(\Omega), \quad (e_{ij})_x \in M^{s,n-s}_{\mathrm{loc}}(\bar\Omega), \quad i,j = 1,\ldots,n,$$

for some $s \in]2, n]$, and

$$\sum_{i,j=1}^n e_{ij}\xi_i\xi_j \ge \nu|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \text{ a.e. in } \Omega.$$

Moreover, we set

$$\mathcal{G}(\Omega) = \{g \in L^\infty(\Omega) : \operatorname*{ess\,inf}_{\Omega} g > 0\}.$$

Consider the operator L defined by (4) with real coefficients satisfying

$$a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i,j = 1,\ldots,n, \tag{3.1}$$

$$a_i \in M_0^{s,n-s}(\Omega) \quad i = 1,\ldots,n, \quad \text{for some } s \in]2, n], \tag{3.2}$$

$$a = a' + a'' \in \widetilde{M}^t(\Omega), \quad a' \in M_0^t(\Omega), \quad \operatorname*{ess\,inf}_{\Omega} a'' > 0, \tag{3.3}$$

where $t = 2$ if $n = 3$, $t > 2$ if $n = 4$, and $t = n/2$ if $n > 4$.

We also set

$$L_0 u = -\sum_{i,j=1}^n a_{ij} u_{x_i x_j}, \quad u \in W^2(\Omega).$$

Furthermore, suppose that $((a_{ij}))$ satisfies the following condition:

(α) there exist $\nu \in \mathbb{R}_+$ and $((e_{ij})) \in E(\nu, \Omega)$ such that

$$g = \frac{\sum_{i,j=1}^n e_{ij} a_{ij}}{\sum_{i,j=1}^n a_{ij}^2} \in \mathcal{G}(\Omega), \quad \operatorname*{ess\,sup}_{\Omega} \sum_{i,j=1}^n (e_{ij} - g a_{ij})^2 < \nu^2.$$

We also need a non-negative function

$$\beta \in \widetilde{M}^t(\Omega) \quad \text{such that} \quad \beta_x \le \beta\gamma, \tag{3.4}$$

where $\gamma \in \widetilde{M}^{s,n-s}(\Omega)$ for some $s \in]2, n]$.

THEOREM 3.1. *Suppose that Ω has property (Φ) with the homeomorphisms Φ_h of class C^2, the coefficients of the operator L satisfy (3.1)–(3.3) and β is given by (3.4). Also assume that $((a_{ij}))$ satisfies condition (α) with*

$$(e_{ij})_x \in \widetilde{M}^{s,n-s}_{\mathrm{loc}}(\bar\Omega), \quad i,j = 1,\ldots,n,$$

and

$$\begin{aligned} &\lim_{|x|\to\infty} a_{ij}(x) = a_{ij}^0 \in \mathbb{R}, \quad i,j = 1,\ldots,n, \\ &\lim_{|x|\to\infty} e_{ij}(x) = e_{ij}^0 \in \mathbb{R}, \quad i,j = 1,\ldots,n. \end{aligned} \tag{3.5}$$

Moreover, set

$$g_0 = \lim_{|x|\to\infty} g(x)$$

and denote by ϱ a non-negative function on $\mathbb{R}_+$ such that

$$\begin{aligned} &\operatorname*{ess\,sup}_{\Omega\setminus B_r} \sum_{i,j=1}^n |g a_{ij} - g_0 a_{ij}^0| \le \varrho(r) \quad \forall r \in \mathbb{R}_+, \\ &\lim_{r\to\infty} \varrho(r) = 0 \end{aligned} \tag{3.6}$$

and by σ_r the modulus of continuity of $\zeta_r \sum_{i,j=1}^n (e_{ij})_x$. Then there exist a bounded open subset Ω_0 of Ω and a constant c depending only on Ω, a_i, a', a'', n, ν, β, a_{ij}^0, e_{ij}^0, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|e_{ij}\|_{L^\infty(\Omega)}$, ϱ, σ_r such that

$$\begin{aligned} &\|u\|_{W^2(\Omega)} \le c(|Lu + \lambda g^{-1}\beta u|_{2,\Omega} + |u|_{2,\Omega_0}) \\ &\qquad\qquad \forall u \in W^2(\Omega) \cap W_0^1(\Omega),\ \forall \lambda \ge 0. \end{aligned} \tag{3.7}$$

Proof. We proceed as in [CLM$_2$], Theorem 4.1, remarking that property (Φ) implies the existence of a $\tau \in \mathbb{R}_+$ such that, for any $x \in \mathbb{R}^n$, either $B(x,\tau) \cap \partial\Omega = \emptyset$ or $B(x,\tau) \cap \partial\Omega \ne \emptyset$ and $B(x,\tau) \subset U_h$ for some $h \in \mathbb{N}$.

We consider a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\varphi|_{B_{1/2}} = 1, \quad \operatorname{supp}\varphi \subset B_1,$$

and, for $x \in \Omega$,

$$\psi = \psi^x : y \in \mathbb{R}^n \to \varphi\left(\frac{x-y}{\tau}\right)$$

where τ is defined above.

So, if $u \in W^2(\Omega) \cap W_0^1(\Omega)$, then $v = \psi u \in W^2(\Omega) \cap W_0^1(\Omega)$ and either $\operatorname{supp} v \Subset \Omega$ or $\operatorname{supp} v \subset U_h$ for some $h \in \mathbb{N}$.

As a consequence of the results in Section 7 of [CLM$_1$] (see (7.5)–(7.7), (7.9), (7.10), Lemmas 7.1 and 7.2 in [CLM$_1$]), for any $\lambda \ge 0$ we get

$$(\nu^2 - \varepsilon^2)|v_{xx}|^2_{2,\Omega} \le \int_\Omega \Big(-\sum_{i,j=1}^n e_{ij} v_{x_i x_j} + \lambda \beta v\Big)^2 + c_1(\varepsilon)|f v_x|^2_{2,\Omega},$$

where

$$f = 1 + \gamma + \sum_{i,j=1}^n (e_{ij})_x.$$

So, if we set

$$h = \operatorname*{ess\,sup}_\Omega \Big[\sum_{i,j=1}^n |e_{ij} - g a_{ij}|^2\Big]^{1/2},$$

for $\varepsilon < \nu - h$ we have

$$\begin{aligned}(\nu - \varepsilon)|v_{xx}|_{2,\Omega} &\le \Big|\sum_{i,j=1}^n (e_{ij} - g a_{ij}) v_{x_i x_j} - g L_0 v + \lambda \beta v\Big|_{2,\Omega} + c_1^{1/2}(\varepsilon)|f v_x|_{2,\Omega} \\ &\le h|v_{xx}|_{2,\Omega} + \|g\|_{L^\infty(\Omega)}|L_0 v + \lambda g^{-1}\beta v|_{2,\Omega} + c_1^{1/2}(\varepsilon)|f v_x|_{2,\Omega},\end{aligned}$$

from which, by condition (α),

$$|v_{xx}|_{2,\Omega} \le c_2(|L_0 v + \lambda g^{-1}\beta v|_{2,\Omega} + |f v_x|_{2,\Omega}).$$

Fix $r \in \mathbb{R}_+$ and set $w = \zeta_r u$. Then, by applying the above inequality with $v = \psi w$, we get

$$|(\psi w)_{xx}|_{2,\Omega} \le c_2(|L_0(\psi w) + \lambda g^{-1}\beta \psi w|_{2,\Omega} + |f(\psi w)_x|_{2,\Omega}). \tag{3.8}$$

The first term of the right hand side can be bounded as follows:

$$\begin{aligned}|L_0(\psi w) + \lambda g^{-1}\beta\psi w|_{2,\Omega} &\le |\psi(L_0 w + \lambda g^{-1}\beta w)|_{2,\Omega} \\ &\quad + 2\|a_{ij}\|_{L^\infty(\Omega)}|\psi_x w_x|_{2,\Omega} + \|a_{ij}\|_{L^\infty(\Omega)}|\psi_{xx} w|_{2,\Omega} \\ &\le c_3(|L_0 w + \lambda g^{-1}\beta w|_{2,\Omega(x,\tau)} + |w_x|_{2,\Omega(x,\tau)} + |w|_{2,\Omega(x,\tau)}).\end{aligned}$$

The second term can be estimated by means of Lemma 1.1.

Hence from (3.8) we deduce the inequality

$$\|w_{xx}\|_{2,\Omega(x,\tau/2)} \le c_4(|L_0 w + \lambda g^{-1}\beta w|_{2,\Omega(x,\tau)} + |w_x|_{2,\Omega(x,\tau)} + |w|_{2,\Omega(x,\tau)}).$$

Therefore, applying Lemma 1.1 of [CLM$_1$], we obtain

$$\|(\zeta_r u)_{xx}\|_{2,\Omega} \le c_5(|L_0(\zeta_r u) + \lambda g^{-1}\beta\zeta_r u|_{2,\Omega} + |(\zeta_r u)_x|_{2,\Omega} + |\zeta_r u|_{2,\Omega})$$

from which, since $a'' \in \widetilde{M}^t(\Omega)$, by Lemma 1.2, we get

$$\|\zeta_r u\|_{W^2(\Omega)} \le c_6(|L_0(\zeta_r u) + (a'' + \lambda g^{-1}\beta)\zeta_r u|_{2,\Omega} + |\zeta_r u|_{2,\Omega}). \tag{3.9}$$

On the other hand, by Theorem 4.2 of [CLM$_2$], there exists a bounded open subset Ω_0' of Ω such that

$$\begin{aligned}&\|(1-\zeta_r)u\|_{W^2(\Omega)} \\ &\quad\le c_7\Big(\Big|-g_0\sum_{i,j=1}^n a^0_{ij}((1-\zeta_r)u)_{x_i x_j} + (ga'' + \lambda\beta)(1-\zeta_r)u\Big|_{2,\Omega} \\ &\qquad + |(1-\zeta_r)u|_{2,\Omega_0'}\Big) \\ &\quad\le c_7(\varrho(r)|((1-\zeta_r)u)_{xx}|_{2,\Omega} + \|g\|_{L^\infty(\Omega)} \\ &\qquad \times L_0((1-\zeta_r)u) + (a'' + \lambda g^{-1}\beta)(1-\zeta_r)u|_{2,\Omega} + |u|_{2,\Omega_0'}).\end{aligned}$$

We deduce, by (3.6), that there exists $r_0 \in \mathbb{R}_+$ such that

$$\begin{aligned}&\|(1-\zeta_{r_0})u\|_{W^2(\Omega)} \\ &\quad\le c_8(|L_0((1-\zeta_{r_0})u) + (a'' + \lambda g^{-1}\beta)(1-\zeta_{r_0})u|_{2,\Omega} + |u|_{2,\Omega_0'}).\end{aligned} \tag{3.10}$$

From (3.9) and (3.10) we get

$$\|u\|_{W^2(\Omega)} \le c_9(|L_0 u + (a'' + \lambda g^{-1}\beta)u|_{2,\Omega} + |u|_{2,\Omega_0''}),$$

with $\Omega_0'' = \Omega_0' \cup (B_{r_0} \cap \Omega)$.

By using Lemmas 1.1 and 1.2, the result follows. ∎

From Theorem 3.1, proceeding as in [TT$_1$], Corollary 4.2, we have

COROLLARY 3.2. *Under the hypotheses of Theorem* 3.1, *if* $\beta^{-1} \in L^\infty_{\mathrm{loc}}(\overline{\Omega})$, *there exist* $\lambda_0, c \in \mathbb{R}_+$ *such that*

$$\|u\|_{W^2(\Omega)} \le c|Lu + \lambda g^{-1}\beta u|_{2,\Omega} \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega),\ \forall \lambda \in [\lambda_0, \infty[,$$

with c *a constant depending on* Ω, a_i, a', a'', n, ν, β, a^0_{ij}, e^0_{ij}, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|e_{ij}\|_{L^\infty(\Omega)}$, ϱ, σ_r.

4. Regularization of coefficients. Let $\{J_k\}_{k\in\mathbb{N}}$ be a sequence of mollifiers, i.e.

$$J_k \in C_0^\infty(\mathbb{R}^n), \quad J_k \ge 0, \quad \operatorname{supp} J_k \subset B_{1/k}, \quad \int J_k = 1.$$

For $k \in \mathbb{N}$ we set

$$a^k_{ij} = J_k * p(g a_{ij}), \quad i,j = 1,\ldots,n, \tag{4.1}$$

$$e^k_{ij} = J_k * p(e_{ij}), \quad i,j = 1,\ldots,n. \tag{4.2}$$

LEMMA 4.1. *Let* Ω *be an open subset of* $\mathbb{R}^n$ *with property* (Φ). *If* $((e_{ij})) \in E(\nu,\Omega)$ *for some* $\nu \in \mathbb{R}_+$, *then*

$$((p(e_{ij}))) \in E(\nu, \Omega(\eta)),$$

where $\Omega(\eta)$ *is given by Remark* 2.4.

Proof. Let $((e_{ij})) \in E(\nu, \Omega)$. From Remark 2.4 we deduce that for almost every $x \in \Omega(\eta) \setminus \overline{\Omega}$ there exist $x_1 \in U_{k_1} \cap \Omega, \ldots, x_h \in U_{k_h} \cap \Omega$, with $h \in \{1, \ldots, N\}$, such that

$$p(e_{ij})(x) = \theta_{k_1}(x)e_{ij}(x_1) + \ldots + \theta_{k_h}(x)e_{ij}(x_h) \quad \text{for } i,j = 1, \ldots, n,$$

where $e_{ij}(x_1), \ldots, e_{ij}(x_h)$ are symmetric and positive with lower bound ν. Then

$$p(e_{ij})(x) = \theta_{k_1}(x)e_{ji}(x_1) + \ldots + \theta_{k_h}(x)e_{ji}(x_h) = p(e_{ji})(x), \tag{4.3}$$

i.e. $((p(e_{ij})))$ is symmetric a.e. in $\Omega(\eta)$, and

$$\begin{aligned} \sum_{i,j=1}^{n} p(e_{ij})(x)\xi_i\xi_j &= \theta_{k_1}(x) \sum_{i,j=1}^{n} e_{ij}(x_1)\xi_i\xi_j + \ldots + \theta_{k_h}(x) \sum_{i,j=1}^{n} e_{ij}(x_h)\xi_i\xi_j \\ &\geq (\theta_{k_1}(x) + \ldots + \theta_{k_h}(x))\nu|\xi|^2 = \nu|\xi|^2, \end{aligned} \tag{4.4}$$

i.e. $((p(e_{ij})))$ is uniformly positive definite a.e. in $\Omega(\eta)$ with lower bound ν.

From Corollary 2.3 we know that

$$e_{ij} \in L^\infty(\Omega) \Rightarrow p(e_{ij}) \in L^\infty(\mathbb{R}^n),$$

and moreover

$$\|p(e_{ij})\|_{L^\infty(\mathbb{R}^n)} \leq c\|e_{ij}\|_{L^\infty(\Omega)} \tag{4.5}$$

for $i,j = 1, \ldots, n$, with $c \in \mathbb{R}_+$ depending only on n and the constants implied in (Φ).

From Corollary 2.7 we deduce that

$$e_{ij} \in L^\infty(\Omega),\ (e_{ij})_x \in M^{s,n-s}_{\mathrm{loc}}(\overline{\Omega}) \Rightarrow (p(e_{ij}))_x \in M^{s,n-s}_{\mathrm{loc}}(\mathbb{R}^n). \tag{4.6}$$

The result follows from (4.3)–(4.6). ∎

LEMMA 4.2. *Let Ω be an open subset of $\mathbb{R}^n$ with property (Φ), and $((a_{ij}))$ satisfy condition (α) with associated $((e_{ij})) \in E(\nu, \Omega)$. Then there exists $k_1 \in \mathbb{N}$ such that, for every $k \geq k_1$, $((a^k_{ij}))$ satisfies condition (α) with associated $((e^k_{ij})) \in E(\nu, \Omega)$.*

Proof. Let $((e_{ij})) \in E(\nu, \Omega)$; so by Lemma 4.1, $((p(e_{ij}))) \in E(\nu, \Omega(\eta))$. Let $k_1 = k_1(\eta) \in \mathbb{N}$ be such that

$$x \in \Omega, y \in B_{1/k_1} \Rightarrow x - y \in \Omega(\eta).$$

Suppose $k \geq k_1$. From the properties of $p(e_{ij})$ stated in Lemma 4.1, we have

$$e^k_{ji} = J_k * p(e_{ji}) = J_k * p(e_{ij}) = e^k_{ij}, \tag{4.7}$$

and so $((e^k_{ij}))$ is symmetric. Moreover,

$$\sum_{i,j=1}^{n} e^k_{ij}\xi_i\xi_j = J_k * \Big(\sum_{i,j=1}^{n} p(e_{ij})\xi_i\xi_j\Big) \geq (J_k * 1)\nu|\xi|^2 = \nu|\xi|^2, \tag{4.8}$$

whence $((e^k_{ij}))$ is uniformly positive definite with lower bound ν.

From (4.5) we get

$$\|e^k_{ij}\|_{L^\infty(\Omega)} \leq \|p(e_{ij})\|_{L^\infty(\mathbb{R}^n)} \leq c\|e_{ij}\|_{L^\infty(\Omega)}, \tag{4.9}$$

and so $e^k_{ij} \in L^\infty(\Omega)$. For any $r \in \mathbb{R}_+$ it turns out that

$$\zeta_r(e^k_{ij})_x \in C^\infty_0(\mathbb{R}^n), \tag{4.10}$$

and a fortiori $\zeta_r(e^k_{ij})_x \in M^{s,n-s}(\Omega)$ for any $s \in\,]2,n]$.

By collecting (4.7)–(4.10) we can conclude that

$$((e^k_{ij})) \in E(\nu, \Omega) \quad \text{for } k \geq k_1.$$

Suppose that $((a_{ij}))$ satisfies condition (α) with associated $((e_{ij})) \in E(\nu, \Omega)$. For $x \in \Omega$ we have

$$e^k_{ij}(x) - a^k_{ij}(x) = \int_{\mathbb{R}^n} J_k(y)p(e_{ij} - ga_{ij})(x-y)\,dy$$

and so, if $k \geq k_1$,

$$\Big(\sum_{i,j=1}^{n} |e^k_{ij}(x) - a^k_{ij}(x)|^2\Big)^{1/2} \leq \operatorname*{ess\,sup}_{\Omega(\eta)} \Big(\sum_{i,j=1}^{n} |p(e_{ij} - ga_{ij})|^2\Big)^{1/2}.$$

By proceeding as in Corollary 2.5 and applying the triangle inequality, we get

$$\Big(\sum_{i,j=1}^{n} |p(e_{ij} - ga_{ij})|^2\Big)^{1/2} \leq \operatorname*{ess\,sup}_{\Omega} \Big(\sum_{i,j=1}^{n} |e_{ij} - ga_{ij}|^2\Big)^{1/2} \quad \text{a.e. in } \Omega(\eta),$$

and so

$$\sup_{\Omega} \Big(\sum_{i,j=1}^{n} |e^k_{ij} - a^k_{ij}|^2\Big)^{1/2} \leq \operatorname*{ess\,sup}_{\Omega} \Big(\sum_{i,j=1}^{n} |e_{ij} - ga_{ij}|^2\Big)^{1/2} < \nu. \tag{4.11}$$

From (4.9) and (4.11) we deduce that

$$\|a^k_{ij}\|_{L^\infty(\Omega)} \leq \|e^k_{ij}\|_{L^\infty(\Omega)} + \nu \leq c\|e_{ij}\|_{L^\infty(\Omega)} + \nu. \tag{4.12}$$

Setting

$$\nu_1 = \nu - \operatorname*{ess\,sup}_{\Omega} \Big(\sum_{i,j=1}^{n} |e_{ij} - ga_{ij}|^2\Big)^{1/2} > 0,$$

from (4.8) and (4.11) we have

$$\begin{aligned}(4.13)\qquad \sum_{i,j=1}^{n} a_{ij}^k\xi_i\xi_j &= \sum_{i,j=1}^{n} e_{ij}^k\xi_i\xi_j - \sum_{i,j=1}^{n}(e_{ij}^k - a_{ij}^k)\xi_i\xi_j\\ &\geq \nu|\xi|^2 - (\nu-\nu_1)|\xi|^2 = \nu_1|\xi|^2,\end{aligned}$$

and so $((a_{ij}^k))$ is uniformly positive definite a.e. in Ω with lower bound ν_1 independent of k.

From (4.9), (4.12), (4.13), it follows that

$$(4.14)\qquad g_k = \frac{\sum_{i,j=1}^{n} e_{ij}^k a_{ij}^k}{\sum_{i,j=1}^{n}(a_{ij}^k)^2} \in L^\infty(\Omega).$$

Furthermore, by (4.11), we obtain

$$\sum_{i,j=1}^{n}\left((e_{ij}^k)^2 + (a_{ij}^k)^2 - 2e_{ij}^k a_{ij}^k\right) < \nu^2,$$

whence, by (4.8) and (4.13),

$$\begin{aligned}\sum_{i,j=1}^{n} e_{ij}^k a_{ij}^k &> \frac{1}{2}\Big(\sum_{i,j=1}^{n}(e_{ij}^k)^2 + \sum_{i,j=1}^{n}(a_{ij}^k)^2 - \nu^2\Big)\\ &\geq \frac{1}{2}(n\nu^2 + n\nu_1^2 - \nu^2) > 0,\end{aligned}$$

which, together with (4.14), yields

$$(4.15)\qquad g_k \in \mathcal{G}(\Omega).$$

Finally, since

$$\sum_{i,j=1}^{n}(e_{ij}^k - g_k a_{ij}^k)^2 \leq \sum_{i,j=1}^{n}(e_{ij}^k - a_{ij}^k)^2,$$

from (4.11) we have

$$(4.16)\qquad \sum_{i,j=1}^{n}(e_{ij}^k - g_k a_{ij}^k)^2 < \nu^2,$$

whence we can conclude that $((a_{ij}^k))$ satisfies (α) with associated $((e_{ij}^k))$. ■

Below and in the sequel we assume that the hypotheses of Lemma 4.2 are satisfied.

REMARK 4.3. If (3.5) holds, then

$$(4.17)\qquad \lim_{|x|\to\infty} e_{ij}^k(x) = e_{ij}^0.$$

Proof. From the second equality of (3.5) and Corollary 2.6 we deduce that

$$(4.18)\qquad \lim_{\substack{|x|\to\infty\\ x\in\Omega(\eta)}} p(e_{ij})(x) = e_{ij}^0.$$

Then, if $k \geq k_1$, for any $\varepsilon > 0$ there exists $r_\varepsilon \in \mathbb{R}_+$ such that

$$\begin{aligned}|e_{ij}^k(x) - e_{ij}^0| &\leq \int_{\mathbb{R}^n} J_k(x-y)|p(e_{ij})(y) - e_{ij}^0|\,dy\\ &< \varepsilon\int_{\mathbb{R}^n} J_k(x-y)\,dy = \varepsilon \qquad \forall x\in\Omega \text{ and } |x| > r_\varepsilon. \ ■\end{aligned}$$

From (3.5) we also deduce that

$$(4.19)\qquad \lim_{\substack{|x|\to\infty\\ x\in\Omega(\eta)}} p(ga_{ij})(x) = g_0 a_{ij}^0.$$

If we set

$$(4.20)\qquad \varrho(r) = \sum_{i,j=1}^{n} \operatorname*{ess\,sup}_{\Omega(\eta)\setminus B_r} |p(ga_{ij}) - g_0 a_{ij}^0|, \qquad r\in\mathbb{R}_+,$$

then (3.6) is satisfied.

By reasoning as in Remark 4.3, we obtain the following

LEMMA 4.4. *If (3.5) holds, then for any $k \geq k_1$ we have*

$$\operatorname*{ess\,sup}_{\Omega\setminus B_{r+1}} \sum_{i,j=1}^{n} |a_{ij}^k - g_0 a_{ij}^0| \leq \varrho(r), \qquad r\in\mathbb{R}_+.$$

Let $\bar r, r \in \mathbb{R}_+$ be such that

$$\zeta_{\bar r}(x+y) \leq \zeta_r(x) \qquad \forall x\in\mathbb{R}^n,\ y\in B_1$$

and let $\widetilde r \in \mathbb{R}_+$ be as in Corollary 2.7.

Let $\xi \in \mathbb{R}_+$ be as in the proof of Corollary 2.8 and define

$$(4.21)\qquad \sigma_{\bar r}(\delta) = \sum_{i,j=1}^{n}\left(\sigma[\zeta_{\bar r}e_{ij},\Omega](\xi\delta) + \sigma[\zeta_{\bar r}(e_{ij})_x,\Omega](\xi\delta)\right).$$

Since $e_{ij}\in L^\infty(\Omega) \subset \widetilde{M}^{s,n-s}(\Omega)$, if $(e_{ij})_x \in \widetilde{M}^{s,n-s}_{\mathrm{loc}}(\overline\Omega)$, then

$$\lim_{\delta\to 0}\sigma_{\bar r}(\delta) = 0.$$

LEMMA 4.5. *If $(e_{ij})_x \in \widetilde{M}^{s,n-s}_{\mathrm{loc}}(\overline\Omega)$, $i,j = 1,\ldots,n$, then for any $k\geq k_1$,*

$$\sigma[\zeta_{\bar r}(e_{ij}^k)_x,\Omega](\delta) \leq c\,\sigma_{\bar r}(\delta), \qquad \delta\in\mathbb{R}_+,\ \bar r\in\mathbb{R}_+,\ i,j = 1,\ldots,n,$$

where c is a constant depending only on n and on the constants appearing in (Φ).

Proof. Let E be a measurable subset of Ω such that $|E(x)| \le \delta$ for any $x \in \Omega$. Setting $p = s$, $\lambda = n - s$, for any $x_0 \in \mathbb{R}^n$ and $t \in]0,1]$, we have

$$\frac{1}{t^{\lambda/p}}\Big(\int_{B(x_0,t)\cap\Omega} |\chi_E(x)\zeta_{\bar r}(x)(e^k_{ij})_x(x)|^p\,dx\Big)^{1/p}$$
$$= \frac{1}{t^{\lambda/p}}\Big(\int_{B(x_0,t)\cap\Omega} \Big|\chi_E(x)\zeta_{\bar r}(x)\int_{B_1} J_k(y)(p(e_{ij}))_x(x-y)\,dy\Big|^p\,dx\Big)^{1/p}$$
$$\le \frac{1}{t^{\lambda/p}}\int_{B_1} J_k(y)\Big(\int_{B(x_0,t)\cap\Omega} |\chi_E(x)\zeta_{\bar r}(x)(p(e_{ij}))_x(x-y)|^p\,dx\Big)^{1/p}dy$$
$$\le \frac{1}{t^{\lambda/p}}\int_{B_1} J_k(y)\Big(\int_{B(x_0-y,t)\cap(\Omega-y)} |\chi_{E-y}\zeta_r(p(e_{ij}))_x|^p\,dx\Big)^{1/p}dy$$
$$\le \sup_{y\in B_1}\|\chi_{E-y}\zeta_r(p(e_{ij}))_x\|_{M^{p,\lambda}(\Omega-y)}.$$

Then, by proceeding as in Corollary 2.8, we get the inequality

$$\|\chi_E(x)\zeta_{\bar r}(x)(e^k_{ij})_x\|_{M^{p,\lambda}(\Omega)} \le c(\sigma[\zeta_{\bar r}e_{ij},\Omega](\xi\delta) + \sigma[\zeta_{\bar r}(e_{ij})_x,\Omega](\xi\delta)),$$

from which we have the assertion. ∎

5. Existence and uniqueness theorem. In this section we shall give an existence and uniqueness theorem for the solution of the problem

$$u \in W^2(\Omega)\cap W^1_0(\Omega),\quad Lu = f,\quad f\in L^2(\Omega), \tag{5.1}$$

for the operator L defined in (4).

THEOREM 5.1. *Suppose that Ω has property (Φ) with the homeomorphisms Φ_h of class C^2, and the coefficients of the operator L satisfy the assumptions of Theorem* 3.1. *Then the operator*

$$L : u \in W^2(\Omega)\cap W^1_0(\Omega) \to Lu \in L^2(\Omega)$$

is a Fredholm operator with index zero. Furthermore, if

$$a_0 = \operatorname*{ess\,inf}_{\Omega} a > 0, \tag{5.2}$$

then problem (5.1) *is uniquely solvable.*

Proof. Set, for each $k \ge k_1$,

$$L_k u = -\sum_{i,j=1}^n a^k_{ij}u_{x_ix_j} + \sum_{i=1}^n g a_i u_{x_i} + gau.$$

From (3.5) we have

$$\lim_{|x|\to\infty}(a^k_{ij})_{x_h}(x) = 0$$

and so, by Remark 1.3, $(a^k_{ij})_x \in M^{s,n-s}_0(\Omega)$.

Since L_k satisfies the hypotheses of Theorem 6.2 of [CLM$_1$], there exists a unique solution u_k of the problem

$$u\in W^2(\Omega)\cap W^1_0(\Omega),\quad L_ku = gf,\quad f\in L^2(\Omega). \tag{5.3}$$

Furthermore, by (4.13), (4.17), (4.12), (4.9) and Lemmas 4.4–4.5, the operator L_k satisfies the hypotheses of Theorem 3.1 uniformly with respect to k. Then there exists a constant c independent of k such that

$$\|u_k\|_{W^2(\Omega)} \le c(|gf|_{2,\Omega} + |u_k|_{2,\Omega_0}).$$

Suppose that $f\in L^2(\Omega)\cap L^\infty(\Omega)$. As above, the operator L_k satisfies the hypotheses of Theorem 2.1 of [CLM$_2$], and so $u_k\in L^\infty(\Omega)$ with

$$\|u_k\|_{L^\infty(\Omega)} \le a_0^{-1}\|gf\|_{L^\infty(\Omega)},$$

whence

$$\|u_k\|_{W^2(\Omega)} \le c_1(|f|_{2,\Omega} + \|f\|_{L^\infty(\Omega)}) \tag{5.4}$$

where c_1 is a constant independent of k.

From (5.4) it follows that there exists a subsequence $(u_{k_n})_{k\in\mathbb{N}}$ of $(u_k)_{k\in\mathbb{N}}$ weakly convergent in $W^2(\Omega)$ to a function $u\in W^2(\Omega)\cap W^1_0(\Omega)$, and therefore also $L_ku_{k_n}\to gLu$ in the sense of distributions. It follows that u is a solution of problem (5.1) if $f\in L^2(\Omega)\cap L^\infty(\Omega)$.

Denote by $R(L)$ the range of the operator

$$L : u\in W^2(\Omega)\cap W^1_0(\Omega)\to Lu\in L^2(\Omega).$$

Then $L^2(\Omega)\cap L^\infty(\Omega)\subset R(L)$. On the other hand, as a consequence of (3.7) and known results, $R(L)$ is a closed subspace of $L^2(\Omega)$; but $L^2(\Omega)\cap L^\infty(\Omega)$ is dense in $L^2(\Omega)$, and so $R(L) = L^2(\Omega)$. Hence it is sufficient to show that L is a Fredholm operator with index zero.

For this purpose we consider the problem

$$\begin{cases} L_0u + \sum_{i=1}^n a_iu_{x_i} + (a'' + \lambda g^{-1}\beta)u = f, & f\in L^2(\Omega),\\ u\in W^2(\Omega)\cap W^1_0(\Omega), \end{cases} \tag{5.5}$$

with $\lambda\ge 0$ and $\beta(x) = (1+|x|^2)^{-1}, x\in\mathbb{R}^n$.

By means of the same technique used above for problem (5.1), we can show the existence of solutions for problem (5.5), as $a''+\lambda g^{-1}\beta > 0$. Moreover, since β satisfies (3.4) and $\beta^{-1}\in L^\infty_{\rm loc}(\overline{\Omega})$, from Corollary 3.2 we deduce that the solution is unique, provided that λ is large enough.

So the operator $L_0 + \sum_{i=1}^n a_i\partial/\partial x_i + (a''+\lambda g^{-1}\beta)$ is a Fredholm operator with index zero.

Finally, since $a',\beta\in M^t_0(\Omega)$, by Lemma 1.2 the operator

$$u\in W^2(\Omega)\to (a' - \lambda g^{-1}\beta)u\in L^2(\Omega)$$

is compact, and from known results we have the assertion. ∎

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Maximal functions and smoothness spaces in $L_p(\mathbb{R}^d)$

by

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Abstract. We study smoothness spaces generated by maximal functions related to the local approximation errors of integral operators. It turns out that in certain cases these smoothness classes coincide with the spaces $C_p^\alpha(\mathbb{R}^d)$, $0 < p \le \infty$, introduced by DeVore and Sharpley [DS] by means of the so-called sharp maximal functions of Calderón and Scott. As an application we characterize the $C_p^\alpha(\mathbb{R}^d)$ spaces in terms of the coefficients of wavelet decompositions.

1. Introduction. Maximal operators play an important role in various aspects of harmonic analysis and approximation theory, such as interpolation and differentiation. A paradigm is the so-called *sharp maximal function*, of Calderón and Scott [CS], given by

$$f_\alpha^\sharp(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1+\alpha/d}} \int_Q |f - f_Q|, \quad 0 < \alpha < 1, \tag{1.1}$$

where $f_Q := |Q|^{-1} \int_Q f$ is the average of f over Q, and Q ranges over all cubes containing x. When $\alpha > 0$, $f_\alpha^\sharp$ is related to classical differentiation; for instance it is well known that

$$f \in \mathrm{Lip}_\alpha(\mathbb{R}^d) \Leftrightarrow f_\alpha^\sharp \in L_\infty(\mathbb{R}^d), \quad 0 < \alpha < 1, \tag{1.2}$$

where Lip_α is the Lipschitz space of smoothness α.

The extension of (1.1) to functions of higher smoothness was given by DeVore and Sharpley [DS]. For every $\alpha \ge 1$ they replaced the average f_Q by a best polynomial approximation from $\Pi_{[\alpha]}$ (the space of polynomials of degree at most $[\alpha]$) and they introduced the spaces $C_p^\alpha := C_p^\alpha(\mathbb{R}^d)$.

For $0 < p \le \infty$ and $\alpha > 0$, C_p^α is defined to be the collection of all functions $f \in L_p := L_p(\mathbb{R}^d)$ such that

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