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Schauder theorems for linear elliptic and parabolic problems with unbounded coefficients in $\mathbb{R}^n$

by

ALESSANDRA LUNARDI (Parma)

Abstract. We study existence, uniqueness, and smoothing properties of the solutions to a class of linear second order elliptic and parabolic differential equations with unbounded coefficients in $\mathbb{R}^n$. The main results are global Schauder estimates, which hold in spite of the unboundedness of the coefficients.

1. Introduction. We consider a class of second order elliptic operators in $\mathbb{R}^n$,

$$\begin{aligned}(1.1)\qquad \mathcal{A}u(x) &= \sum_{i,j=1}^{n} q_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} p_i(x)D_iu(x) + r(x)u(x),\\ &= \mathrm{Tr}(Q(x)D^2u(x)) + \langle P(x), Du(x)\rangle + r(x)u(x),\end{aligned}$$

where the coefficients are regular enough and may grow not more than exponentially as $|x| \to \infty$. The main assumptions are that r is bounded from above,

$$(1.2)\qquad \sup_{x\in\mathbb{R}^n} r(x) = r_0 < \infty, \quad \forall x \in \mathbb{R}^n,$$

that the problem is uniformly elliptic,

$$(1.3)\qquad \sum_{i,j=1}^{n} q_{ij}(x)\xi_i\xi_j \ge \nu(x)|\xi|^2, \quad \forall x,\xi \in \mathbb{R}^n, \text{ with } \inf_{x\in\mathbb{R}^n} \nu(x) = \nu_0 > 0,$$

and that P satisfies a dissipativity condition,

$$(1.4)\quad \sum_{i,j=1}^{n} D_ip_j(x)\xi_i\xi_j \le p(x)|\xi|^2, \quad \forall x,\ \xi \in \mathbb{R}^n, \text{ with } \sup_{x\in\mathbb{R}^n} p(x) = p_0 < \infty.$$

Moreover, we assume that there exist $\lambda_0 \ge r_0$ and a regular function φ such that

$$(1.5)\qquad \lim_{|x|\to\infty} \varphi(x) = \infty, \quad \sup_{x\in\mathbb{R}^n} (\mathcal{A}\varphi(x) - \lambda_0\varphi(x)) < \infty.$$

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In the case where P is the gradient of a smooth function F, (1.4) means simply that D^2F is bounded from above. Sufficient conditions in order that (1.5) holds are easily given; in its turn (1.5) yields a maximum principle for the solution of (1.6) and (1.7).

We study existence, uniqueness and regularity of the solutions to

$$\lambda u(x) - \mathcal{A}u(x) = f(x), \quad x \in \mathbb{R}^n, \tag{1.6}$$

$$\begin{cases} u_t(t,x) - \mathcal{A}u(t,x) = g(t,x), & 0 < t \le T,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.7}$$

where f, g, u_0 are bounded and continuous (or Hölder continuous) functions, and $\lambda > r_0$.

The main results of this paper are Schauder type theorems for both (1.6) and (1.7), as follows.

THEOREM 1. *Let $0 < \theta < 1$. If $\lambda > r_0$, then for every $f \in C^\theta(\mathbb{R}^n)$ problem* (1.6) *has a unique solution $u \in C^{2+\theta}(\mathbb{R}^n)$, and there exists $C > 0$, independent of f, such that*

$$\|u\|_{C^{2+\theta}(\mathbb{R}^n)} \le C\|f\|_{C^\theta(\mathbb{R}^n)}.$$

THEOREM 2. *Let $T > 0$, $0 < \theta < 1$, and $g \in C([0,T] \times \mathbb{R}^n)$ be such that $g(t,\cdot) \in C^\theta(\mathbb{R}^n)$ for every t and $\sup_{0\le t\le T} \|g(t,\cdot)\|_{C^\theta(\mathbb{R}^n)} < \infty$. Let moreover $u_0 \in C^{2+\theta}(\mathbb{R}^n)$. Then problem* (1.7) *has a unique bounded solution u belonging to $C^{1,2}([0,T] \times \mathbb{R}^n)$, and there is $C > 0$, independent of g, u_0, such that*

$$\sup_{0\le t\le T} \|u(t,\cdot)\|_{C^{2+\theta}(\mathbb{R}^n)} \le C(\|u_0\|_{C^{2+\theta}(\mathbb{R}^n)} + \sup_{0\le t\le T} \|g(t,\cdot)\|_{C^\theta(\mathbb{R}^n)}).$$

While the literature about elliptic and parabolic problems with bounded coefficients is very rich, there are much less results about equations with unbounded coefficients (we mean of course global results, since the local ones depend only on the smoothness of the coefficients). Even existence and uniqueness of the solution to (1.6) and (1.7) for smooth data is not trivial, in general.

If $\det Q(x) \equiv 1$, a fundamental solution was proved to exist by Ito ([13]) under smoothness assumptions on the coefficients, and the only growth condition (1.2). This allowed proving existence (not in general uniqueness) of a classical solution to (1.7) for continuous L^1 data g, u_0.

Later, Bodanko ([5]), Aronson and Besala ([1], [4]) allowed r to grow as $|x|^\lambda$ if the ellipticity constant $\nu(x)$ grows not faster than $|x|^{2-\lambda}$, $0 \le \lambda \le 2$, while P cannot grow faster than $|x|$. They proved existence and uniqueness of the solution to (1.7) for data in suitable weighted spaces.

In [9] Schrödinger operators (of the type $u \mapsto \sum_{i,j=1}^n D_i(a_{ij}D_ju) - Vu$, with measurable and bounded a_{ij}) are considered, from the point of view of symmetric Markov semigroups, under some "positivity" assumptions on V. The results are then transferred to associated operators (again with unbounded coefficients) in suitable weighted spaces.

In all the above papers Hölder regularity is not considered. Optimal Hölder regularity results for (1.6) were proved by Cannarsa and Vespri in [6], with different techniques and stronger hypotheses guaranteeing analyticity of the associated semigroups: they assumed that Q is bounded and that the growth of r balances in a certain sense the growth of P. More recently, in [11] we proved Schauder estimates for the Ornstein–Uhlenbeck operator, where the coefficients q_{ij} are constant and the coefficients p_i are linear. Then we considered the case of bounded q_{ij} and Lipschitz continuous p_i ([17]), and lastly the case of coefficients with polynomial growth has been studied by stochastic methods by Cerrai in [7], [8].

The initial value problem (1.7) arises naturally in important fields of applied mathematics such as stochastic control and filtering theory. See for instance [3], [12], [2], [19]. The connection with stochastic control is obvious: if we perturb a dynamical system governed by ODE's,

$$\begin{cases} X' = P(X), & t > 0, \\ X(0) = x, \end{cases}$$

by a white noise with coefficients depending on the solution,

$$\begin{cases} dX = P(X)dt + \sqrt{2Q(X)}dW_t, & t > 0, \\ X(0) = x; \end{cases}$$

then the solution $X(t,x)$ is related to problem (1.7) with $r \equiv 0$ by the equality $u(t,x) = \mathbb{E}(u_0(X(t,x)))$.

Our study begins from problem (1.7), with $g \equiv 0$ and $T = \infty$. We prove that for every continuous and bounded u_0 problem (1.7) has a unique classical solution, which is bounded in $[0,\tau] \times \mathbb{R}^n$ for every $\tau > 0$. The associated semigroup $T(t)$ enjoys very nice smoothing properties: it maps continuously $C(\mathbb{R}^n)$ (the space of continuous and bounded functions in $\mathbb{R}^n$) into $C^3(\mathbb{R}^n)$ (the space of thrice differentiable functions in $\mathbb{R}^n$ with continuous and bounded derivatives up to the third order), and moreover there are C, $\omega \in \mathbb{R}$ such that

$$\|T(t)u_0\|_\infty + t^{1/2}\sum_{i=1}^n \|D_iT(t)u_0\|_\infty + t\sum_{i,j=1}^n \|D_{ij}T(t)u_0\|_\infty + t^{3/2}\sum_{i,j,l=1}^n \|D_{ijl}T(t)u_0\|_\infty \le Ce^{\omega t}\|u_0\|_\infty, \quad t > 0. \tag{1.8}$$

In general such a semigroup is not analytic and it is not strongly continuous in $C(\mathbb{R}^n)$ (not even in $\mathrm{BUC}(\mathbb{R}^n)$, as proved in [11]), so that the usual semigroup theory is not of help in the study of problem (1.7).

The main point is to find *a priori* estimates of the type (1.8). Then existence of the solution may be shown by an approximation procedure, which however is not straightforward (see Section 3).

The *a priori* estimates (1.8) are proved in an elementary way, applying the maximum principle to the equation satisfied by

$$z = u^2 + \alpha t\sum_{i=1}^{n}(D_i u)^2 + \alpha^2 t^2 \sum_{i,j=1}^{n}(D_{ij}u)^2 + \alpha^3 t^3 \sum_{i,j,l=1}^{n}(D_{ijl}u)^2,$$

with $\alpha > 0$. Indeed, an easy although lengthy computation shows that z satisfies

$$\text{(1.9)}\qquad \begin{cases} z_t(t,x) - \mathcal{A}z(t,x) - (1+2r_0)z(t,x) = g(t,x), & t>0,\ x\in\mathbb{R}^n,\\ z(0,x) = (u_0(x))^2, & x\in\mathbb{R}^n,\end{cases}$$

where the continuous function g has nonpositive values in $[0,1]\times\mathbb{R}^n$ provided α is suitably small. Thanks to assumption (1.5), the classical maximum principle may be adapted to our equations with unbounded coefficients, and it gives

$$\|z(t,\cdot)\|_{L^\infty((0,1)\times\mathbb{R}^n)} \le e^{\omega t}\|u_0\|_\infty^2$$

for some $\omega\in\mathbb{R}$. By using then the semigroup law, (1.8) follows easily.

Arguing similarly, we can also prove that

$$\text{(1.10)}\qquad \|T(t)u_0\|_{C^3(\mathbb{R}^n)} \le Ce^{\omega t}\|u_0\|_{C^3(\mathbb{R}^n)}, \quad t>0,$$

for every $u_0\in C^3(\mathbb{R}^n)$. By interpolation we get

$$\text{(1.11)}\qquad \|T(t)\|_{L(C^\theta(\mathbb{R}^n),C^\alpha(\mathbb{R}^n))} \le \frac{Ce^{\omega t}}{t^{(\alpha-\theta)/2}}, \quad t>0,\ 0\le\theta\le\alpha\le 3,$$

which coincides with the well known Hölder estimates in the case of bounded coefficients. Then we apply the interpolation procedure of [15]: given three Banach spaces $Y_0\supset Y_1\supset Y_2$ and a semigroup $T(t)$ such that

$$\|T(t)\|_{L(Y_0,Y_i)} \le Ce^{\omega t}/t^{\gamma_i}, \quad t>0,\ i=1,2,$$

with $\omega\ge 0$, $0\le\gamma_1<1<\gamma_2$, the domain of the "generator" (in a suitable sense) of $T(t)$ in Y_0 is continuously embedded in the interpolation space

$$(Y_1,Y_2)_{\beta,\infty}$$

with $\beta = (1-\gamma_1)/(\gamma_2-\gamma_1)$.

Choosing $Y_0 = C^\theta(\mathbb{R}^n)$, $Y_1 = C^\alpha(\mathbb{R}^n)$, and $Y_2 = C^{2+\alpha}(\mathbb{R}^n)$, with $0<\theta<\alpha<1$, we find that the domain of the realization of $\mathcal{A}$ in $C^\theta(\mathbb{R}^n)$ is continuously embedded in

$$(C^\alpha(\mathbb{R}^n), C^{2+\alpha}(\mathbb{R}^n))_{1-(\alpha-\theta)/2,\infty} = C^{2+\theta}(\mathbb{R}^n).$$

This is nothing but an optimal Schauder regularity result for problem (1.6): indeed, it implies that if $f\in C^\theta(\mathbb{R}^n)$ then the solution u is in $C^{2+\theta}(\mathbb{R}^n)$. A similar interpolation method for abstract parabolic problems gives the optimal Schauder regularity and estimates of Theorem 2.

2. Maximum principles and a priori estimates. We state below some probably well known generalizations of the classical maximum principle to parabolic and elliptic problems in $\mathbb{R}^n$.

PROPOSITION 2.1. *Let the data* $q_{ij}:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ *and* $r:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ *satisfy*

$$\text{(2.1)}\qquad \sum_{i,j=1}^{n} q_{ij}(t,x)\xi_i\xi_j\ge 0, \quad r(t,x)\le r_0, \quad 0\le t\le T,\ x,\xi\in\mathbb{R}^n,$$

and assume that there exists a C^2 function $\varphi:\mathbb{R}^n\to\mathbb{R}$ satisfying (1.5) *with $\lambda_0\ge r_0$. Let $z:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ be a bounded classical solution of*

$$\text{(2.2)}\qquad \begin{cases} z_t = \displaystyle\sum_{i,j=1}^{n} q_{ij}D_{ij}z + \sum_{i=1}^{n} p_iD_iz + rz + g(t,x), & 0\le t\le T,\ x\in\mathbb{R}^n,\\ z(0,x) = z_0(x), & x\in\mathbb{R}^n,\end{cases}$$

where

$$g(t,x)\le 0, \quad 0\le t\le T,\ x\in\mathbb{R}^n.$$

If $\sup z>0$, *then*

$$\sup_{x\in\mathbb{R}^n} z(t,x)\le e^{\lambda_0 t}\sup_{x\in\mathbb{R}^n} z_0(x), \quad 0\le t\le T.$$

Similarly, if

$$g(t,x)\ge 0, \quad 0\le t\le T,\ x\in\mathbb{R}^n,$$

and $\inf z<0$, *then*

$$\inf_{x\in\mathbb{R}^n} z(t,x)\ge e^{\lambda_0 t}\inf_{x\in\mathbb{R}^n} z_0(x), \quad 0\le t\le T.$$

In particular, if $g\equiv 0$ then

$$\|z(t,\cdot)\|_\infty\le e^{\lambda_0 t}\|z_0\|_\infty, \quad 0\le t\le T.$$

Proof. Let $\lambda\ge\lambda_0$, $\lambda>r_0$ and set $v(t,x)=z(t,x)e^{-\lambda t}$. Then

$$\begin{cases} v_t(t,x) = \mathcal{A}v(t,x) - \lambda v(t,x) + g(t,x)e^{-\lambda t}, & 0\le t\le T,\ x\in\mathbb{R}^n,\\ v(0,x) = z_0(x), & x\in\mathbb{R}^n.\end{cases}$$

Assume that $\sup z>0$, so that $\sup v>0$. We claim that

$$\text{(2.3)}\qquad \sup_{0\le t\le T,\,x\in\mathbb{R}^n} v(t,x)\le \sup z_0.$$

This means that $\sup_{0\le t\le T, x\in\mathbb{R}^n} z(t,x)e^{-\lambda t} \le \sup z_0$ so that, by letting $\lambda \to \lambda_0$, the statement follows.

Consider the sequence

$$v_k(t,x) = v(t,x) - \frac{1}{k}\varphi(x).$$

Then $\lim_{k\to\infty} \sup v_k = \sup v$. Moreover, for k sufficiently large, v_k has a positive maximum at some point (t_k, x_k). If $t_k = 0$ for all k then $\sup v_k \le \sup z_0 - \inf \varphi/k$ and (2.3) follows; if $t_k > 0$ then $D_t v_k(t_k,x_k) \ge 0$, $\mathcal{A}v_k(t_k,x_k) - r_0 v_k(t_k,x_k) \le 0$, and $g(t_k,x_k)e^{-\lambda t_k} \le 0$, so that

$$(\lambda - r_0)\max v_k \le \frac{1}{k}(\mathcal{A}\varphi(x_k) - \lambda\varphi(x_k)),$$

which is impossible for k large enough. Therefore (2.3) holds, so that the first statement holds. The second statement may be proved similarly. ∎

A similar maximum principle holds for elliptic equations.

PROPOSITION 2.2. *Let $z \in \bigcap_{p\ge1} W^{2,p}_{\rm loc}(\mathbb{R}^n)$ be a bounded solution of*

$$\lambda z - \sum_{i,j=1}^n q_{ij}D_{ij}z - \sum_{i=1}^n p_i D_i z - rz = f(x), \quad x \in \mathbb{R}^n, \tag{2.4}$$

where the data $q_{ij}, p_i, r, f : \mathbb{R}^n \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy

$$\sum_{i,j=1}^n q_{ij}\xi_i\xi_j \ge 0, \quad r \le r_0, \quad f \text{ continuous and bounded},$$

and there exist $\lambda_0 \ge r_0$ and $\varphi \in C^2(\mathbb{R}^n)$ such that (1.5) holds. Then for every $\lambda \ge \lambda_0$, $\lambda > r_0$ we have

$$\sup_{x\in\mathbb{R}^n} |z(x)| \le \frac{1}{\lambda - r_0} \sup_{x\in\mathbb{R}^n} |f(x)|.$$

Proof. In [14, §3.1] it has been proved that if $z \in \bigcap_{p\ge1} W^{2,p}_{\rm loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is such that $\mathcal{A}u$ is continuous, then at any relative maximum (respectively, minimum) point x_0 we have $\mathcal{A}u(x_0) \le 0$ (respectively, $\mathcal{A}u(x_0) \ge 0$). The rest of the proof is quite analogous to the proof of Proposition 2.1 and it is omitted. ∎

REMARK 2.3. Sufficient conditions for (1.5) to hold are easily given. If for instance $\varphi(x) = |x|^2$, then (1.5) is satisfied if there is λ_0 such that

$$\sup_{x\in\mathbb{R}^n} (2\,\mathrm{Tr}\,Q(x) + 2\langle P(x), x\rangle + (r(x) - \lambda_0)|x|^2) < \infty.$$

We may now derive the *a priori* estimates which are a crucial step of the paper.

THEOREM 2.4. *Let q_{ij}, p_i, and r be smooth functions satisfying conditions (1.2)–(1.4) and such that there exists φ satisfying (1.5). Moreover, assume that there exists $K > 0$ such that for every $x \in \mathbb{R}^n$,*

$$\begin{aligned} &\text{(i)} && |D^\beta q_{ij}(x)| \le K\nu(x), && |\beta| = 1,2,3,\\ &\text{(ii)} && |D^\beta p_i(x)| \le K(1 + |p(x)|), && |\beta| = 2,3,\\ &\text{(iii)} && |D^\beta r(x)| \le K(1 + |r(x)|), && |\beta| = 1,2,3. \end{aligned} \tag{2.5}$$

Let u_0 be continuous and bounded, and let u be a bounded classical solution to (1.7) with $g \equiv 0$ such that $t \mapsto \|D_i u(t,\cdot)\|_{L^\infty(K)}$ is in $L^1(0,1)$ for every $i = 1,\dots,n$ and for every compact subset $K \subset \mathbb{R}^n$. Then u is smooth for $t > 0$ and for every $T > 0$ there is $C = C(\nu_0, r_0, \lambda_0, K, n, T) > 0$ such that

$$\begin{aligned} &\text{(i)} && \|u(t,\cdot)\|_\infty \le e^{\lambda_0 t}\|u_0\|_\infty, && 0 < t \le T,\\ &\text{(ii)} && \|t^{k/2}D^\beta u(t,\cdot)\|_\infty \le C\|u_0\|_\infty, && 0 < t \le T,\ |\beta| = 1,2,3. \end{aligned} \tag{2.6}$$

Proof. Proposition 2.1 applied to equation (1.7) with $g \equiv 0$ gives immediately (2.6)(i). From the general regularity theory of parabolic problems it follows that u is smooth for $t > 0$ and moreover $\lim_{t\to0} t^{\beta/2}D^\beta u(t,x) = 0$ for every $x \in \mathbb{R}^n$, $|\beta| = 1,2,3$. Define a function $z(t,x)$ by

$$z = u^2 + \alpha t|Du|^2 + \frac{\alpha^2t^2}{2}|D^2u|^2 + \frac{\alpha^3t^3}{3}|D^3u|^2, \quad t > 0,\ x \in \mathbb{R}^n,$$

where $\alpha > 0$ is to be chosen later and $|Du|^2 = \sum_{i=1}^n (D_iu)^2$, $|D^2u|^2 = \sum_{i,j=1}^n (D_{ij}u)^2$, $|D^3u|^2 = \sum_{i,j,l=1}^n (D_{ijl}u)^2$. An elementary computation gives

$$\begin{cases} z_t(t,x) - \mathcal{A}z(t,x) = g(t,x) + r(x)z(t,x), & 0 < t \le T,\ x \in \mathbb{R}^n,\\ z(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where

$$g(t,x) = \sum_{i=1}^5 g_i(t,x),$$

$$\begin{aligned} g_1 = &-2\sum_{i,j=1}^n q_{ij}D_iuD_ju - 2\alpha t\sum_{i,j,l=1}^n q_{ij}D_{il}uD_{jl}u\\ &- \alpha^2t^2\sum_{i,j,l,m=1}^n q_{ij}D_{ilm}uD_{jlm}u - \frac{\alpha^3t^3}{3}\sum_{i,j,l,m,k=1}^n q_{ij}D_{ilmk}uD_{jlmk}u, \end{aligned}$$

$$g_2 = \alpha\sum_{i=1}^n |Du|^2 + \alpha^2t|D^2u|^2 + \frac{\alpha^3t^2}{2}|D^3u|^2,$$

$$g_3 = \sum_{i,j,l=1}^{n} D_l q_{ij}\Big(2\alpha t D_l u D_{ij}u + 2\alpha^2 t^2 \sum_{m=1}^{n} D_{lm}u D_{ijm}u$$
$$+ \alpha^3 t^3 \sum_{m,k=1}^{n} D_{lmk}u D_{ijmk}u\Big)$$
$$+ \sum_{i,j,l,m=1}^{n} D_{lm}q_{ij}\Big(\alpha^2 t^2 D_{lm}u D_{ij}u + \alpha^3 t^3 \sum_{k=1}^{n} D_{lmk}u D_{ijk}u\Big)$$
$$+ \frac{\alpha^3 t^3}{3}\sum_{i,j,l,m,k=1}^{n} D_{lmk}q_{ij} D_{lmk}u D_{ij}u,$$

$$g_4 = \sum_{i,l=1}^{n} D_l p_i\Big(2\alpha t D_l u D_i u + 2\alpha^2 t^2 \sum_{m=1}^{n} D_{lm}u D_{im}u$$
$$+ \alpha^3 t^3 \sum_{m,k=1}^{n} D_{lmk}u D_{imk}u\Big)$$
$$+ \sum_{i,l,m=1}^{n} D_{lm}p_i\Big(\alpha^2 t^2 D_{lm}u D_i u + \alpha^3 t^3 \sum_{k=1}^{n} D_{lmk}u D_{ik}u\Big)$$
$$+ \frac{\alpha^3 t^3}{3}\sum_{i,l,m,k=1}^{n} D_{lmk}p_i D_{lmk}u D_i u,$$

$$g_5 = \sum_{l=1}^{n} D_l r\Big(2\alpha t u D_l u + 2\alpha^2 t^2 \sum_{m=1}^{n} D_{lm}u D_m u + \alpha^3 t^3 \sum_{m,k=1}^{n} D_{lmk}u D_{mk}u\Big)$$
$$+ \sum_{l,m=1}^{n} D_{lm}r\Big(\alpha^2 t^2 u D_{lm}u + \alpha^3 t^3 \sum_{k=1}^{n} D_{lmk}u D_k u\Big)$$
$$+ \frac{\alpha^3 t^3}{3} u \sum_{l,m,k=1}^{n} D_{lmk}r D_{lmk}u.$$

We claim that if α is suitably small then

$$(2.7)\qquad g(t,x) + r(x)z(t,x) \le (1+2r_0)z(t,x), \qquad 0 < t \le T,\ x \in \mathbb{R}^n.$$

Then Proposition 2.1 applied to equation (1.7) with the modified operator

$$\widetilde{\mathcal{A}} = \mathcal{A} + (1+2r_0)I$$

and λ_0 replaced by $\omega_0 = \max\{\lambda_0, 3r_0+1\}$ gives $z(t,x) \le e^{\omega_0 t}\|u_0\|_\infty$ for $0 < t \le T$, $x \in \mathbb{R}^n$, and estimates (2.6)(ii) follow.

Let us estimate g. Concerning g_1 we have

$$g_1 \le -2\nu(x)\Big(|Du|^2 + \alpha t|D^2u|^2 + \frac{\alpha^2 t^2}{2}|D^3u|^2 + \frac{\alpha^3 t^3}{6}|D^4 u|^2\Big),$$

so that taking $\alpha \le \nu_0$ we get

$$g_1 + g_2 \le -\nu(x)\Big(|Du|^2 + \alpha t|D^2u|^2 + \frac{\alpha^2 t^2}{2}|D^3u|^2 + \frac{\alpha^3 t^3}{3}|D^4u|^2\Big).$$

Let us estimate g_3. By assumption (2.5)(i) we have, for every $\varepsilon > 0$,

$$|g_3(t,x)| \le K\nu(x)\Big\{2\alpha t \sum_{i,j,l=1}^{n}\Big(\frac{(D_l u)^2}{2\varepsilon} + \frac{\varepsilon}{2}(D_{ij}u)^2\Big)$$
$$+ 2\alpha^2 t^2 \sum_{i,j,l,m=1}^{n}\Big(\frac{(D_{lm}u)^2}{2\varepsilon} + \frac{\varepsilon}{2}(D_{ijm}u)^2\Big)$$
$$+ \alpha^3 t^3 \sum_{i,j,l,m,k=1}^{n}\Big(\frac{(D_{lmk}u)^2}{2\varepsilon} + \frac{\varepsilon}{2}(D_{ijmk}u)^2\Big)$$
$$+ \alpha^2 t^2 \sum_{i,j,l,m=1}^{n}\Big(\frac{(D_{ij}u)^2}{2} + \frac{1}{2}(D_{lm}u)^2\Big)$$
$$+ \alpha^3 t^3 \sum_{i,j,l,m,k=1}^{n}\Big(\frac{(D_{ijk}u)^2}{2} + \frac{1}{2}(D_{lmk}u)^2\Big)$$
$$+ \frac{\alpha^3 t^3}{3}\sum_{i,j,l,m,k=1}^{n}\Big(\frac{(D_{ij}u)^2}{2} + \frac{1}{2}(D_{lmk}u)^2\Big)\Big\}.$$

Take ε small ($\varepsilon = 1/(8nK)$) in such a way that

$$K\nu(x)\Big(2\nu_0 t\frac{\varepsilon}{2}n|D^2u|^2 + 2\nu_0^2 t^2\frac{\varepsilon}{2}n|D^3u|^2 + \nu_0^3 t^3\frac{\varepsilon}{2}n|D^4u|^2\Big)$$
$$\le \nu(x)\Big(\frac{1}{4}\nu_0 t|D^2u|^2 + \frac{1}{4}\frac{\nu_0^2 t^2}{2}|D^3u|^2 + \frac{1}{2}\frac{\nu_0^3 t^3}{3}|D^4u|^2\Big)$$

and then take α small enough in such a way that

$$K\nu(x)\alpha T n^2|Du|^2/\varepsilon \le \frac{\nu(x)}{2}|Du|^2,$$
$$K\nu(x)\Big(\alpha T n^2\Big(\frac{1}{\varepsilon}+1\Big) + \frac{\alpha^2 T^2 n^3}{6}\Big)|D^2u|^2 \le \frac{1}{4}\nu(x)|D^2u|^2,$$
$$K\nu(x)\alpha T n^2\Big(\frac{1}{2\varepsilon} + \frac{7}{6}\Big)|D^3u|^2 \le \frac{1}{4}\frac{\nu(x)}{2}|D^3u|^2.$$

Then

$$g_1 + g_2 + g_3 \le -\frac{1}{2}\nu(x)\Big(|Du|^2 + \alpha t|D^2u|^2 + \frac{\alpha^2 t^2}{2}|D^3u|^2 + \frac{\alpha^3 t^3}{3}|D^4u|^2\Big).$$

Let us estimate g_4: for every $\varepsilon > 0$ we have

$$\begin{aligned} g_4(t,x) \le{}& 2\alpha t p(x)|Du|^2 + 2\alpha^2t^2p(x)|D^2u|^2 + \alpha^3t^3p(x)|D^3u|^2 \\ &+ K(1+|p(x)|)\Big\{\alpha^2t^2\sum_{l,m,i=1}^n\Big(\frac{\varepsilon}{2}(D_{lm}u)^2 + \frac{1}{2\varepsilon}(D_iu)^2\Big) \\ &+ \alpha^3t^3\sum_{l,m,i,k=1}^n\Big(\frac{\varepsilon}{2}(D_{lmk}u)^2 + \frac{1}{2\varepsilon}(D_{ik}u)^2\Big) \\ &+ \frac{\alpha^3t^3}{3}\sum_{l,m,i,k=1}^n\Big(\frac{\varepsilon}{2}(D_{lmk}u)^2 + \frac{1}{2\varepsilon}(D_iu)^2\Big)\Big\}. \end{aligned}$$

If $p(x) \ge 0$ take $\varepsilon = 1$ to get

$$\begin{aligned} g_4(t,x) \le{}& \Big(2\alpha t p_0 + K(1+p_0)\Big(\frac{n^2}{2}\alpha^2t^2 + \frac{n^3}{6}\alpha^3t^3\Big)\Big)|Du|^2 \\ &+ \Big(2\alpha^2t^2p_0 + K(1+p_0)\Big(\frac{n}{2}\alpha^2t^2 + \frac{n^2}{2}\alpha^3t^3\Big)\Big)|D^2u|^2 \\ &+ \Big(\alpha^3t^3p_0 + K(1+p_0)\Big(\frac{n}{2}\alpha^3t^3 + \frac{n}{6}\alpha^3t^3\Big)\Big)|D^3u|^2. \end{aligned}$$

Taking α small enough we get

$$g_4 \le \frac{\nu_0}{2}\Big(|Du|^2 + \alpha t|D^2u|^2 + \frac{\alpha^2t^2}{2}|D^3u|^2\Big),$$

so that

$$g_1 + g_2 + g_3 + g_4 \le 0. \tag{2.8}$$

If $p(x) \le 0$ take ε small ($\varepsilon = 3/(4Kn^2)$) in such a way that

$$2\alpha^2t^2p(x) + K|p(x)|n^2\alpha^2t^2\varepsilon/2 \le \alpha^2t^2p(x),$$

$$\alpha^3t^3p(x) + K|p(x)|(n\alpha^3t^3\varepsilon/2 + n^2\alpha^3t^3\varepsilon/6) \le \alpha^3t^3p(x)/2,$$

and then take α small enough such that

$$\begin{aligned} 2\alpha Tp(x) + K(1+|p(x)|)\Big(\frac{\alpha^2T^2n^2}{2\varepsilon} + \frac{\alpha^3T^3n^3}{6\varepsilon}\Big) &\le \frac{\nu_0}{2}, \\ \alpha Tp(x) + \frac{K\alpha Tn^2\varepsilon}{2} + K(1+|p(x)|)\frac{\alpha^2T^2n^2}{2\varepsilon} &\le \frac{\nu_0}{2}, \\ \alpha Tp(x) + K\Big(\frac{\alpha Tn\varepsilon}{2} + \frac{\alpha Tn\varepsilon}{6}\Big) &\le \frac{\nu(x)}{4}. \end{aligned}$$

Then also in this case (2.8) holds.

Let us estimate g_5. For every $\varepsilon > 0$ we have

$$\begin{aligned} g_5(t,x) \le K(1+|r(x)|)\Big\{&2\alpha t\sum_{l=1}^n\Big(\frac{\varepsilon}{2}(D_lu)^2 + \frac{1}{2\varepsilon}u^2\Big) \\ &+ 2\alpha^2t^2\sum_{l,m=1}^n\Big(\frac{\varepsilon}{2}(D_{lm}u)^2 + \frac{1}{2\varepsilon}(D_mu)^2\Big) \\ &+ \alpha^3t^3\sum_{l,m,k=1}^n\Big(\frac{\varepsilon}{2}(D_{lmk}u)^2 + \frac{1}{2\varepsilon}(D_{mk}u)^2\Big) \\ &+ \alpha^2t^2\sum_{l,m=1}^n\Big(\frac{\varepsilon}{2}(D_{lm}u)^2 + \frac{1}{2\varepsilon}u^2\Big) \\ &+ \alpha^3t^3\sum_{l,m,k=1}^n\Big(\frac{\varepsilon}{2}(D_{lmk}u)^2 + \frac{1}{2\varepsilon}(D_ku)^2\Big) \\ &+ \frac{\alpha^3t^3}{3}\sum_{l,m,k=1}^n\Big(\frac{\varepsilon}{2}(D_{lmk}u)^2 + \frac{1}{2\varepsilon}u^2\Big)\Big\}. \end{aligned}$$

Let ε be so small ($\varepsilon = 1/(14K)$) that

$$K\varepsilon(\nu_0t|Du|^2 + 3\nu_0^2t^2|D^2u|^2/2 + 7\nu_0^3t^3|D^3u|^2/6) \le \frac{z}{2},$$

and then let α be so small that

$$\frac{K\alpha T}{\varepsilon}((n+\alpha Tn^2/2+\alpha^2T^2n^3/6)u^2 + (n+\alpha Tn^2/2)|Du|^2 + n|D^2u|^2/2) \le \frac{z}{2}.$$

Then

$$|g_5| \le (1+|r(x)|)z,$$

so that if $r(x) \le 0$ then $g_5 + rz \le z$, and if $r(x) \ge 0$ then $g_5 + rz \le (1+2r_0)z$. Recalling (2.8) we get (2.7), and the statement follows. ∎

REMARK 2.5. In the case where assumptions (2.5) are replaced by the less restrictive ones:

$$\text{(2.9)}\qquad \begin{aligned} &\text{(i)} && |D^\beta q_{ij}(x)| \le K\nu(x), && |\beta| = 1, \\ &\text{(ii)} && |D^\beta r(x)| \le K(1+|r(x)|), && |\beta| = 1, \end{aligned}$$

the procedure of Theorem 2.4 gives bounds for u and Du. It is sufficient to replace the function z by

$$\widetilde{z} = u^2 + \alpha t|Du|^2,$$

and to apply the maximum principle of Proposition 2.1 to the equations satisfied by u and $\widetilde{z}$, to get

$$\text{(2.10)}\qquad \begin{aligned} &\text{(i)} && \|u(t,\cdot)\|_\infty \le e^{r_0t}\|u_0\|_\infty, && 0 < t \le T, \\ &\text{(ii)} && \|t^{1/2}D^\beta u(t,\cdot)\|_\infty \le C\|u_0\|_\infty, && 0 < t \le T,\ |\beta| = 1. \end{aligned}$$

Arguing again as in the proof of Theorem 2.4 the following *a priori* estimates may be proved.

THEOREM 2.6. *Let the assumptions of Theorem* 2.4 *be satisfied. Let* u_0 *be a* C^3 *bounded function having bounded derivatives up to the third order, and let* u *be a classical solution to* (1.7) *with* $g\equiv 0$. *Then for every* $T>0$ *there is* $C=C(\nu_0,r_0,\lambda_0,K,n,T)>0$ *such that*

$$\|D^\beta u(t,\cdot)\|_\infty \le C\|u_0\|_{C^3(\mathbb{R}^n)}, \quad 0<t\le T,\ |\beta|=1,2,3. \tag{2.11}$$

3. The semigroup associated with $\mathcal{A}$. Even if the very strong *a priori* estimates (2.6) hold, it is not clear whether the problem

$$\begin{cases} u_t(t,x)-\mathcal{A}u(t,x)=0, & t>0,\ x\in\mathbb{R}^n,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{3.1}$$

has a solution. This section is devoted to proving existence, uniqueness and regularity of the solution of (3.1) and then of (1.7). The proof is in three steps: first we consider the case where P is Lipschitz continuous and it has bounded second and third order derivatives, then the case where P is Lipschitz continuous, and then the general case. In any case we shall prove the following results.

THEOREM 3.1. *For every* $u_0\in C(\mathbb{R}^n)$ *problem* (3.1) *has a unique bounded classical solution* u. *In addition,* u *is smooth for* $t>0$ *and it satisfies* (2.6). *If* $u_0\in C^3(\mathbb{R}^n)$, *then* u *satisfies* (2.11).

Thanks to Theorem 3.1 we may define a semigroup of linear operators in $C(\mathbb{R}^n)$ by

$$(T(t)u_0)(x)\doteq u(t,x), \quad t\ge 0,\ x\in\mathbb{R}^n,\ u_0\in C(\mathbb{R}^n),$$

where u is the unique bounded solution of problem (3.1). By taking $T=1$ in estimates (2.6), (2.11), and using the semigroup law, it follows that there exists $\omega\in\mathbb{R}$ such that

$$\begin{aligned} &\|T(t)\|_{L(C(\mathbb{R}^n),C^k(\mathbb{R}^n))}\le \frac{Ce^{\omega t}}{t^{k/2}}, \quad k=1,2,3;\\ &\|T(t)\|_{L(C^3(\mathbb{R}^n))}\le Ce^{\omega t}, \quad t>0, \end{aligned} \tag{3.2}$$

so that by interpolation

$$\|T(t)\|_{L(C^\theta(\mathbb{R}^n),C^\alpha(\mathbb{R}^n))}\le \frac{C_{\theta,\alpha}e^{\omega t}}{t^{(\alpha-\theta)/2}}, \quad 0\le\theta\le\alpha\le 3. \tag{3.3}$$

Therefore, as far as spaces of continuous functions and Hölder spaces are concerned, $T(t)$ has the same behavior of semigroups generated by elliptic operators with bounded coefficients.

Unfortunately, in general $T(t)$ is not strongly continuous in $C(\mathbb{R}^n)$ (not even in the space $\mathrm{BUC}(\mathbb{R}^n)$ of uniformly continuous and bounded functions), and it is not analytic (not even in $L^p(\mathbb{R}^n)$), as the counterexamples in [11] and [18] show. So we cannot use the standard semigroup theory to study the inhomogeneous problem (1.7), but we shall use the strong smoothing properties of $T(t)$.

We shall see that if g has some mild smoothness property then the (unique) solution of (1.7) is given by the variation of constants formula

$$u(t,x)=(T(t)u_0)(x)+\int_0^t (T(t-s)g(s,\cdot))(x)\,ds, \quad 0\le t\le T. \tag{3.4}$$

We define the function spaces which will be involved in the proof.

DEFINITION 3.2. Let $\alpha,\beta\ge 0$ and $a<b$. $C^{\alpha,\beta}([a,b]\times\mathbb{R}^n)$ denotes the space of continuous functions u such that $t\mapsto u(t,x)\in C^\alpha([a,b])$ for every $x\in\mathbb{R}^n$, $x\mapsto u(t,x)\in C^\beta(\mathbb{R}^n)$ for every $t\in[a,b]$, and

$$\|u\|_{C^{\alpha,\beta}([a,b]\times\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n}\|u(\cdot,x)\|_{C^\alpha([a,b])}+\sup_{t\in[a,b]}\|u(t,\cdot)\|_{C^\beta(\mathbb{R}^n)}<\infty.$$

If $\alpha>0$, we denote by $C^{0,\beta}_\alpha((a,b]\times\mathbb{R}^n)$ the space of continuous functions $g:(a,b]\times\mathbb{R}^n\to\mathbb{R}$ such that $g(t,\cdot)\in C^\beta(\mathbb{R}^n)$ for every $t\in(a,b]$ and

$$\|g\|_{C^{0,\beta}_\alpha((a,b]\times\mathbb{R}^n)}=\sup_{a<t\le b}(t-a)^\alpha\|g(t,\cdot)\|_{C^\beta(\mathbb{R}^n)}<\infty.$$

PROPOSITION 3.3. *Let* $\alpha,\theta\in(0,1)$, $\theta<\beta\le 2+\theta$ *and let* $u_0\in C(\mathbb{R}^n)$ *and* $g\in C^{0,\beta}_\alpha((0,T]\times\mathbb{R}^n)$. *Then the function* u *defined by* (3.4) *belongs to* $C([0,T]\times\mathbb{R}^n)\cap C^{1,2}((0,T]\times\mathbb{R}^n)\cap C^{0,2+\theta}_{1+\theta/2}((0,T]\times\mathbb{R}^n)$, *and it is the unique bounded solution of* (1.7). *Moreover, there is* $C>0$ *such that for* $0<t\le T$,

$$\|u(t,\cdot)\|_\infty+t^{1+\theta/2}\|u(t,\cdot)\|_{C^{2+\theta}}\le C(\|u_0\|_\infty+\sup_{0<t\le T}\|g(t,\cdot)\|_{C^\beta(\mathbb{R}^n)}).$$

If in addition $u_0\in C^{2+\theta}(\mathbb{R}^n)$ *and* $t\mapsto g(t,\cdot)\in C^{0,\beta}([0,T]\times\mathbb{R}^n)$, *then* $u\in C^{1,2}([0,T]\times\mathbb{R}^n)\cap C^{0,2+\theta}([0,T]\times\mathbb{R}^n)$, *and there is* $C>0$ *such that*

$$\|u(t,\cdot)\|_{C^{2+\theta}}\le C(\|u_0\|_{C^{2+\theta}}+\|g\|_{C^{0,\beta}([0,T]\times\mathbb{R}^n)}), \quad 0\le t\le T.$$

Note that the last statement of the proposition is weaker than the statement of Theorem 2, since $\beta>\theta$. To prove Theorem 2 we will need a more refined technique. However, we will need the result of Proposition 3.3 as a preliminary step. In the proof we shall use the next two technical lemmas.

LEMMA 3.4. *If* $\{\varphi_j\}_{j\in\mathbb{N}}$ *is a bounded sequence in* $C(\mathbb{R}^n)$ *converging to a function* φ *uniformly on every compact subset of* $\mathbb{R}^n$, *then for every* $T>0$ *and for every compact set* $K\subset\mathbb{R}^n$ *we have*

$$\lim_{j\to\infty}\sup_{0\le t\le T}\|T(t)(\varphi_j-\varphi)_{|K}\|_{L^\infty(K)}=0.$$

The second lemma deals with Hölder continuity of certain integrals depending on a parameter.

LEMMA 3.5. *Let $\theta \in (0,3)$, not an integer, let I be a real interval, and let $\varphi : I \to C^{\theta}(\mathbb{R}^n)$ be such that for every $x \in \mathbb{R}^n$ the real function $t \mapsto \varphi(t)(x)$ is continuous in I, and $\|\varphi(t)\|_{C^\theta} \le c(t)$ with $c \in L^1(I)$. Then the function*

$$f(x) = \int_I \varphi(t)(x)\,dt, \qquad x \in \mathbb{R}^n,$$

belongs to $C^{\theta}(\mathbb{R}^n)$, and

$$\|f\|_{C^\theta} \le \|c\|_{L^1(I)}.$$

Proof. We recall that $C^{\theta}(\mathbb{R}^n)$ is the space of functions $f \in C(\mathbb{R}^n)$ such that

$$[f]_\theta = \sup_{x,h\in\mathbb{R}^n, h\ne 0} |h|^{-\theta}\Big|\sum_{l=0}^{3}(-1)^l f(x+lh)\Big| < \infty,$$

and the norm

$$f \mapsto \|f\|_\infty + [f]_\theta$$

is equivalent to the C^θ norm. See e.g. [T, Sect. 2.7.2]. If $\varphi : I \to C^{\theta}(\mathbb{R}^n)$ is such that $t \mapsto \varphi(t)(x)$ is continuous for every $x \in \mathbb{R}^n$ and $\|\varphi(t)\|_{C^\theta} \le c(t)$ with $c \in L^1(I)$, then for every $x, h \in \mathbb{R}^n$ we have

$$\Big|\sum_{l=0}^{3}(-1)^l \int_I \varphi(t)(x+lh)\,dt\Big| \le \int_I \Big|\sum_{l=0}^{3}(-1)^l\varphi(t)(x+lh)\Big|\,dt \le \int_I Kc(t)\,dt\,|h|^\theta,$$

so that $f(x) = \int_I \varphi(t)(x)\,dt$ belongs to $C^\theta(\mathbb{R}^n)$, and the statement follows. ∎

The next subsections will be devoted to proving Theorem 3.1, Proposition 3.3, and Lemma 3.4.

3.1. *The case where P is Lipschitz continuous with bounded second and third order derivatives.* Throughout the whole subsection we shall assume that (1.2)–(1.5) hold, and moreover that P is Lipschitz continuous with bounded second and third order derivatives.

Proof of Theorem 3.1. It is not difficult to construct a family of approximating problems with bounded coefficients which satisfy the assumptions of Theorem 2.4. Indeed, for every $k \in \mathbb{N}$ let $\varphi_k : \mathbb{R} \to \mathbb{R}$ be any smooth function such that

$$\varphi_k(x) = \begin{cases} x & \text{for } |x| \le k, \\ k+1 & \text{for } x \ge k+1, \\ -k-1 & \text{for } x \le -k-1, \end{cases}$$

$$0 \le \varphi_k'(x) \le 1, \qquad |\varphi_k''(x)| \le 2, \qquad |\varphi_k'''(x)| \le 2,$$

and set

$$\Phi_k(x) = (\varphi_k(x_1), \ldots, \varphi_k(x_n)), \qquad x \in \mathbb{R}^n,$$

$$Q_k(x) = Q(\Phi_k(x)), \qquad P_k(x) = P(\Phi_k(x)), \qquad r_k(x) = r(\Phi_k(x)), \qquad x \in \mathbb{R}^n.$$

Then Q_k, P_k, r_k are smooth and bounded, with bounded derivatives up to the third order; moreover, they satisfy estimates (1.4) with $\nu(x)$ replaced by $\nu(\Phi_k(x)) \le \nu_0$, $p(x)$ replaced by $p(\Phi_k(x))$, and $r(x)$ replaced by $r(\Phi_k(x))$. By the standard theory of parabolic equations with bounded coefficients the problem

$$\begin{cases} D_t u_k(t,x) = \operatorname{Tr}(Q_k(x)D^2u(t,x)) \\ \qquad\qquad + \langle P_k(x), Du(t,x)\rangle + r_k(x)u(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \\ u_k(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases}$$

has a unique classical bounded solution u_k, and $t^{|\beta|/2}|D^\beta u_k(t,x)|$, $|\beta| = 1,2,3$, is bounded in $(0,1)\times\mathbb{R}^n$. By Theorem 2.4, u_k satisfies estimates (2.6), with constants independent of k. Therefore the sequence u_k is bounded in $C^{1,3}([\varepsilon,T]\times K)$ for every $0<\varepsilon<T<\infty$, and for every compact $K \subset \mathbb{R}^n$. Let us show that $\{u_k : k \in \mathbb{N}\}$ is equicontinuous in $[0,T]\times K$ for every $T > 0$ and for every compact $K \subset \mathbb{R}^n$.

Fix any $R > 0$ and let θ be a smooth cutoff function such that

$$(3.5)\qquad 0 \le \theta(x) \le 1, \qquad \theta \equiv 1 \quad \text{on } B(0,R), \qquad \theta \equiv 0 \quad \text{outside } B(0,R+1).$$

The function $v_k(t,x) = u_k(t,x)\theta(x)$ is continuous and bounded, and for $k \ge R+1$ it satisfies

$$(3.6)\qquad \begin{cases} \dfrac{\partial}{\partial t} v_k = \mathcal{A}v_k + \psi_k, & 0 < t \le T,\ x \in B(0,R+1), \\ v_k(t,x) = 0, & 0 < t \le T,\ x \in \partial B(0,R+1), \\ v_k(0,x) = u_0(x)\theta(x), & x \in B(0,R+1), \end{cases}$$

where

$$(3.7)\qquad \psi_k = -2\sum_{i,j=1}^{n} q_{ij}D_i\theta D_j u_k(t,x) - u_k(t,x)\Big(\sum_{i,j=1}^{n} q_{ij}D_{ij}\theta + \sum_{i=1}^{n} p_i D_i\theta\Big).$$

For every k, ψ_k is continuous in $(0,T]\times B(0,R+1)$ and $\sqrt{t}\,\psi_k$ is bounded. Therefore,

$$(3.8)\qquad v_k(t,\cdot) = T_R(t)(u_0\theta) + \int_0^t T_R(t-s)\psi_k(s,\cdot)\,ds, \qquad 0 < t \le T,$$

where $T_R(t)$ is the analytic semigroup generated by the realization of $\mathcal{A}$ with Dirichlet boundary condition in $C(B(0,R+1))$. Since $\|\psi_k(t,\cdot)\|_\infty \le Ct^{-1/2}$ for every $t \in (0,T]$, with constant C independent of k, by the standard theory of parabolic equations in bounded sets $(t,x) \mapsto \int_0^t T_R(t-s)\psi_k(s,\cdot)\,ds$ belongs to $C^{\beta,2\beta}([0,T]\times B(0,R+1))$ for every $\beta \in (0,1/2)$, with norm independent of k. Therefore the sequence v_k is equicontinuous in $[0,T]\times B(0,R+1)$, and since $u_k(t,x) = v_k(t,x)$ for $|x| \le R$, the sequence u_k is equicontinuous in $[0,T]\times B(0,R)$. Since R is arbitrary, u_k is equicontinuous in $[0,T]\times K$ for every compact $K\subset\mathbb{R}^n$.

A subsequence converges to u in $C([0,T]\times K)\cap C^{1-\alpha,3-\alpha}([\varepsilon,T]\times K)$ for every $\alpha\in(0,1)$, $0<\varepsilon<T<\infty$ and for every compact $K\subset\mathbb{R}^n$. Such a u is a solution of problem (1.7) with $g\equiv 0$, and it is the unique bounded solution thanks to Proposition 2.1. Since the data are smooth, u is smooth for $t>0$. Since u is the pointwise limit of a sequence of functions satisfying (2.6), it also satisfies (2.6). If in addition $u_0\in C^3(\mathbb{R}^n)$ then every u_k satisfies (2.11) and hence u satisfies (2.11).

All the statements of Theorem 3.1 follow, with the exception of the estimates on the third order space derivatives. They are recovered by applying once again Theorems 2.4 and 2.6. ∎

Proof of Lemma 3.4. Set $f_j=\varphi_j-\varphi$, and fix any $R>0$. Let θ be a smooth cutoff function satisfying (3.5).

For every $k\in\mathbb{N}$ let $T_k(t)$ be the analytic semigroup generated by the realization of the operator

$$\mathcal{A}_k=\mathrm{Tr}(QD^2\cdot)+\langle P_k,D\cdot\rangle+r_kI$$

in $C(\mathbb{R}^n)$ (see the proof of Theorem 3.1). The function

$$v_{j,k}(t,x)=(T_k(t)f_j)(x)\theta(x)e^{-\lambda t}$$

is continuous and bounded, and for $k\ge R+1$ it satisfies

$$\frac{\partial}{\partial t}v_{j,k}=\mathcal{A}v_{j,k}-\lambda v_{j,k}+\psi_{j,k},\qquad v_{j,k}(0,\cdot)=f_j\theta,$$

where

$$\psi_{j,k}=-2e^{-\lambda t}\sum_{i,j=1}^n q_{ij}D_i\theta D_j(T_k(t)f_j)-e^{-\lambda t}(T_k(t)f_j)\Big(\sum_{i,j=1}^n q_{ij}D_{ij}\theta+\sum_{i=1}^n p_iD_i\theta\Big).$$

Now, $t\mapsto\psi_{j,k}(t,\cdot)$ belongs to $C((0,T];X)\cap L^1(0,T;X)$ for every j and k. Therefore,

$$v_{j,k}(t,\cdot)=e^{-\lambda t}T_k(t)(f_j\theta)+\int_0^t e^{-\lambda(t-s)}T_k(t-s)\psi_{j,k}(s,\cdot)\,ds,\qquad 0<t\le T.$$

Since $\|\psi_{j,k}(t,\cdot)\|_\infty\le Ct^{-1/2}$ for every $t\in(0,T]$, with constant C independent of j and k, we have

$$\|v_{j,k}(t,\cdot)\|_\infty\le e^{(-\lambda+r_0)t}\|f_j\theta\|_\infty+CT^{1/2}\int_0^1\frac{e^{(-\lambda+r_0)(1-\sigma)}}{\sigma^{1/2}}\,d\sigma.$$

Fix any $\varepsilon>0$ and let λ_0 be so large that $CT^{1/2}\int_0^1 e^{(-\lambda_0+r_0)(1-\sigma)}\sigma^{-1/2}\,d\sigma\le\varepsilon$. Then

$$\|(T_k(t)f_j)\theta\|_\infty\le e^{r_0t}\|f_j\theta\|_\infty+\varepsilon e^{\lambda_0 t}.$$

Since θ vanishes outside $B(0,R+1)$ and f_j goes to 0 uniformly on every compact set, for j large enough we have $\|f_j\theta\|_\infty\le\varepsilon$. Therefore,

$$\lim_{j\to\infty}\sup_{0\le t\le T,\ k\in\mathbb{N}}\|T_k(t)(\varphi_j-\varphi)_{|B(0,R)}\|_{L^\infty(B(0,R))}=0.$$

Since the function $T(t)(\varphi_j-\varphi)_{|B(0,R)}$ is the uniform limit of a subsequence of $T_k(t)(\varphi_j-\varphi)_{|B(0,R)}$, we get

$$\lim_{j\to\infty}\sup_{0\le t\le T}\|T(t)(\varphi_j-\varphi)_{|B(0,R)}\|_{L^\infty(B(0,R))}=0.$$

As R is arbitrary the statement follows. ∎

Proof of Proposition 3.3. The proof is similar to the one given in [17], with minor modifications; we write it down for the reader's convenience.

Let us prove that u is continuous in $[0,T]\times\mathbb{R}^n$. We already know that $(t,x)\mapsto T(t)u_0(x)$ is continuous. By Lemma 3.4 the function $(t,s,x)\mapsto(T(t-s)g(s,\cdot))(x)$ is continuous in $\{(t,s):0\le s\le t\le T\}\times\mathbb{R}^n$, so that

$$v(t,x)=\int_0^t(T(t-s)g(s,\cdot))(x)\,ds,\qquad 0\le t\le T,\ x\in\mathbb{R}^n,$$

is well defined and continuous.

Let us prove that $v(t,\cdot)\in C^{2+\theta}(\mathbb{R}^n)$ for every $t\in(0,T]$. By estimates (3.3) we have $\|T(t-s)\|_{L(C^\beta(\mathbb{R}^n),C^{2+\theta}(\mathbb{R}^n))}\le C(t-s)^{1-(\beta-\theta)/2}$ for $0\le s<t\le T$. Therefore,

$$\|T(t-s)g(s)\|_{C^{2+\theta}(\mathbb{R}^n)}\le\frac{C}{s^\alpha(t-s)^{1-(\beta-\theta)/2}}\|g\|_{C^{0,\beta}_\alpha((0,T]\times\mathbb{R}^n)}.$$

Since $s\mapsto(s^\alpha(t-s)^{1-(\beta-\theta)/2})^{-1}$ is in $L^1(0,t)$, by Lemma 3.5 we have $v(t,\cdot)\in C^{2+\theta}(\mathbb{R}^n)$ for every $t\in(0,T]$, and

$$\|v(t,\cdot)\|_{C^{2+\theta}}\le C\int_0^t\frac{ds}{s^\alpha(t-s)^{1-(\beta-\theta)/2}}=\frac{C'}{t^{\alpha+(\beta-\theta)/2}}\|g\|_{C^{0,\beta}_\alpha((0,T]\times\mathbb{R}^n)}.$$

Therefore, $t\mapsto v(t,\cdot)$ is bounded in $[\varepsilon,T]$ with values in $C^{2+\theta}(K)$ for every $\varepsilon\in(0,T)$. Since v is continuous, it belongs to $C([0,T];C(K))$ for every compact set $K\subset\mathbb{R}^n$. By [14, Prop. 1.1.3(iii), 1.1.4(iii)], $t\mapsto v(t,\cdot)$ belongs to $C([\varepsilon,T];C^2(K))$. Therefore, v and its first and second order space derivatives are continuous and bounded in $[\varepsilon,T]\times\mathbb{R}^n$. Concerning the regularity with respect to t, we know that for every $\varphi\in C(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$ the function $t\mapsto(T(t)\varphi)(x)$ is continuously differentiable in $(0,\infty)$, with $(\partial/\partial)t(T(t)\varphi)(x)=(\mathcal{A}T(t)\varphi)(x)$. Therefore for every $s\in[0,T]$ the function $t\mapsto(T(t-s)g(s,\cdot))(x)$ is continuously differentiable in $(s,T]$, and

$(\partial/\partial)t(T(t-s)g(s,\cdot))(x) = (\mathcal{A}T(t-s)g(s,\cdot))(x)$. By (3.3), if $|x| \le R$ we have

$$|(\mathcal{A}T(t-s)g(s,\cdot))(x)| \le \frac{C}{s^\alpha}\left(\frac{1}{(t-s)^{1-\beta/2}} + \frac{R}{(t-s)^{1/2-\beta/2}}\right)\|g\|_{C^{0,\beta}_\alpha((0,T]\times\mathbb{R}^n)}$$

so that $(\partial/\partial t)(T(t-s)g(s,\cdot))(x)$ is in $L^1(0,t)$. Then v is continuously differentiable with respect to time and

$$v_t = \int_0^t \mathcal{A}T(t-s)g(s)\,ds + g \quad \text{in } (0,T]\times\mathbb{R}^n.$$

Moreover, for every multiindex β with $|\beta| = 1,2$ and for $t \in (0,T]$ the function $s \mapsto D^\beta(T(t-s)g(s,\cdot))_{|K}$ belongs to $C(0,T;C(K)) \cap L^1((0,T);C(K))$ for every compact set $K \subset \mathbb{R}^n$. The realization of the derivative D^β is a closed operator in $C(K)$, so that

$$D^\beta \int_0^t T(t-s)g(s,\cdot)(x)\,ds = \int_0^t D^\beta T(t-s)g(s,\cdot)(x)\,ds$$

for every $x \in K$. Therefore

$$v_t = \mathcal{A}\int_0^t T(t-s)g(s)\,ds + g \quad \text{in } (0,T]\times\mathbb{R}^n.$$

By (3.3), the function $t \mapsto t^{1+\theta/2}T(t)u_0$ is bounded with values in $C^{2+\theta}(\mathbb{R}^n)$. Moreover, $(t,x) \mapsto (T(t)u_0)(x)$ belongs to $C^{1,2}((0,T]\times\mathbb{R}^n)$. Then the function

$$u = T(t)u_0 + v, \quad 0 \le t \le T,\ x \in \mathbb{R}^n,$$

belongs to $C^{1,2}((0,T]\times\mathbb{R}^n)$, satisfies (1.7), and

$$\sup_{0<t\le T} t^{1+\theta/2}\|u(t,\cdot)\|_{C^{2+\theta}(\mathbb{R}^n)} \le K(\|u_0\|_\infty + \sup_{0<t\le T} t^\alpha \|g(t,\cdot)\|_{C^\beta(\mathbb{R}^n)}).$$

If in addition g is bounded with values in $C^\beta(\mathbb{R}^n)$ then $v(t,\cdot)$ is bounded up to $t=0$ with values in $C^{2+\theta}(\mathbb{R}^n)$, since by (3.3),

$$\|T(t)\|_{L(C^\beta(\mathbb{R}^n),C^{2+\theta}(\mathbb{R}^n))} \le C(t-s)^{-1+(\beta-\theta)/2} \in L^1(s,t).$$

Again by (3.3), if $u_0 \in C^{2+\theta}(\mathbb{R}^n)$ then also $t \mapsto T(t)u_0$ is bounded with values in $C^{2+\theta}(\mathbb{R}^n)$. Then, arguing as above, one sees that $u \in C^{1,2}([0,T]\times\mathbb{R}^n)$. ∎

3.2. *The case where P is Lipschitz continuous*. Throughout this subsection we shall assume that (1.2)–(1.4) hold, that there exists φ satisfying (1.5), and that P is Lipschitz continuous. Then the procedure of the previous subsection cannot be followed to prove Theorem 3.1 because the truncated functions P_k do not satisfy in general the coercivity condition (1.4). To overcome this difficulty we shall argue as in [17, Prop. 2.3].

Proof of Theorem 3.1. Let φ be any mollifier, and set

$$\widetilde{P}(x) = \int_{\mathbb{R}^n} P(x-y)\varphi(y)\,dy, \quad x \in \mathbb{R}^n. \tag{3.9}$$

Then $\widetilde{P}$ is smooth, with bounded derivatives of all orders. Moreover,

$$|\widetilde{P}(x) - P(x)| \le \int_{\mathbb{R}^n} |P(x-y) - P(x)|\varphi(y)\,dy \le [P]_{\mathrm{Lip}} \int_{\mathbb{R}^n} |y|\varphi(y)\,dy,$$

so that $\widetilde{P} - P$ is bounded.

Set

$$\widetilde{\mathcal{A}}u(x) = \mathrm{Tr}(Q(x)D^2u(x)) + \langle \widetilde{P}(x), Du(x)\rangle + ru, \tag{3.10}$$

and let $\widetilde{T}(t)$ be the associated semigroup, which exists thanks to the results of the previous subsection. By the change of the unknown function $v(t,x) = e^{-\lambda t}u(t,x)$, with $\lambda > 0$, problem (3.1) is transformed into

$$\begin{cases} v_t = \mathcal{A}v - \lambda v, & 0 < t \le T,\ x \in \mathbb{R}^n, \\ v(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.11}$$

which is equivalent to

$$\begin{cases} v_t - \widetilde{\mathcal{A}}v = -\lambda v + \langle \widetilde{P} - P, Dv\rangle, & 0 < t \le T,\ x \in \mathbb{R}^n, \\ v(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.12}$$

By setting

$$(\Phi g)(t,x) = \int_0^t e^{-\lambda(t-s)}\widetilde{T}(t-s)g(s,\cdot)\,ds(x),$$

for every $g \in C([0,T]\times\mathbb{R}^n)$, problem (3.12) is equivalent to

$$v(t,x) = (\Gamma v)(t,x) = e^{-\lambda t}\widetilde{T}(t)u_0(x) + [\Phi(\langle \widetilde{P} - P, Dv\rangle)](t,x), \quad 0 \le t \le T,\ x \in \mathbb{R}^n. \tag{3.13}$$

We are going to prove that Γ maps the space

$$Y = C([0,T]\times\mathbb{R}^n) \cap C^{0,2+\theta}_{1+\theta/2}((0,T]\times\mathbb{R}^n)$$

into itself, and it is a contraction if λ is large enough. For every $u \in Y$, each derivative D_iu belongs to $C^{0,\beta}_{(1+\beta)/2}((0,T]\times\mathbb{R}^n)$ and $t \mapsto D_iu(t,\cdot)$ is in $C((0,T);C^\beta(K))$ for every $\beta \in (0,1)$ and for every compact set K. Indeed, using the well known interpolation estimates for Hölder norms (see e.g. [20, §2.7.2]) we get

$$\|u(t,\cdot)\|_{C^{1+\beta}(\mathbb{R}^n)} \le C\|u(t,\cdot)\|^{(1+\beta)/(2+\theta)}_{C^{2+\theta}(\mathbb{R}^n)}\|u(t,\cdot)\|^{1-(1+\beta)/(2+\theta)}_{L^\infty(\mathbb{R}^n)} \le \frac{C}{t^{(1+\beta)/2}}\|u\|_Y, \quad 0 < t \le T,$$

so that $D_i u \in C^{0,\beta}_{(1+\beta)/2}((0,T]\times\mathbb{R}^n)$. Moreover, since $t \mapsto u(t,\cdot)$ is bounded in $[\varepsilon, T]$ with values in $C^{2+\theta}(K)$ and continuous in $[\varepsilon,T]$ with values in $C(K)$ for every compact set K, by [14, Prop. 1.1.3(iii), 1.1.4(iii)] it belongs to $C([\varepsilon,T];C^\alpha(K))$ for every $\alpha < 2+\theta$. Consequently, $D_i u \in C((0,T);C^\beta(K))$.

Since $P - \widetilde{P}$ is Lipschitz continuous and bounded, the function

$$g(s,x) = \langle P(x) - \widetilde{P}(x), Du(s,x)\rangle, \quad 0 < s \le T,$$

belongs to $C^{0,\beta}_{(1+\beta)/2}((0,T]\times\mathbb{R}^n)$ and $s \mapsto g(s,\cdot)$ is in $C((0,T];C^\beta(K))$ for every $\beta \in (0,1)$.

Fix once and for all any $\beta \in (\theta, 1)$. By Proposition 3.3 applied to the semigroup $e^{-\lambda t}\widetilde{T}(t)$, the operator Γ maps Y into itself. We revisit the estimates of Proposition 3.3 to prove that Γ is a contraction for λ large enough. Arguing as in the proof of Proposition 3.3 we see that for u, v in Y we have

$$\|(\Gamma u)(t,\cdot) - (\Gamma v)(t,\cdot)\|_{C^{2+\theta}(\mathbb{R}^n)} \le \int_0^t \frac{Ce^{(-\lambda+r_0)(t-s)}}{s^{(1+\beta)/2}(t-s)^{1-(\beta-\theta)/2}}\,ds\,\|g\|_{C^{0,\beta}_{(1+\beta)/2}((0,T]\times\mathbb{R}^n))}, \quad 0 < t \le T,$$

where now

$$g(s,x) = \langle P(x) - \widetilde{P}(x), (Du(s,x) - Dv(s,x))\rangle, \quad 0 < s \le T,\ x \in \mathbb{R}^n.$$

It is not hard to see that

$$C_1(\lambda) = \sup_{0<t\le T} t^{1+\theta/2}\int_0^t \frac{e^{(-\lambda+r_0)(t-s)}}{s^{(1+\beta)/2}(t-s)^{1-(\beta-\theta)/2}}\,ds$$

goes to 0 as λ goes to ∞. Moreover, for every $u \in Y$, each derivative $D_i u$ is bounded by $C/\sqrt{t}$. Indeed, from [20, §2.7.2] we get for $0 < t \le T$,

$$\|u(t,\cdot)\|_{C^1(\mathbb{R}^n)} \le C\|u(t,\cdot)\|^{1/(2+\theta)}_{C^{2+\theta}(\mathbb{R}^n)}\|u(t,\cdot)\|^{1-1/(2+\theta)}_\infty \le \frac{C}{t^{1/2}}\|u\|_Y,$$

so that $\sqrt{t}\,D_i u(t,x)$ is bounded. It follows that for every $x \in \mathbb{R}^n$ we have

$$|(\Gamma u)(t)(x) - (\Gamma v)(t)(x)| \le C\int_0^t \frac{e^{(-\lambda+r_0)(t-s)}}{s^{1/2}}\,ds\,\|g\|_Y, \quad 0 < t \le T.$$

Again, it is easy to see that

$$C_2(\lambda) = \sup_{0<t\le T} t^{1+\theta/2}\int_0^t \frac{e^{(-\lambda+r_0)(t-s)}}{s^{1/2}}\,ds$$

goes to 0 as λ goes to ∞. Therefore, for λ large enough Γ is a contraction in Y, so that it has a unique fixed point $v \in Y$.

From Proposition 3.3 we know that v belongs also to $C^{1,2}((0,T]\times\mathbb{R}^n)$ and it satisfies (3.12) pointwise. Therefore, $u(t,x) = e^{\lambda t}v(t,x)$ belongs to

$C^{1,2}((0,T]\times\mathbb{R}^n)$, it satisfies (3.1), and

$$\|u\|_\infty + \sup_{0<t\le T} t^{1+\theta/2}\|u(t,\cdot)\|_{C^{2+\theta}(\mathbb{R}^n)} \le e^{\lambda T}\|v\|_Y \le e^{\lambda T}C\|u_0\|_\infty.$$

In the case where $u_0 \in C^{2+\theta}(\mathbb{R}^n)$ the operator Γ is defined in the space

$$\widetilde{Y} = C([0,T]\times\mathbb{R}^n) \cap C^{0,2+\theta}([0,T]\times\mathbb{R}^n).$$

Arguing as above it is not difficult to show that it maps $\widetilde{Y}$ into itself and it is a contraction if λ is large enough.

The statement follows. ■

Proof of Lemma 3.4. Let $f_j = \varphi_j - \varphi$ and set

$$v_j(t,x) = e^{-\lambda t}(T(t)f_j)(x), \quad 0 \le t \le T,\ x \in \mathbb{R}^n.$$

We know from the proof of Theorem 3.1 that if λ is sufficiently large we have

$$v_j(t,\cdot) = e^{-\lambda t}\widetilde{T}(t)f_j + \int_0^t e^{-\lambda(t-s)}\widetilde{T}(t-s)\langle \widetilde{P} - P, Dv_j(s,\cdot)\rangle\,ds, \quad 0 \le t \le T,$$

where $\widetilde{P}$ is defined in (3.9). From estimates (3.2) we get $|\langle \widetilde{P} - P, Dv_j(s,\cdot)\rangle| \le Ce^{(\omega-\lambda)s}s^{-1/2}$, with constants C, ω independent of j, so that

$$|v_j(t,x)| \le |e^{-\lambda t}(\widetilde{T}(t)f_j)(x)| + Ct^{1/2}\int_0^1 \frac{e^{(r_0-\lambda)(1-\sigma)}e^{(\omega-\lambda)\sigma}}{\sigma^{1/2}}\,d\sigma, \quad 0 \le t \le T,\ x \in \mathbb{R}^n.$$

Fix $\varepsilon > 0$ and take λ large enough in order that

$$C_3(\lambda) = CT^{1/2}\int_0^1 \frac{e^{(r_0-\lambda)(1-\sigma)}e^{(\omega-\lambda)\sigma}}{\sigma^{1/2}}\,ds \le \varepsilon.$$

Then

$$|(T(t)f_j)(x)| \le |(\widetilde{T}(t)f_j)(x)| + \varepsilon e^{\lambda t}, \quad 0 \le t \le T,\ x \in \mathbb{R}^n,$$

and since $(\widetilde{T}(t)f_j)_{|K}$ goes to 0 uniformly on every compact set $K \subset \mathbb{R}^n$ as $j \to \infty$, the statement follows. ■

The proof of Proposition 3.3 is based uniquely on Lemma 3.4 (see the previous subsection). Since Lemma 3.4 holds, so does the statement of Proposition 3.3.

3.3. *The case where P satisfies* (1.4) *with* $p_0 = 0$. In this subsection we assume that (1.2)–(1.4) hold with $p_0 = 0$, and that there exists φ satisfying (1.5).

Proof of Theorem 3.1. We introduce the Yosida approximations of P,

$$P_k(x) = P(x_k), \quad k \in \mathbb{N},$$

where x_k is the unique solution of

$$x_k - \frac{1}{k}P(x_k) = x.$$

Then $P_k(x) \to P(x)$ as $k \to \infty$, uniformly for $x \in K$ for every compact $K \subset \mathbb{R}^n$, and it satisfies the dissipativity assumption

$$\langle DP_k(x)\xi, \xi\rangle \le 0, \quad x, \xi \in \mathbb{R}^n.$$

Moreover, every P_k is Lipschitz continuous (with Lipschitz constant possibly blowing up as $k \to \infty$). Anyway, the problem

$$(3.14)\qquad \begin{cases} D_t u_k - \mathrm{Tr}(Q(x)D^2u_k) \\ \qquad - \langle P_k(x), Du_k\rangle - ru_k = 0, \quad 0 < t < T,\ x \in \mathbb{R}^n, \\ u_k(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases}$$

has a unique bounded classical solution u_k which is smooth for $t > 0$. By Remark 2.5 it satisfies

$$(3.15)\qquad \begin{aligned} &\|u_k(t,\cdot)\|_\infty \le C\|u_0\|_\infty, \\ &\|Du_k(t,\cdot)\|_\infty \le \frac{C}{\sqrt{t}}\|u_0\|_\infty, \quad 0 < t \le T. \end{aligned}$$

This is enough to prove existence of the solution to (3.1) through standard arguments. Indeed, fix any $R > 0$ and let θ be a smooth cutoff function satisfying (3.5). For $k \ge R+1$ the function $v_{k,R} = \theta u_k$ satisfies

$$(3.16)\qquad \begin{cases} D_t v_{k,R} - \mathrm{Tr}(QD^2 v_{k,R} - rv_{k,R}) = \psi_k, \\ v_{k,R}(0,x) = u_0(x)\theta(x), \quad x \in B(0,R+1), \\ v_{k,R}(t,x) = 0, \quad 0 < t < T,\ x \in \partial B(0,R+1), \end{cases}$$

where

$$\psi_k(t,x) = \theta\langle P_k(x), Du_k\rangle - \sum_{i,j=1}^n q_{ij}(u_k D_{ij}\theta) + 2D_i u_k D_j\theta, \qquad 0 < t < T,\ x \in B(0,R+1).$$

The right hand side of the differential equation is bounded by $C/\sqrt{t}$, with constant C independent of k. By the usual theory of parabolic problems we see that $v_{k,R}$ is bounded in $C^{1-\alpha/2,2-\alpha}([\varepsilon,T] \times B(0,R+1))$ for every $\varepsilon > 0$, by a constant independent of k. Since $v_{k,R} = u_k$ in $B(0,R)$, it follows that u_k is bounded in $C^{1-\alpha/2,2-\alpha}([\varepsilon,T] \times B(0,R))$ for every $\varepsilon > 0$, by a constant independent of k, so that each derivative $D_i u_k$ is bounded in $C^{1/2-\alpha/2,1-\alpha}([\varepsilon,T] \times B(0,R))$ for every $\varepsilon > 0$, by a constant independent of k. Considering again problem (3.16) with R replaced by $R-1$ and applying again the standard theory of parabolic problems we deduce (taking into account that DP_k is bounded on $B(0,R)$ by a constant independent of k) that $v_{k,R-1}$ is bounded in $C^{1-\alpha/2,2-\alpha}([\varepsilon,T] \times B(0,R))$ for every $\alpha \in (0,1)$, $\varepsilon \in (0,T)$, by a constant independent of k, so that u_k is bounded in $C^{1+\beta/2,2+\beta}([\varepsilon,T] \times B(0,R-1))$ for every $\varepsilon \in (0,T)$ and $\beta \in (0,1)$, by a constant independent of k. Moreover, the functions u_k are equicontinuous in $[0,T] \times B(0,R-1)$. The usual diagonal procedure gives a subsequence which converges to a function u in $C([0,T] \times B(0,R)) \cap C^{1,2}([\varepsilon,T] \times B(0,R))$ for every $\varepsilon \in (0,T)$ and $R \in \mathbb{N}$. Such a function u is a bounded classical solution of (3.1). Since the data are smooth, u is smooth for $t > 0$. Theorem 2.4 may be applied to conclude that u satisfies (2.6). ∎

The proof of Lemma 3.4 is quite similar to the one of Subsection 3.1 and it is left to the reader. Once Lemma 3.4 is established, the proof of Proposition 3.3 is the same as in Subsection 3.1.

3.4. *The case where P satisfies* (1.4). Finally, consider the case where P satisfies (1.4) with p_0 arbitrary. The function

$$\widetilde{P}(x) = P(x) - p_0 x$$

satisfies (1.4) with p_0 replaced by 0, so that by the results of Subsection 3.3 the statements hold if P is replaced by $\widetilde{P}$. Arguing now as in Subsection 3.3 we get Theorem 3.1 and Lemma 3.4, and consequently Proposition 3.3. ∎

4. The relations between $T(t)$ and $\mathcal{A}$. As we remarked in the previous section, the semigroup $T(t)$ is not necessarily strongly continuous nor analytic in $X = C(\mathbb{R}^n)$, even upon replacing X by $\mathrm{BUC}(\mathbb{R}^n)$. So, we cannot speak of "generator" of $T(t)$ in the standard sense. Nevertheless, a realization of the operator $\mathcal{A}$ in X is naturally associated with $T(t)$, as follows. For every $\lambda > \lambda_0$ consider the operator

$$(4.1)\qquad (R(\lambda)f)(x) = \int_0^\infty e^{-\lambda t}(T(t)f)(x)\,dt, \quad x \in \mathbb{R}^n.$$

The integral may not converge in $L(X)$, but for every compact set $K \subset \mathbb{R}^n$ the function $t \mapsto T(t)f$ is continuous with values in $C(K)$, so that $(R(\lambda)f)(x)$ is well defined and continuous. Therefore, $R(\lambda) \in L(X)$, and since $\|T(t)f\|_\infty \le e^{\lambda_0 t}\|f\|_\infty$ we have

$$\|R(\lambda)\|_{L(X)} \le \frac{1}{\lambda - \lambda_0}.$$

Moreover, $R(\lambda)$ satisfies the resolvent identity because $T(t)$ is a semigroup, and it is one-to-one because for every $x \in \mathbb{R}^n$, $(R(\lambda)f)(x)$ is the anti-Laplace transform of the real continuous function $t \mapsto (T(t)\varphi)(x)$, which takes the value $f(x)$ at $t = 0$. Therefore there exists a closed operator

$$A : D(A) \to X, \quad D(A) = \mathrm{Range}\,R(\lambda) \quad \text{for } \lambda > \lambda_0,$$

such that $R(\lambda) = R(\lambda, A)$ for $\lambda > \lambda_0$. We are going to show that A is a realization of $\mathcal{A}$ in X. The proof is similar to the corresponding ones in [17], [14].

PROPOSITION 4.1. *We have*

$$D(A) = \Big\{ f \in \bigcap_{p\ge 1} W^{2,p}_{\rm loc}(\mathbb{R}^n)\cap X : \mathcal{A}f \in X\Big\}, \qquad Af = \mathcal{A}f \quad \forall f\in D(A).$$

Moreover, $D(A)$ *is continuously embedded in* $C^\theta(\mathbb{R}^n)$ *for every* $\theta \in (0,2)$, *and there is* $C(\theta) > 0$ *such that*

$$\|f\|_{C^\theta(\mathbb{R}^n)} \le C(\theta)\|f\|_\infty^{1-\theta/2}\|f\|_{D(A)}^{\theta/2}, \qquad \forall f\in D(A).$$

Proof. First we prove the inclusion $\subset$. Let ω be the constant appearing in estimates (3.2) and (3.3). If $\phi \in C^1(\mathbb{R}^n)$ and $\lambda > \omega$, estimates (3.3) with $\theta = 1$ and Lemma 3.5 imply that $R(\lambda)\phi \in C^{2+\alpha}(\mathbb{R}^n)$ for every $\alpha \in (0,1)$. Using the equality $(\mathcal{A}T(t)\phi)(x) = (\partial/\partial t)(T(t)\phi)(x)$ for all x we get

$$\lambda R(\lambda)\phi - \mathcal{A}R(\lambda)\phi = \phi,$$

so that

$$\mathcal{A}R(\lambda)\phi = AR(\lambda)\phi.$$

From the general theory of elliptic differential equations with regular coefficients, for every $p > 1$ and $R > 0$ we get

$$\|R(\lambda)\phi\|_{W^{2,p}(B(0,R))} \le C\|\phi\|_{L^p(B(0,R+1))}, \tag{4.2}$$

with C independent of ϕ, possibly depending on R.

Let $f \in D(A)$, $\lambda > \omega$, and set $\phi = \lambda f - Af$. Let $\{\phi_n\} \subset C^1(\mathbb{R}^n)$ be a sequence converging to ϕ in $C(K)$ for every compact set $K \subset \mathbb{R}^n$. Set $f_n = R(\lambda)\phi_n$. Then $f_n \in C^2(\mathbb{R}^n)$ and $f_n \to f$, $Af_n \to Af$ in $C(K)$ as $n\to\infty$, thanks to Lemma 3.4. Applying estimate (4.2) to $\phi_n - \phi_m$ we see that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{2,p}(B(0,R))$ for every $p > 1$ and $R > 0$, so that $f \in W^{2,p}_{\rm loc}(\mathbb{R}^n)$ and the equality $\mathcal{A}f_n = Af_n$ for all n implies $\mathcal{A}f = Af$. The inclusion $\subset$ is thus proved.

Let now $f \in \bigcap_{p\ge1} W^{2,p}_{\rm loc}(\mathbb{R}^n)\cap X$ be such that $\mathcal{A}f \in X$. Fix $\lambda > \omega$ and set $\phi = \lambda f - \mathcal{A}f$, $g = R(\lambda)\phi$. Our aim is to show that $f = g$. From the first part of the proof we know that $g \in \bigcap_{p\ge1} W^{2,p}_{\rm loc}(\mathbb{R}^n)\cap X$ and that $\lambda g - \mathcal{A}g = \phi$. Therefore, $\lambda(f-g) - \mathcal{A}(f-g) = 0$. By Proposition 2.2, $f = g$, so that $f \in D(A)$.

Let us prove that $D(A) \subset C^\theta(\mathbb{R}^n)$ for every $\theta \in (0,2)$. Let $0 \ne f \in D(A)$, $\lambda > \omega$ and set $\varphi = \lambda f - Af$. Then

$$f(x) = \int_0^\infty e^{-\lambda t}(T(t)\varphi)(x)\,dt,$$

so that by Lemma 3.5 and estimates (3.3), $f \in C^\theta(\mathbb{R}^n)$, and for $\theta \ne 1$,

$$\|f\|_{C^\theta(\mathbb{R}^n)} \le C\frac{\Gamma(1-\theta/2)}{(\lambda-\omega)^{1-\theta/2}}\|\varphi\|_\infty \le C\frac{\Gamma(1-\theta/2)}{(\lambda-\omega)^{1-\theta/2}}(\lambda\|f\|_\infty + \|Af\|_\infty).$$

Taking the minimum for $\lambda > \omega$ we get

$$\|f\|_{C^\theta(\mathbb{R}^n)} \le C_1(\theta)\|f\|_\infty^{1-\theta/2}\|f\|_{D(A)}^{\theta/2} + C_2(\theta)\|f\|_\infty$$

and the statement follows. ■

5. Proof of Theorems 1 and 2. Once we have established the fundamental estimates (3.3) and the representation formula

$$u(x) = \int_0^\infty e^{-\lambda t}(T(t)f)(x)\,dt, \qquad x\in\mathbb{R}^n,$$

for the solution of the elliptic problem (1.6), the Schauder regularity theorems 1 and 2 are easily proved by the method of [15]. We cannot apply directly the results of [15] which were stated with X replaced by $\mathrm{BUC}(\mathbb{R}^n)$ and with slightly more restrictive assumptions on $T(t)$.

Proof of Theorem 1. Uniqueness of the classical bounded solution of (1.6) follows from Proposition 2.2.

Let $f \in C^\theta(\mathbb{R}^n)$ and $\lambda > \lambda_0$. Then $\lambda \in \varrho(A)$, so that the function

$$u(x) = \int_0^\infty e^{-\lambda t}(T(t)f)(x)\,dt$$

is well defined and belongs to $D(A)$. By Proposition 4.1, $u \in W^{2,p}_{\rm loc}(\mathbb{R}^n)$ for every $p \ge 1$ and

$$\lambda u - \mathcal{A}u = f,$$

moreover $u \in C^\theta(\mathbb{R}^n)$ and

$$\|u\|_{C^\theta(\mathbb{R}^n)} \le C(\theta)\|u\|_\infty^{1-\theta/2}\|u\|_{D(A)}^{\theta/2} \le \frac{C(\theta)}{(\lambda-\lambda_0)^{1-\theta/2}}\|f\|_\infty.$$

Let now $\eta > \omega$, ω being the constant in estimates (3.2) and (3.3). Then

$$\eta u - \mathcal{A}u = f + (\eta-\lambda)u = \varphi$$

with

$$\|\varphi\|_{C^\theta(\mathbb{R}^n)} \le \Big(1 + \frac{|\eta-\lambda|C(\theta)}{(\lambda-\lambda_0)^{1-\theta/2}}\Big)\|f\|_{C^\theta(\mathbb{R}^n)}.$$

Since $\omega \ge \lambda_0$, we have $\eta \in \varrho(A)$, so that

$$u(x) = \int_0^\infty e^{-\eta t}(T(t)\varphi)(x)\,dt.$$

Let us prove that for every $\alpha \in (\theta, 1)$,

$$u \in (C^\alpha(\mathbb{R}^n), C^{2+\alpha}(\mathbb{R}^n))_{1-(\alpha-\theta)/2,\infty} = C^{2+\theta}(\mathbb{R}^n).$$

(The last equality is well known, see e.g. [20, Thm. 1, §2.7.2]).

We recall that if $Y_2 \subset Y_1$ are Banach spaces then

$$(Y_1, Y_2)_{\gamma,\infty} = \{u \in Y_1 : \|u\|_{\gamma,\infty} = \sup_{0<\xi<1} \xi^{-\gamma} K(\xi, u) < \infty\},$$

where

$$K(\xi, u) = \inf_{u=a+b} (\|a\|_{Y_1} + \xi \|b\|_{Y_2}).$$

For every $\xi > 0$ set

$$a(x) = \int_0^\xi e^{-\eta t}(T(t)\varphi)(x)\, dt, \quad b(x) = \int_\xi^\infty e^{-\eta t}(T(t)\varphi)(x)\, dt, \quad x \in \mathbb{R}^n.$$

Then $u = a + b$, and by Lemma 3.5 and estimates (3.3),

$$\|a\|_{C^\alpha(\mathbb{R}^n)} \le C \int_0^\xi t^{-(\alpha-\theta)/2}\, dt\, \|\varphi\|_{C^\theta(\mathbb{R}^n)} = C' \xi^{1-(\alpha-\theta)/2} \|\varphi\|_{C^\theta(\mathbb{R}^n)},$$

$$\|b\|_{C^\alpha(\mathbb{R}^n)} \le C \int_\xi^\infty t^{-1-(\alpha-\theta)/2}\, dt\, \|\varphi\|_{C^\theta(\mathbb{R}^n)} = C'' \xi^{-(\alpha-\theta)/2} \|\varphi\|_{C^\theta(\mathbb{R}^n)},$$

so that taking $\gamma = 1 - (\alpha - \theta)/2$ we get $u \in (C^\alpha(\mathbb{R}^n), C^{2+\alpha}(\mathbb{R}^n))_{\gamma,\infty}$ and

$$\|u\|_{\gamma,\infty} \le (C' + C'')\|\varphi\|_{C^\theta(\mathbb{R}^n)} \le \frac{|\eta - \lambda| C(\theta)}{(\lambda - \lambda_0)^{1-\theta/2}} (C' + C'')\|f\|_{C^\theta(\mathbb{R}^n)},$$

and the statement is proved. ∎

Proof of Theorem 2. From Proposition 3.3 we know that problem (1.7) has a unique solution $u \in C^{1,2}([0, T] \times \mathbb{R}^n)$, given by the variation of constants formula

$$u(t, x) = (T(t)u_0)(x) + v(t, x),$$

where

$$v(t, x) = \int_0^t T(t - s)g(s, \cdot)(x)\, ds.$$

Thanks to estimates (3.3) the first term $T(t)u_0$ belongs to $C^{2+\theta}(\mathbb{R}^n)$ for every t and

$$\|T(t)u_0\|_{C^{2+\theta}(\mathbb{R}^n)} \le C\|u_0\|_{C^{2+\theta}(\mathbb{R}^n)}, \quad 0 \le t \le T.$$

To estimate the second term we use the arguments of Theorem 1. Fix any $t \in [0, T]$, and for every $\xi \in (0, 1)$ set

$$a(x) = \begin{cases} \int_{t-\xi}^t (T(t-s)g(s,\cdot)(x)\, ds & \text{if } \xi \le t, \\ \int_0^t (T(t-s)g(s,\cdot))(x)\, ds & \text{if } \xi > t; \end{cases}$$

$$b(x) = \begin{cases} \int_0^{t-\xi} (T(t-s)g(s,\cdot))(x)\, ds & \text{if } \xi \le t, \\ 0 & \text{if } \xi > t. \end{cases}$$

By Lemma 3.5 and estimates (3.3), for $\theta < \alpha < 1$ we have

$$\|a\|_{C^\alpha(\mathbb{R}^n)} \le C \int_{\max(t-\xi,0)}^t (t-s)^{-(\alpha-\theta)/2}\, ds \sup_{0\le s\le T} \|g(s,\cdot)\|_{C^\theta(\mathbb{R}^n)}$$
$$\le C' \xi^{1-(\alpha-\theta)/2} \sup_{0\le s\le T} \|g(s,\cdot)\|_{C^\theta(\mathbb{R}^n)},$$

and

$$\|b\|_{C^{2+\alpha}(\mathbb{R}^n)} \le C \int_0^{\max(t-\xi,0)} (t-s)^{-1-(\alpha-\theta)/2}\, ds \sup_{0\le s\le T} \|g(s,\cdot)\|_{C^\theta(\mathbb{R}^n)}$$
$$\le C'' \xi^{-(\alpha-\theta)/2} \sup_{0\le s\le T} \|g(s,\cdot)\|_{C^\theta(\mathbb{R}^n)},$$

so that $v(t,\cdot) \in (C^\alpha(\mathbb{R}^n), C^{2+\alpha}(\mathbb{R}^n))_{1-(\alpha-\theta)/2,\infty} = C^{2+\theta}(\mathbb{R}^n)$, and

$$\sup_{0\le t\le T} \|v(t)\|_{C^{2+\theta}(\mathbb{R}^n)} \le (C' + C'') \sup_{0\le t\le T} \|g(t,\cdot)\|_{C^\theta(\mathbb{R}^n)}.$$

The statement follows. ∎

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Dipartimento di Matematica
Università di Parma
Via D'Azeglio 85/A
43100 Parma, Italy
E-mail: lunardi@prmat.math.unipr.it

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