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Commutators of quasinilpotents and invariant subspaces

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Abstract. It is proved that the set $\mathcal{Q}$ of quasinilpotent elements in a Banach algebra is an ideal, i.e. equal to the Jacobson radical, if (and only if) the condition $[\mathcal{Q},\mathcal{Q}] \subseteq \mathcal{Q}$ (or a similar condition concerning anticommutators) holds. In fact, if the inner derivation defined by a quasinilpotent element p maps $\mathcal{Q}$ into itself then $p \in \operatorname{Rad}\mathcal{A}$. Higher commutator conditions of quasinilpotents are also studied. It is shown that if a Banach algebra satisfies such a condition, then every quasinilpotent element has some fixed power in the Jacobson radical. These results are applied to topologically transitive representations. As a consequence, it is proved that a closed algebra of polynomially compact operators satisfying a higher commutator condition must have an invariant nest of closed subspaces, with "gaps" of bounded dimension. In particular, if $[\mathcal{Q},\mathcal{Q}] \subseteq \mathcal{Q}$, then the algebra must be triangularizable. An example is given showing that this may fail for more general algebras.

1. Introduction. Let $\mathcal{A}$ be a Banach algebra. The set of quasinilpotent elements in $\mathcal{A}$ is denoted by $\mathcal{Q}$, that is, $\mathcal{Q} = \{a \in \mathcal{A} : \varrho(a) = 0\}$ where $\varrho(a)$ is the spectral radius of a. The Jacobson radical $\operatorname{Rad}\mathcal{A}$ of $\mathcal{A}$ is often denoted by $\mathcal{R}$; in Banach algebras, $\mathcal{R} = \{a \in \mathcal{A} : ab \in \mathcal{Q} \text{ for all } b \in \mathcal{A}\}$. The algebra $\mathcal{A}$ is called *semisimple* if $\mathcal{R} = \{0\}$; if $\mathcal{A} = \mathcal{R}$, $\mathcal{A}$ is called a *radical algebra*, and in this case every element of $\mathcal{A}$ is quasinilpotent. It is obvious that $\mathcal{R}$ is always a subset of $\mathcal{Q}$ and the converse holds when $\mathcal{A}$, or more generally $\mathcal{A}/\mathcal{R}$, is commutative. Some simple cases where these sets differ are the Banach algebras $M_n(\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$, or more generally $\mathcal{B}(X)$ (resp. $\mathcal{K}(X)$), the algebra of all bounded (resp. compact) linear operators on a Banach space X; these are semisimple algebras but still have plenty of quasinilpotent elements.

The commutator $ab - ba$ of a and b is denoted by $[a,b]$ or by $D_b(a)$. We write $\{a,b\}$ for the anticommutator $ab + ba$. The set $\mathcal{Q}$ is said to be *closed under commutation* (resp. *anticommutation*) if $[a,b]$ (resp $\{a,b\}$) is quasinilpotent whenever a and b are in $\mathcal{Q}$.

In [16] it is proved that $\mathcal{Q}$ equals $\mathcal{R}$ if and only if $\mathcal{Q}$ is closed under addition or multiplication. This result, as well as the result of Zemánek [19],

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that $\mathcal{A}/\mathcal{R}$ is commutative if and only if spectral radius is subadditive or submultiplicative, demonstrates the close connection between the algebraic and the topological structure in Banach algebras. A further connection of these structures will be established if the following questions are answered.

QUESTION A. If $\mathcal{Q}$ is closed under commutation, is it equal to $\mathcal{R}$?

QUESTION B. If $\mathcal{Q}$ is closed under anticommutation, is it equal to $\mathcal{R}$?

Question A is absolutely natural if one examines the proofs of Lemmas 1 and 2 in [16]. On the other hand, motivation for question B also comes from the Engel–Jacobson theorem about finite-dimensional algebras [11]. Notice that for Banach algebras, the condition $\{\mathcal{A},\mathcal{A}\} \subseteq \mathcal{Q}$ obviously implies that $\mathcal{A}$ is a radical algebra, so that $\mathcal{Q} = \mathcal{R}$. In addition, a slight change in the proof of Lemma 1 of [16] yields that if $\{\mathcal{A},\mathcal{Q}\} \subseteq \mathcal{Q}$ then $\mathcal{Q} = \mathcal{R}$.

An affirmative answer to both questions A and B will be given in Theorem 2, which is one of the main results of this paper. In addition, a new proof of the main result of [16] will also be given in the same theorem.

Concerning higher commutators, observe that the algebra $M_2(\mathbb{C})$ satisfies $[q,[q,p]] \in \mathcal{Q}$ for all $p,q \in \mathcal{Q}$. Here, (quasi-)nilpotents are not in the radical, but their squares are. Generally, if $\mathcal{A}$ satisfies a higher commutator condition then every quasinilpotent has some fixed power in $\operatorname{Rad}\mathcal{A}$ (Theorem 3).

There is also a very interesting interaction between conditions such as $[\mathcal{A},\mathcal{A}] \subseteq \mathcal{Q}$ or $\mathcal{Q} = \mathcal{R}$ and the notion of simultaneous triangularization of algebras of operators, especially of compact operators (see [4], [6], [7] and [10]). We improve some of these results by showing that the conditions $[\mathcal{Q},\mathcal{Q}] \subseteq \mathcal{Q}$ or $\{\mathcal{Q},\mathcal{Q}\} \subseteq \mathcal{Q}$ imply simultaneous triangularizability of closed algebras of polynomially compact operators. This means (precise definitions will be given in Section 3) that such algebras have "nests" of invariant subspaces with at most one-dimensional "gaps". Similarly, such algebras satisfying higher commutator conditions admit "block triangularization" with blocks of bounded dimension (see Theorem 11).

2. General results. We first obtain an affirmative answer to Questions A and B (Theorem 2). Note that higher commutator conditions do not imply the equality $\mathcal{Q} = \mathcal{R}$ (consider for example $M_n(\mathbb{C})$). Algebras satisfying such conditions are characterized in Theorem 3.

Notice that the basic tool in this area ([9], [16], [18]) has so far been Jacobson's density theorem ([3], [12]). Our proof will be based on the following improvement due to Sinclair [15]:

THEOREM. *Let $\mathcal{A}$ be a Banach algebra and π an algebraically irreducible representation of $\mathcal{A}$ on a Banach space X.*

(i) (Jacobson) *If $x_1,\ldots,x_n$ are linearly independent and $y_1,\ldots,y_n$ are any vectors in X, then there exists $a \in \mathcal{A}$ such that $\pi(a)x_i = y_i$, $i = 1,\ldots,n$.*

(ii) (Sinclair) *If additionally $y_1,\ldots,y_n$ are linearly independent, then $a \in \mathcal{A}$ may be chosen invertible.*

Whenever necessary, we adjoin a unit, denoted by $\mathbf{1}$, to a Banach algebra. We will need the observation that if T is a quasinilpotent operator on a Banach space X and $x \in X$, then the set $\{x, Tx, T^2x, \ldots\}\backslash\{0\}$ is linearly independent. For if $\sum_{k=0}^{n} \lambda_k T^k x = 0$, then writing

$$\sum_{k=0}^{n} \lambda_k t^k = t^m(t - z_1)(t - z_2)\ldots(t - z_{n-m})$$

where $z_i \neq 0$ we obtain $T^m x = 0$, since $T - z_i\mathbf{1}$ is invertible.

Using Sinclair's theorem we will prove the following

PROPOSITION 1. *Let $\mathcal{A}$ be a Banach algebra. If $p \in \mathcal{Q}$ and p is not contained in* $\operatorname{Rad}\mathcal{A}$ *then there exists $q \in \mathcal{Q}$ such that pq, $p+q$, $\{p,q\}$ and $[p,q]$ are not quasinilpotent.*

Proof. Since $p \notin \operatorname{Rad}\mathcal{A}$ there exists an algebraically irreducible representation π of $\mathcal{A}$ on a Banach space X and a vector $x \in X$ such that if $x_k = \pi(p^{k-1})x$ then the set $S = \{x_1, x_2, x_3, x_4\}\backslash\{0\}$ is linearly independent in X and contains x_1, x_2. Using Sinclair's theorem we find an invertible element $a \in \mathcal{A}$ such that $\pi(a)x_1 = x_2$, $\pi(a)x_2 = x_1$ and $\pi(a)y = y$ for $y \in S\backslash\{x_1, x_2\}$. Let q be the quasinilpotent $a^{-1}pa$. Then we find that pq, $p+q$, $[p,q]$, and $\{p,q\}$ are not quasinilpotent, since their images under π have a nonzero eigenvalue. Indeed, noting that $\pi(q)x_i = \pi(p)x_i$ for $i = 3,4$, one verifies that x_2 is an eigenvector for $\pi(pq)$, that $x_1 - x_2$ is an eigenvector for $\pi(p+q)$ and for $\pi(\{p,q\})$ and that $x_1 - x_4$ is an eigenvector for $\pi([p,q])$. ■

An immediate consequence is the following

THEOREM 2. *Let $\mathcal{A}$ be a Banach algebra. The following are equivalent:*

(a) $\mathcal{Q} + \mathcal{Q} \subseteq \mathcal{Q}$,
(b) $\mathcal{Q}\mathcal{Q} \subseteq \mathcal{Q}$,
(c) $[\mathcal{Q}, \mathcal{Q}] \subseteq \mathcal{Q}$,
(d) $\{\mathcal{Q}, \mathcal{Q}\} \subseteq \mathcal{Q}$,
(e) $\mathcal{Q} = \operatorname{Rad}\mathcal{A}$.

REMARKS. (i) The equivalence of (a), (b) and (e) is due to Z. Słodkowski, W. Wojtyński and J. Zemánek [16], who use subharmonicity of spectral radius [17] as well as Jacobson's density theorem. In a previous version of the present paper, these tools were used to prove the equivalence of (c) to (e). The new method simultaneously yields all equivalences and is more transparent.

(ii) In [16] it is shown that if an element $p \in \mathcal{Q}$ satisfies $D_p(\mathcal{A}) \subseteq \mathcal{Q}$ then $p \in \operatorname{Rad} \mathcal{A}$. Proposition 1 improves this result to the implication $D_p(\mathcal{Q}) \subseteq \mathcal{Q} \Rightarrow p \in \operatorname{Rad} \mathcal{A}$. Note that, if $p \in \mathcal{A}$ is not assumed to be quasinilpotent, the hypothesis $D_p(\mathcal{Q}) \subseteq \mathcal{Q}$ or even $D_p(\mathcal{Q}) \subseteq \operatorname{Rad} \mathcal{A}$ does not imply $D_p(\mathcal{A}) \subseteq \operatorname{Rad} \mathcal{A}$, since there exist nonabelian Banach algebras for which $\mathcal{Q} = \{0\}$ (see [2]).

(iii) In Sinclair's theorem the element $a \in \mathcal{A}$ can in fact be chosen of the form $a = e^b$ for some $b \in \mathcal{A}$. Therefore the quasinilpotent element q in our proof is in fact homotopic to p via the homotopy $q(t) = e^{-tb} p e^{tb}$ ($t \in [0,1]$).

If $q^n \in \mathcal{R}$, then it is easy to see that $D_q^n(\mathcal{Q}) \subseteq \mathcal{Q}$. We prove that the converse also holds for a quasinilpotent element. The example of the Banach algebra $M_n(\mathbb{C})$ shows that this conclusion is best possible.

THEOREM 3. *Let $\mathcal{A}$ be a Banach algebra, p a quasinilpotent element in $\mathcal{A}$ and n a positive integer. If $D_p^n(\mathcal{Q}) \subseteq \mathcal{Q}$ then $p^n \in \operatorname{Rad} \mathcal{A}$.*

Proof. Suppose that $p^n \notin \operatorname{Rad} \mathcal{A}$. We will construct a quasinilpotent $q \in \mathcal{A}$ such that $D_p^n(q)$ is not quasinilpotent. Since p is quasinilpotent and p^n is not contained in $\operatorname{Rad} \mathcal{A}$ there exists an irreducible representation π on a Banach space X and an x in X such that if $x_k = \pi(p^{k-1})x$, then the set $S = \{x_k : 1 \le k \le 5n+1\} \setminus \{0\}$ is linearly independent and contains $x_1, \ldots, x_{n+1}$. By Sinclair's theorem we find an invertible $a \in \mathcal{A}$ such that $\pi(a)x_1 = x_{n+1}$, $\pi(a)x_{n+1} = x_1$ and $\pi(a)y = y$ for $y \in S \setminus \{x_1, x_{n+1}\}$; hence $\pi(a)x_k = x_k$ for $1 < k \le 5n+1$ and $k \ne n+1$. Define $q = a^{-1}p^n a$. Then q is quasinilpotent and $\pi(q)x_1 = x_{2n+1}$, $\pi(q)x_{n+1} = x_1$ and $\pi(q)x_k = \pi(p^n)x_k$ for $1 < k \le 5n+1$ and $k \ne n+1$.

An easy calculation (using the Leibniz formula for inner derivations) shows that $D_{\pi(p)}^n(\pi(q))x_1 = x_1 + \mu x_{2n+1} + \lambda x_{3n+1}$ for some scalars λ and μ. However, since $\pi(p)x_{2n+m} = x_{2n+m+1}$ and $\pi(qp)x_{2n+m} = \pi(pq)x_{2n+m}$ for all m, one verifies that $D_{\pi(p)}^n(\pi(q))x_{2n+m} = 0$ and hence $D_{\pi(p)}^n(\pi(q))(x_1 + \mu x_{2n+1} + \lambda x_{3n+1}) = x_1 + \mu x_{2n+1} + \lambda x_{3n+1}$. Thus the operator $D_{\pi(p)}^n(\pi(q))$ is not quasinilpotent. The conclusion is that $D_p^n(q)$ is not quasinilpotent either. ∎

The next result, which will be used in the following section, improves a theorem of Grabiner [5] on Banach algebras consisting of nilpotents.

PROPOSITION 4. *Let $\mathcal{A}$ be a Banach algebra consisting of algebraic elements. Then there exists $m \in \mathbb{N}$ such that for each $a \in \mathcal{A}$ there is a nonzero polynomial p, of degree at most m, with $p(a) = 0$.*

Proof. The Banach algebra $\mathcal{A}/\operatorname{Rad} \mathcal{A}$ consists of elements with finite spectrum, and so by Theorem 5.4.2 of [3] must be finite-dimensional. Thus there exists $n \in \mathbb{N}$ such that for each $a \in \mathcal{A}$ there is a nonzero polynomial p, of degree at most n, with $p(a) \in \operatorname{Rad} \mathcal{A}$.

Now a quasinilpotent algebraic element must be nilpotent. Thus the Banach algebra $\mathcal{R} = \operatorname{Rad} \mathcal{A}$ consists of nilpotents, and thus by Grabiner's theorem [5] must have bounded index of nilpotence. Hence there exists $k \in \mathbb{N}$ such that for each $q \in \operatorname{Rad} \mathcal{A}$ we have $q^k = 0$. It follows that for each $a \in \mathcal{A}$ there is a nonzero polynomial p of degree at most $m = nk$ with $p(a) = 0$. ∎

An operator A on a Banach space X is said to be *polynomially compact* when there is a polynomial $p \ne 0$ such that $p(A)$ is compact.

COROLLARY 5. *Let $\mathcal{A}$ be a norm closed algebra of polynomially compact operators on a Banach space X. Then there exists $n \in \mathbb{N}$ such that for every $A \in \mathcal{A}$ there is a nonzero polynomial p of degree at most n such that $p(A)$ is compact.*

Proof. Consider the Banach algebra $\mathcal{B} = \mathcal{A}/\mathcal{A} \cap \mathcal{K}(X)$. $\mathcal{B}$ consists of algebraic elements so from the previous proposition it has bounded index, n say. So for every $A \in \mathcal{A}$ there exists a nonzero polynomial p with $\deg p \le n$ such that $p(A + \mathcal{K}(X)) = 0$, which means that $p(A)$ is compact. ∎

3. Applications to invariant subspace theory. In the last fifteen years, much work has been done on simultaneous triangularizability of sets (especially algebras) of operators (see [4], [6], [7], [10], [11] and their references). We recall that a *nest* $\mathcal{N}$ on a Banach space X is a complete totally ordered set of closed subspaces of X containing $\{0\}$ and X. A set $\mathcal{S}$ of bounded operators acting on X is called *simultaneously triangularizable* when there exists an $\mathcal{S}$-invariant nest which is maximal as a nest.

An immediate corollary of Theorem 2 is the following.

PROPOSITION 6. *Let $\mathcal{A}$ be a norm closed algebra of polynomially compact operators on a Banach space. If $[\mathcal{Q}, \mathcal{Q}] \subseteq \mathcal{Q}$ or $\{\mathcal{Q}, \mathcal{Q}\} \subseteq \mathcal{Q}$ then $\mathcal{A}$ is simultaneously triangularizable.*

Proof. For such algebras Radjabalipour [10] has shown that the equality $\mathcal{Q} = \mathcal{R}$ implies that $\mathcal{A}$ is simultaneously triangularizable. Thus the result follows from Theorem 2. ∎

REMARK. (i) A different proof of the first implication (and also of Radjabalipour's theorem) will follow as a special case of Theorem 11 below. Note also that, for norm closed algebras consisting of compact operators, the same result has been proved in [4]. The method used in [4] is quite different and applies to more general operator algebras.

(ii) If $\mathcal{A}$ is a simultaneously triangularizable closed algebra of compact operators, then $\mathcal{Q}$ is an ideal (see, for example, Theorem 2 in [7]). This is no longer true for norm closed algebras of polynomially compact operators, as can be seen from the example in [10].

In [6] it is shown that if $\mathcal{A}$ is an algebra of operators on a Hilbert space H which is closed in the weak operator topology and contains a maximal abelian selfadjoint algebra, then the condition $\mathcal{Q} = \mathcal{R}$ implies simultaneous triangularizability of $\mathcal{A}$. Since by [1] there is no proper ultraweakly closed subalgebra of $\mathcal{B}(H)$ containing a maximal abelian selfadjoint algebra, the theorem is also valid if $\mathcal{A}$ is assumed to be ultraweakly closed. Combining this with Theorem 2, the following is immediately obtained.

Corollary 7. *Let $\mathcal{A}$ be an ultraweakly closed subalgebra of $\mathcal{B}(H)$ which contains a maximal abelian selfadjoint algebra. If $[\mathcal{Q},\mathcal{Q}] \subseteq \mathcal{Q}$ or $\{\mathcal{Q},\mathcal{Q}\} \subseteq \mathcal{Q}$ then $\mathcal{A}$ is simultaneously triangularizable.*

Recall that a (continuous) representation π of an algebra $\mathcal{A}$ on a Banach space X is called *topologically transitive* when $\pi(\mathcal{A})$ has no nontrivial *closed* invariant subspaces. In general, it is not known whether the kernel of such a representation must contain the radical (:= the intersection of the kernels of *algebraically* irreducible representations).

Lemma 8. *Let π be a topologically transitive representation of a Banach algebra $\mathcal{A}$ on some Banach space X such that $\pi(\mathcal{A})$ consists of polynomially compact operators. Then* $\operatorname{Rad}\mathcal{A} \subseteq \ker\pi$.

Proof. Let $\mathcal{J} \subseteq \mathcal{A}$ be the closed ideal $\mathcal{J} = \pi^{-1}(\mathcal{K}(X))$. By hypothesis, for each $a \in \mathcal{A}$ there is a nonzero polynomial p with $p(\pi(a))$ compact, that is, $p(a) \in \mathcal{J}$. Hence the Banach algebra $\mathcal{B} = \mathcal{A}/\mathcal{J}$ consists of algebraic elements. By Proposition 4 there exists $k \in \mathbb{N}$ such that p can be chosen of degree at most k. If $a \in \operatorname{Rad}\mathcal{A}$ we conclude that $B = \pi(a)^k$ must be compact. Suppose that $B \neq 0$. Since $\pi(\mathcal{A})$ is transitive and the operator B is compact, using Lomonosov's lemma [8], one finds $c \in \mathcal{A}$ such that the operator $\pi(c)B$ has a nonzero eigenvalue. Since the map π is a morphism, the spectrum of ca^k must be nonzero, and so $a^k \notin \operatorname{Rad}\mathcal{A}$. This contradiction shows that $\pi(a)^k = 0$.

Thus the algebra $\pi(\operatorname{Rad}\mathcal{A})$ consists of nilpotent operators of index at most k. By Theorem 4.1 of [6], $\pi(\operatorname{Rad}\mathcal{A})$ must have an invariant subspace. Since $\pi(\operatorname{Rad}\mathcal{A})$ is an ideal of the transitive algebra $\pi(\mathcal{A})$, it must be zero. ∎

Proposition 9. *Let $\mathcal{A}$ be a Banach algebra each element of which has totally disconnected spectrum. Let π be a topologically transitive representation of $\mathcal{A}$ on some Banach space X such that $\pi(\mathcal{A})$ consists of polynomially compact operators. If $\dim(X) \geq n$, then there exist n orthogonal idempotents $e_i \in \mathcal{A}$ such that the operators $\pi(e_i)$ have finite nonzero rank.*

Proof. If $\mathcal{J} = \pi^{-1}(\mathcal{K}(X))$ then, as in the proof of Lemma 8, the Banach algebra $\mathcal{B} = \mathcal{A}/\mathcal{J}$ consists of algebraic elements.

Case (i): *Suppose that* $\pi(\mathcal{J}) = \{0\}$. Then π induces a representation, ϕ say, of $\mathcal{B}$ on X which is (topologically) transitive. By Lemma 8, $\phi(\operatorname{Rad}\mathcal{B}) = \{0\}$.

Thus ϕ induces a (topologically) transitive representation ψ of the finite-dimensional algebra $\mathcal{B}/\operatorname{Rad}\mathcal{B}$ on X. Transitivity shows that $\dim(X) < \infty$, and now Burnside's theorem ([12] Corollary 8.6), shows that $\psi(\mathcal{B}/\operatorname{Rad}\mathcal{B}) = \mathcal{B}(X)$, so $\pi(\mathcal{A}) = \mathcal{B}(X)$. If $\xi_1, \ldots, \xi_n \in X$ are linearly independent, there exists $\pi(a) \in \pi(\mathcal{A}) = \mathcal{B}(X)$ with $\pi(a)\xi_k = k\xi_k$ for $k = 1, \ldots, n$ and the corresponding Riesz projections e_i of a satisfy our requirements.

Case (ii): *Suppose that* $\pi(\mathcal{J}) \neq \{0\}$. This is a nonzero ideal of $\pi(\mathcal{A})$ and hence must act transitively on X. Now let $\mathcal{D}$ be the closure of $\pi(\mathcal{J})$. This is a transitive algebra consisting of compact operators, so by Lomonosov's lemma there exists $A \in \mathcal{D}$ having a nonzero eigenvalue. The Riesz projection of A which corresponds to this eigenvalue is a nonzero, finite rank operator in $\mathcal{D}$. So $\mathcal{D}$ is a norm closed, transitive algebra, containing a finite rank operator. By Corollary 2.5 of [14], if ξ_k, $k = 1, \ldots, n$, are linearly independent vectors in X, then there exists T_1 in $\mathcal{D}$ such that $T_1\xi_k = k\xi_k$. Using the continuity of spectrum on the space of compact operators [3], one finds $\pi(a)$ in $\pi(\mathcal{J})$ having n nonzero, distinct eigenvalues. These will also be in the (totally disconnected) spectrum of a in $\mathcal{A}$ and the corresponding Riesz projections $e_i \in \mathcal{A}$ satisfy our requirements. ∎

Remark 1. In this proposition, as well as in the theorem below, the assumption that each element of $\mathcal{A}$ has totally disconnected spectrum cannot be omitted. Indeed, B. Aupetit [2] has constructed a nonabelian algebra $\mathcal{A}_A$ of 2×2 matrices over the bidisk algebra $A(\mathbb{D}\times\mathbb{D})$ which contains no nonzero quasinilpotents, and hence no nontrivial idempotents. This consists of all matrices

$$F = \begin{pmatrix} f & g \\ f_0\alpha(g) & \alpha(f) \end{pmatrix}$$

with $f, g \in A(\mathbb{D}\times\mathbb{D})$, where $f_0(z,w) = z + w$ and $(\alpha f)(z,w) = f(w,z)$. If $\pi(F) = F(0,1)$, clearly π is a transitive representation of $\mathcal{A}_A$ on a space of dimension 2.

Theorem 10. *Let $\mathcal{A}$ be a Banach algebra each element of which has totally disconnected spectrum. Let π be a topologically transitive representation of $\mathcal{A}$ on some Banach space X such that $\pi(\mathcal{A})$ consists of polynomially compact operators. If $n \in \mathbb{N}$ is such that $D_q^n(\mathcal{Q}) \subseteq \mathcal{Q}$ for all $q \in \mathcal{Q}$ then* $\dim(X) \leq n$.

Proof. Suppose that $\dim(X) \geq n+1$. Then, by Proposition 9, there exist $n+1$ orthogonal idempotents $e_i \in \mathcal{A}$ such that each $\pi(e_i)$ has finite nonzero rank. Let $\xi_1, \ldots, \xi_{n+1} \in X$ be unit vectors such that $\pi(e_i)\xi_i = \xi_i$, $i = 1, \ldots, n+1$.

CLAIM. *$\pi(e_i)(\pi(\mathcal{A})\xi_{i+1}) = \pi(e_i)(X)$ for all $i = 1, \ldots, n$.*

Proof of Claim. Since $\pi(\mathcal{A})\xi_{i+1}$ contains ξ_{i+1}, transitivity of $\pi(\mathcal{A})$ shows that $\overline{\pi(\mathcal{A})\xi_{i+1}} = X$. Hence

$$\pi(e_i)(X) = \pi(e_i)(\overline{\pi(\mathcal{A})\xi_{i+1}}) \subseteq \overline{\pi(e_i)(\pi(\mathcal{A})\xi_{i+1})} = \pi(e_i)(\pi(\mathcal{A})\xi_{i+1})$$

because the space $\pi(e_i)(\pi(\mathcal{A})\xi_{i+1})$ is finite-dimensional, hence closed.

It follows that there exist a_i, $i = 1, \ldots, n$, in $\mathcal{A}$ with $\pi(e_i)\pi(a_i)\xi_{i+1} = \xi_i$. Let $b_i = e_i a_i e_{i+1}$. The elements b_i are nilpotent of index two and $\pi(b_i)\xi_{i+1} = \xi_i$. Now notice that the sum $b = b_1 + \ldots + b_n$ is nilpotent; indeed, if $f_k = e_1 + \ldots + e_k$ and $f_0 = 0$, then $bf_{n+1} = b$ and $(1 - f_{k-1})bf_k = 0$ for $k = 1, \ldots, n+1$. However, $\pi_N(b)^n \xi_{n+1} = \xi_1$, so $\pi_N(b)^n \neq 0$, contradicting Lemma 8, since $b^n \in \mathcal{R}$ by the hypothesis and Theorem 3. ∎

Let $\mathcal{A}$ be an algebra of bounded operators on a Banach space X and $\mathcal{N}$ a nest of $\mathcal{A}$-invariant closed subspaces of X. For $N \in \mathcal{N}$, we denote by N_- the closed linear span of all subspaces in $\mathcal{N}$ which are properly contained in N. The operator induced by $A \in \mathcal{A}$ on the "gap" N/N_- is denoted by $\pi_N(A)$; that is, $\pi_N(A)\ (x+N_-) = A(x)+N_-$ for all $x \in N$. Note that if $\mathcal{N}$ is maximal with respect to being $\mathcal{A}$-invariant, then each π_N is a topologically transitive representation of $\mathcal{A}$ on N/N_-.

As a corollary of Theorem 10 we obtain

THEOREM 11. *Let $\mathcal{A}$ be an algebra of polynomially compact operators on some Banach space X which is complete with respect to some algebra norm which dominates the operator norm. Suppose that $D_q^n(\mathcal{Q}) \subseteq \mathcal{Q}$ for all $q \in \mathcal{Q}$. If $\mathcal{N}$ is any maximal $\mathcal{A}$-invariant nest, then $\dim(N/N_-) \leq n$ for all $N \in \mathcal{N}$.*

This theorem applies, for example, when $\mathcal{A}$ is a closed subalgebra of some von Neumann–Schatten class, or, more generally, of some complete normed ideal of compact operators on a Banach space.

We give an example to show that the completeness assumption cannot be omitted, even for algebras of compact operators:

EXAMPLE 12. *There exists an algebra of compact operators on a separable Hilbert space which contains no nonzero quasinilpotents and yet is not simultaneously triangularizable.*

Proof. Let $\mathcal{A}_0$ be the Banach algebra of all continuous functions $F : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to M_2(\mathbb{C})$ which are holomorphic in $\mathbb{D} \times \mathbb{D}$. Consider the direct sum

$$\pi(F) = \sum_{(n,m)\in\mathbb{N}^2}^{\oplus} F(1/n, 1/m)$$

as an operator on a separable Hilbert space. It is easy to see that π is a morphism, and $\|\pi(F)\| \leq \sup\{\|F(1/n,1/m)\| : (n,m) \in \mathbb{N}^2\} \leq \|F\|_\infty$. Moreover, π is 1-1, as can easily be seen from the identity principle. Indeed, if $\pi(F) = 0$ then the analytic function $z \to F(z, 1/m)$ vanishes at each $z = 1/n$ and hence identically, and so for each z the analytic function $w \to F(z,w)$ vanishes at each $w = 1/m$ and hence identically.

Notice that if $\pi(F)$ is quasinilpotent then F must be quasinilpotent since π preserves spectrum, and hence $F^2 = 0$. (Indeed, each $F(1/n, 1/m)$ is quasinilpotent in $M_2(\mathbb{C})$ and hence $F(1/n,1/m)^2 = 0$. By the identity principle, $F^2 = 0$ identically.)

Let $\mathcal{C}_0 \subseteq \mathcal{A}_0$ be the closed subalgebra consisting of all $F \in \mathcal{A}_0$ which vanish on $\mathbb{D} \times \{0\} \cup \{0\} \times \mathbb{D}$. We claim that $\pi(\mathcal{C}_0)$ consists of compact operators. For this, it suffices to observe that if $F \in \mathcal{C}_0$ the sum

$$\pi(F) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} F(1/n, 1/m)$$

converges in norm. Indeed, by the uniform continuity of F, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|z_1 - z_2| < \delta$ implies $\|F(z_1, w) - F(z_2, w)\| < \varepsilon$ for all $w \in \mathbb{D}$ and thus $\sup_m \|F(1/n, 1/m) - F(0, 1/m)\| \leq \varepsilon$ for $n > \delta^{-1}$. Since $F(0, 1/m) = 0$ for all $m \in \mathbb{N}$, we obtain $\lim_n \sup_m \|F(1/n, 1/m)\| = 0$; similarly $\lim_m \sup_n \|F(1/n, 1/m)\| = 0$ and the claim follows.

Now Aupetit's algebra $\mathcal{A}_A \subseteq \mathcal{A}_0$ mentioned in Remark 1 contains no nonzero quasinilpotents. Since $\mathcal{C}_0$ is an ideal of $\mathcal{A}_0$, the corresponding subalgebra $\mathcal{A}$ of $\mathcal{C}_0$ has the same property. This consists of all matrices

$$F = \begin{pmatrix} f & g \\ f_0\alpha(g) & \alpha(f) \end{pmatrix}$$

where $f, g \in A(\mathbb{D} \times \mathbb{D})$ vanish on $\mathbb{D} \times \{0\} \cup \{0\} \times \mathbb{D}$, $f_0(z,w) = z + w$ and $(\alpha f)(z,w) = f(w,z)$. Thus $\pi(\mathcal{A})$ is an algebra of compact operators with no nonzero quasinilpotents. It is evident that the compression of $\pi(\mathcal{A})$ to each of its two-dimensional invariant subspaces is transitive (in fact, it equals $M_2(\mathbb{C})$) and thus $\pi(\mathcal{A})$ is not simultaneously triangularizable. ∎

As remarked earlier, the converse of Theorem 11 fails, even for $n = 1$, for algebras containing noncompact operators (see [10]); it is true when $\mathcal{A} \subseteq \mathcal{K}(X)$:

PROPOSITION 13. *Let $\mathcal{A}$ be a norm closed algebra of compact operators and $\mathcal{N}$ a maximal $\mathcal{A}$-invariant nest. If* $\dim(N/N_-) \le n$ *for all* $N \in \mathcal{N}$, *then* $Q^n \in \operatorname{Rad}\mathcal{A}$ *(and so* $D^n_Q(\mathcal{Q}) \subseteq \mathcal{Q}$*) for all* $Q \in \mathcal{Q}$.

Proof. Suppose that there exists $Q \in \mathcal{Q}$ such that $A = Q^n \notin \operatorname{Rad}\mathcal{A}$. Then there exists $B \in \mathcal{A}$ such that the spectrum of BA is nonzero. As BA is compact, there exists a maximal nest, $\mathcal{N}_1$ say, of closed subspaces, containing $\mathcal{N}$, which triangularizes BA (see [13]). Thus the (scalar) operator induced by BA on some "gap" of $\mathcal{N}_1$ is nonzero; it follows that BA cannot vanish in the corresponding "gap" of $\mathcal{N}$, and so neither can A.

Thus there exists $N \in \mathcal{N}$ such that $\pi_N(Q^n)$ is not zero. Let $x \in N/N_-$ be such that $\pi_N(Q^n)x \ne 0$. As remarked in Section 2, and since the operator $\pi(Q)$ is quasinilpotent, the vectors $x, \pi_N(Q)x, \ldots, \pi_N(Q^n)x$ are linearly independent and hence $\dim(N/N_-) \ge n+1$. ∎

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References

[1] W. B. Arveson, *Operator algebras and invariant subspaces*, Ann. of Math. (2) 100 (1974), 433–532.
[2] B. Aupetit, *Propriétés spectrales des algèbres de Banach*, Lecture Notes in Math. 735, Springer, 1979.
[3] —, *A Primer on Spectral Theory*, Springer, 1990.
[4] B. A. Barnes and A. Katavolos, *Properties of quasinilpotents in some operator algebras*, Proc. Roy. Irish Acad. Sect. A 93 (1993), 155–170.
[5] S. Grabiner, *The nilpotency on Banach nil algebras*, Proc. Amer. Math. Soc. 21 (1969), 510.
[6] D. Hadwin, E. Nordgren, M. Radjabalipour, H. Radjavi and P. Rosenthal, *On simultaneous triangularization of collections of operators*, Houston J. Math. 17 (1991), 581–602.
[7] A. Katavolos and H. Radjavi, *Simultaneous triangularization of operators on a Banach space*, J. London Math. Soc. (2) 41 (1990), 547–554.
[8] V. I. Lomonosov, *Invariant subspaces for the family of operators which commute with a completely continuous operator*, Funktsional. Anal. i Prilozhen. 7 (3) (1973) (in Russian); English transl.: Functional Anal. Appl. 7 (1973), 213–214.
[9] V. Pták, *Commutators in Banach algebras*, Proc. Edinburgh Math. Soc. 22 (1979), 207–211.
[10] M. Radjabalipour, *Simultaneous triangularization of algebras of polynomially compact operators*, Canad. Math. Bull. 34 (1991), 260–264.
[11] H. Radjavi, *The Engel–Jacobson theorem revisited*, J. Algebra 111 (1987), 427–430.
[12] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer, 1973.
[13] J. R. Ringrose, *Compact Nonselfadjoint Operators*, Van Nostrand, New York, 1971.

[14] P. Rosenthal, *Applications of Lomonosov's lemma to non-self-adjoint operator algebras*, Proc. Roy. Irish Acad. Sect. A 74 (1974), 271–281.
[15] A. M. Sinclair, *Automatic Continuity of Linear Operators*, London Math. Soc. Lecture Note Ser. 21, Cambridge Univ. Press, 1976.
[16] Z. Słodkowski, W. Wojtyński and J. Zemánek, *A note on quasinilpotent elements of a Banach algebra*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977), 131–134.
[17] E. Vesentini, *On the subharmonicity of the spectral radius*, Boll. Un. Mat. Ital. 4 (1968), 427–429.
[18] P. Vrbová, *A remark concerning commutativity modulo the radical in Banach algebras*, Comment. Math. Univ. Carolin. 22 (1981), 145–148.
[19] J. Zemánek, *Spectral radius characterizations of commutativity in Banach algebras*, Studia Math. 61 (1977), 257–268.

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