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Dominated ergodic theorems in rearrangement invariant spaces

by

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Abstract. We study conditions under which Dominated Ergodic Theorems hold in rearrangement invariant spaces. Consequences for Orlicz and Lorentz spaces are given. In particular, our results generalize the classical theorems for the spaces L_p and the classes $L\log^n L$.

1. Results. The aim of this paper is to generalize the classical Dominated Ergodic Theorems for L_1-L_∞-contractions, which hold in the spaces L_p (see, for example, [K85], p. 52, and references there), to the class of rearrangement invariant Banach spaces. We will show that this generalization can be obtained by using standard techniques connected with the Hardy–Littlewood maximal function. Corresponding consequences for Orlicz and Lorentz spaces include the classical results for the spaces L_p.

Recall some definitions and notations. Let μ be Lebesgue measure on $[0,1]$. For a measurable function f on $(0,1)$, its *decreasing rearrangement* is defined by the formula

$$(1.1)\qquad f^*(t)=\inf\{y>0:\mu\{s:|f(s)|>y\}\le t\},\qquad 0<t\le 1.$$

Clearly f^* is decreasing right-continuous, and has the same distribution as $|f|$.

A Banach space $\mathbf{E}$ of measurable functions on $(0,1)$ is called *rearrangement invariant* (r.i.) if the following conditions hold:

(i) $g\in\mathbf{E}$ and $|f|\le|g|$ imply that $f\in\mathbf{E}$ and $\|f\|_{\mathbf{E}}\le\|g\|_{\mathbf{E}}$;
(ii) $g\in\mathbf{E}$ and $f^*=g^*$ imply that $f\in\mathbf{E}$ and $\|f\|_{\mathbf{E}}=\|g\|_{\mathbf{E}}$.

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One can find the theory of r.i. spaces in [KPS82] and [LT79]. The class of r.i. spaces contains the spaces $L_p = L_p(0,1), 1 \le p \le \infty$, Orlicz and Lorentz spaces. It is known ([KPS82], Ch. 2) that $L_\infty \subset \mathbf{E} \subset L_1$ for every r.i. space $\mathbf{E}$. We write $\mathbf{E}_1 = \mathbf{E}_2$ if these r.i. spaces coincide as sets. In this case the norms of $\mathbf{E}_1$ and $\mathbf{E}_2$ are equivalent.

If f is integrable, so is f^* and we define

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(x)\,dx, \qquad 0 < t \le 1. \tag{1.2}$$

It is clear that the function $f^{**}(t)$ is decreasing and continuous, so

$$f^{**}(\mu(f^{**} > u)) = u \tag{1.3}$$

for all $u > f^{**}(1)$. Since the functions $|f|$ and f^* have the same distribution, it follows that $f^{**}(1) = \|f\|_{L_1}$.

We have $f^{**}(t) \ge f^*(t), (f^*)^* = f^*$ and therefore (by (ii)) $f^{**} \in \mathbf{E}$ implies $f \in \mathbf{E}$ for every r.i. space $\mathbf{E}$. So,

$$H(\mathbf{E}) := \{f \in L_1(0,1) : f^{**} \in \mathbf{E}\} \subset \mathbf{E}. \tag{1.4}$$

The converse inclusion is not true, for example, in the space $\mathbf{E} = L_1$. It can be verified that $H(\mathbf{E})$ is a r.i. space under the norm

$$\|f\|_{H(\mathbf{E})} = \|f^{**}\|_{\mathbf{E}}. \tag{1.5}$$

A linear operator A is said to be an L_1-L_∞ *contraction* if it is a contraction in L_1 and in L_∞. Let $\mathcal{PC}$ be the set of all *positive* L_1-L_∞ contractions and let $\mathcal{PC}_0$ be the subset of $\mathcal{PC}$ which consists of all operators A_θ of the form

$$A_\theta f(t) = f(\theta(t)),$$

where θ is an invertible ergodic measure preserving transformation of $[0,1]$.

It is obvious that

$$A(\mathbf{E}) = \mathbf{E} \quad \text{and} \quad A(H(\mathbf{E})) = H(\mathbf{E})$$

for every $A \in \mathcal{PC}_0$. However, there exist r.i. spaces $\mathbf{E}$ and $A \in \mathcal{PC}$ such that

$$A(\mathbf{E}) \not\subset \mathbf{E}$$

(see [KPS82], Ch. 2, Sec. 5).

On the other hand, it was proved in [M65] and [C66] that a linear operator A is an L_1-L_∞-contraction if and only if

$$(Af)^{**}(s) \le f^{**}(s), \quad 0 < s < 1,$$

for every $f \in L_1$. Hence,

$$A(H(\mathbf{E})) \subset H(\mathbf{E})$$

for every r.i. space $\mathbf{E}$ and all $A \in \mathcal{PC}$.

For $A \in \mathcal{PC}$ put

$$B_A f(t) = \sup_{n\ge 1} \frac{1}{n}\sum_{k=0}^{n-1} A^k|f|(t). \tag{1.6}$$

By the Dunford–Schwartz ergodic theorem [DS58], Ch. 8, $B_A f < \infty$ a.e. for every $f \in L_1$.

Now we can formulate our results. The proofs will be given in Section 2.

THEOREM 1.1. *Let $\mathbf{E}$ be a r.i. space.*

1) *If $A \in \mathcal{PC}$, then $f \in H(\mathbf{E}) \Rightarrow B_A f \in \mathbf{E}$ and*

$$\|B_A f\|_{\mathbf{E}} \le \|f\|_{H(\mathbf{E})}. \tag{1.7}$$

2) *If $A \in \mathcal{PC}_0$, then $B_A f \in \mathbf{E} \Rightarrow f \in H(\mathbf{E})$.*

We will say that the r.i. space $\mathbf{E}$ has the *Hardy–Littlewood property* ($\mathbf{E} \in$ HLP) if $H(\mathbf{E}) = \mathbf{E}$, that is, $f \in \mathbf{E} \Rightarrow f^{**} \in \mathbf{E}$. We write $\mathbf{E} \in$ DET if $B_A f \in \mathbf{E}$ for every $f \in \mathbf{E}$ and $A \in \mathcal{PC}$. Consider the operator

$$D_r f(t) = \begin{cases} f(t/r), & t \le \min\{1, r\}, \\ 0, & r < t \le 1. \end{cases}$$

COROLLARY 1.1. *The following conditions are equivalent for a r.i. space $\mathbf{E}$:*

1) $\mathbf{E} \in$ DET;
2) $\mathbf{E} \in$ HLP;
3) $\|D_r\|_{\mathbf{E}\to\mathbf{E}} = o(r)$ *as* $r \to \infty$;
4) $$d_{\mathbf{E}} := \int_0^1 \|D_{1/r}\|_{\mathbf{E}\to\mathbf{E}}\,dr < \infty.$$

If these conditions hold then

$$\|B_A f\|_{\mathbf{E}} \le d_{\mathbf{E}}\|f\|_{\mathbf{E}}, \quad f \in \mathbf{E}. \tag{1.8}$$

The equivalence 1)⇔2) follows from Theorem 1.1, and the proof of 2)⇔3)⇔4) can be found, for example, in [KPS82], Ch. 2, Sec. 67.

We now turn to the inequality (1.8). It follows from (1.2) that

$$f^{**}(t) = \int_0^1 f^*(tr)\,dr = \int_0^1 D_{1/r} f^*(t)\,dr.$$

Hence, the triangle inequality implies

$$\|f^{**}\|_{\mathbf{E}} \le \int_0^1 \|D_{1/r}\|_{\mathbf{E}\to\mathbf{E}}\,dr\,\|f^*\|_{\mathbf{E}} = d_{\mathbf{E}}\|f\|_{\mathbf{E}}.$$

Using (1.7) and (1.5) we get (1.8).

Recall now the definitions of Orlicz and Lorentz spaces.

Let M be an *Orlicz function*, i.e. a convex increasing function on $[0,\infty)$ with $M(0)=0$. The *Orlicz space* L_M consists of all f such that $\int_0^1 M(|f(t)|/\lambda)\,dt<\infty$ for some positive λ, and it is a r.i. space under the norm

$$\|f\|_M=\inf\Big\{\lambda:\int_0^1 M(|f(t)|/\lambda)\,dt\le 1\Big\}. \tag{1.9}$$

The *Lorentz space* $L_{p,q}$, $1\le p\le\infty$, $1\le q\le\infty$, consists of all functions f such that

$$\|f\|_{p,q}=\Big(\int_0^1 (f^*(t))^q\,dt^{q/p}\Big)^{1/q}<\infty \tag{1.10}$$

for $q<\infty$, and for $q=\infty$,

$$\|f\|_{p,\infty}=\sup_{t>0} t^{1/p}f^*(t)<\infty. \tag{1.11}$$

It is clear that $L_{p,p}=L_p$. For $q>p$ the functional $\|\cdot\|_{p,q}$ does not satisfy the triangle inequality, but there is a norm $\|\cdot\|_{p,q}^{(1)}$ on $L_{p,q}$ which is equivalent to it ([SW71], Ch. 5). For any positive function ψ one can define the Lorentz space $L_{p,\psi}$ by replacing in (1.10) $dt^{q/p}$ by $\psi(t)dt$.

COROLLARY 1.2. *If $p>1$, then $L_{p,q}\in$ DET for every $1\le q\le\infty$. If, in addition, $1\le q\le p$, then*

$$\|B_Af\|_{p,q}\le\frac{p}{p-1}\|f\|_{p,q} \tag{1.12}$$

for every $A\in\mathcal{PC}$.

This result follows from (1.8) and from the well known and easily verified relation $\|D_r\|_{L_{p,q}\to L_{p,q}}=r^{1/p}$. If $p=q>1$, we get the classical inequality for the spaces L_p (see [K85], p. 52).

There exist Lorentz spaces $L_{p,\psi}$ such that $H(L_{p,\psi})\ne L_{p,\psi}$. Necessary and sufficient conditions under which a general Lorentz space $L_{p,\psi}\in$ HLP (and, hence, $L_{p,\psi}\in$ DET) were recently found in [AM90], [Sa90], [B93], [St93].

We now turn to Orlicz spaces.

COROLLARY 1.3. *The following conditions are equivalent for an Orlicz function M:*

1) $L_M\in$ DET;
2) $L_M\in$ HLP;
3) *there is $p>1$ such that*

$$\inf_{t,x>1}\frac{M(tx)}{t^pM(x)}>0;$$

4) $$\sup_{x>1}\frac{\int_1^x M(x/s)\,ds}{M(x)}<\infty;$$

5) $$\sup_{x>1}\frac{x\int_1^x\frac{M'(s)}{s}\,ds}{M(x)}<\infty.$$

We now describe the subspaces $H(L_M)$ for Orlicz spaces L_M. To this end for any Orlicz function M define

$$M_1(x)=\begin{cases}M(x), & 0\le x\le 1,\\ M(1)+x\int_1^x\frac{M'(t)}{t}\,dt, & x>1.\end{cases} \tag{1.13}$$

Then M_1 is an Orlicz function and $M(x)\le M_1(x)$ for all $x>0$. Hence $L_{M_1}\subset L_M$.

PROPOSITION 1.1. *For every Orlicz function M,*

$$H(L_M)=L_{M_1}.$$

This proposition and Theorem 1.1 imply the following dominated ergodic theorem for Orlicz spaces.

COROLLARY 1.4. *For every $n=0,1,\ldots,$*

1) *if $A\in\mathcal{PC}$, then $f\in L_{M_1}\Rightarrow B_Af\in L_M$;*
2) *if $A\in\mathcal{PC}_0$, then $B_Af\in L_M\Rightarrow f\in L_{M_1}$.*

The first statement can also be derived from the results of [ES92], Ch. 3, where the topic is treated in the context of martingales.

Consider now Zygmund's classes $L\log^n L$, $n=1,2,\ldots$ Recall that $L\log^n L$ consists of all measurable functions f on $[0,1]$ such that

$$\int_0^1 |f(x)|(\log^+|f(x)|)^n\,dx<\infty,$$

where $\log^+x=\max\{0,\log x\}$. We put $L\log^n L=L_1$ for $n=0$.

PROPOSITION 1.2. *For every $n=1,2,\ldots,$*

$$H(L\log^{n-1}L)=L\log^n L.$$

Note that the class $L\log^n L$ is an Orlicz space L_M, where $M(x)$ is equivalent as $x\to\infty$ to $l_n(x)=x(\log^+x)^n$ in the following sense: $l_n(c_1x)\le M(x)\le l_n(c_2x)$ for positive constants c_1, c_2 and all x large enough (see [KR61], Ch. 1).

Thus we obtain the following classical results ([K85], p. 54) as a consequence of Theorem 1.1 and Proposition 1.2.

COROLLARY 1.5. *For every $n=0,1,\ldots,$*

1) *if $A\in\mathcal{PC}$, then $f\in L\log^{n+1}L\Rightarrow B_Af\in L\log^n L$;*
2) *if $A\in\mathcal{PC}_0$, then $B_Af\in L\log^n L\Rightarrow f\in L\log^{n+1}L$.*

See also Section 3 for more examples of r.i. spaces without the DET-property.

2. Proofs. The proof of Theorem 1.1 is based on the following lemmas.

LEMMA 2.1. *Let f and g be nonnegative measurable functions on $(0,1)$ such that for every $t>0$,*

$$\mu\{g>t\} \le \frac{1}{t}\int_{\{g>t\}} f\,d\mu. \tag{2.1}$$

Then for all $s\in(0,1)$,

$$g^*(s) \le f^{**}(s). \tag{2.2}$$

Proof. We begin with the known relation

$$\int_0^s f^*(x)\,dx = \sup_{\{G:\mu G=s\}} \int_G |f|\,d\mu$$

(see [KPS82], Ch. 2, Sec. 2). Putting $s=\mu\{g>t\}$ and rewriting (2.1) in the form $t\le(1/s)\int_{\{g>t\}} f\,d\mu$, we get

$$t\le f^{**}(\mu\{g>t\}) \tag{2.3}$$

for every $t>0$.

Let

$$\mathcal{A}=\{s: s=\mu\{g>t\} \text{ for some } t>0\}$$

and $s_0\in\mathcal{A}$. Then (2.3) implies

$$t_0:=\sup\{t:\mu\{g>t\}=s_0\}\le f^{**}(s_0).$$

It can be easily derived from the definition of g^* that $g^*(s_0-0)=t_0$. Hence

$$g^*(s_0-0)\le f^{**}(s_0),\quad s_0\in\mathcal{A}. \tag{2.4}$$

Take now $s\notin\mathcal{A}$ and put

$$s_0=\sup\{u:u>s,\ g^*(u)=g^*(s)\}.$$

Then $s_0\in\mathcal{A}$ and by (2.4) we have

$$g^*(s)=g^*(s_0-0)\le f^{**}(s_0)\le f^{**}(s).$$

Thus, (2.2) is true for all $s\notin\mathcal{A}$. If $s\in\mathcal{A}$, the estimate follows from (2.4). ■

LEMMA 2.2. *For every nonnegative $f\in L_1$ and all $t>f^{**}(1)$,*

$$\mu\{f^{**}>2t\}\le\frac{1}{t}\int_{\{f>t\}} f\,d\mu\le\mu\{f^{**}>t\}. \tag{2.5}$$

Proof. We suppose, without loss of generality, that $f=f^*$. Set $s=\mu\{f^{**}>2t\}$. Then, according to (1.3), $2t=f^{**}(s)$ and (1.2) implies

$$s=\frac{1}{2t}\int_0^s f^*(x)\,dx. \tag{2.6}$$

If $s\le\mu\{f>t\}$, then

$$s\le\frac{1}{2t}\int_0^{\mu\{f>t\}} f^*(x)\,dx=\frac{1}{2t}\int_{\{f>t\}} f\,d\mu$$

and the first inequality in (2.5) follows.

Suppose $s>\mu\{f>t\}$. Then from (2.6) we have

$$s=\frac{1}{2t}\int_{\{f>t\}} f\,d\mu+\frac{1}{2t}\int_{\mu\{f>t\}}^{s} f(x)\,dx.$$

Since $f(x)\le t$ for $x>\mu\{f>t\}$, the last integral does not exceed $s/2$ and we get the lower bound in (2.5).

We now turn to the upper bound. Define $u=\mu\{f^{**}>t\}$. Then, as above,

$$u=\frac{1}{t}\int_0^u f^*(x)\,dx\ge\frac{1}{t}\int_{\{f>f^*(u)\}} f\,d\mu.$$

We have $t=f^{**}(u)\ge f^*(u)$, which yields the desired estimate

$$u\ge\frac{1}{t}\int_{\{f>t\}} f\,d\mu. \ ■$$

LEMMA 2.3. *For every $A\in\mathcal{PC}$, $f\in L_1$ and all $s\in(0,1)$,*

$$(B_Af)^*(s)\le f^{**}(s).$$

Proof. This follows directly from the classical maximal inequality (see [K85], p. 51)

$$\mu\{B_Af>t\}\le\frac{1}{t}\int_{\{B_Af>t\}}|f|\,d\mu \tag{2.7}$$

and (2.2) with $g=B_Af$. ■

LEMMA 2.4. *For every nonnegative $f\in L_1\setminus L_\infty$ there exists a constant $a=a(f)\in(0,1)$ such that for all $s\in(0,a)$ and all $A\in\mathcal{PC}_0$,*

$$f^{**}(s)\le 2(B_Af)^*(s/2).$$

Proof. Now we use Ornstein's inequality (see [O71] and [De73])

$$\mu\{B_A f > t\} \geq \frac{1}{2t} \int\limits_{\{f>t\}} f\, d\mu \quad (t > \|f\|_{L_1}),$$

which holds for every $A \in \mathcal{PC}_0$. This inequality and (2.5) imply that

$$\mu\{B_A f > t\} \geq \mu\{f^{**} > 2t\}/2$$

for $t > \|f\|_{L_1}$. Since $f \notin L_\infty$, we have $\mu\{f^{**} > t\} \geq \mu\{f^* > t\} > 0$ for every $t > 0$. So, the number $a = \sup\{s : f^{**}(s) > 2\|f\|_{L_1}\}$ is positive and we get the desired estimate. ■

Proof of Theorem 1.1. The implication 1) and the estimate (1.7) follow from Lemma 2.3, while the implication 2) is a direct consequence of Lemma 2.4. ■

Proof of Corollary 1.3. The equivalence 2)⇔3) can be easily derived from Corollary 1.1 and Proposition 2.b.5 of [LT79].

We now turn to 3)⇔4). Denote by $a(p)$ the infimum in 3). The implication 3)⇒4) follows from the inequality

$$M(y/t) \leq \frac{M(y)}{a(p)t^p},$$

which is a direct consequence of 3). To prove the converse implication we need an auxiliary statement.

PROPOSITION 2.1. *If* 4) *of Corollary* 1.3 *holds, then there exists* $\varepsilon > 0$ *such that*

$$C(\varepsilon) := \sup_{x>1} \frac{\int_1^x t^\varepsilon M(x/t)\, dt}{M(x)} < \infty. \tag{2.8}$$

Proof. We use arguments from [B93]. Consider the operator

$$TM(x) = \int\limits_1^x M(x/t)\, dt.$$

Denoting by C the supremum in 4), we have $TM(x) \leq CM(x)$ for all $x > 1$. Since T is positive, by iterating we get

$$T^n M(x) \leq C^n M(x) \quad (x > 1) \tag{2.9}$$

for every $n = 1, 2, \ldots$

Now we compute $T^n M$. Let first $n = 2$. Changing variables and the order of integration we get

$$T^2 M(x) = \int\limits_1^x \Bigg(\int\limits_1^{x/s} M\left(\frac{x}{ts}\right) dt \Bigg) ds = \int\limits_1^x \Bigg(\int\limits_s^x M\left(\frac{x}{y}\right) dy \Bigg) \frac{ds}{s}$$
$$= \int\limits_1^x M\left(\frac{x}{y}\right) \log y\, dy.$$

The same arguments and induction give us

$$T^n M(x) = \frac{1}{(n-1)!} \int\limits_1^x M\left(\frac{x}{y}\right) (\log y)^{n-1}\, dy.$$

From (2.9), for all $x > 1$ and $0 < \varepsilon < C^{-1}$,

$$\sum_{n=1}^\infty \varepsilon^{n-1} T^n M(x) \leq C \sum_{n=1}^\infty C^{n-1}\varepsilon^{n-1} M(x) = C_1 M(x), \tag{2.10}$$

where $C_1 = C\sum_{n=1}^\infty (C\varepsilon)^{n-1} = C(1 - C\varepsilon)^{-1} < \infty$. On the other hand, the first sum in (2.10) is equal to

$$\sum_{n=1}^\infty \int\limits_1^x M\left(\frac{x}{y}\right) \frac{(\log y)^{n-1}\varepsilon^{n-1}}{(n-1)!}\, dy = \int\limits_1^x y^\varepsilon M\left(\frac{x}{y}\right) dy.$$

Thus,

$$\int\limits_1^x y^\varepsilon M\left(\frac{x}{y}\right) dy \leq C_1 M(x)$$

and (2.8) follows. ■

Implication 4)⇒3). By Proposition 2.1 there exists $\varepsilon > 0$ such that for $x > s > 1$,

$$C(\varepsilon)M(x) \geq \int\limits_1^s t^\varepsilon M\left(\frac{x}{t}\right) dt \geq M\left(\frac{x}{s}\right) \int\limits_1^s t^\varepsilon\, dt = M\left(\frac{x}{s}\right) \frac{s^{1+\varepsilon} - 1}{1+\varepsilon}$$

because $M(x/t)$ is decreasing with respect to t. Hence, for $s > 2$,

$$\frac{M(x)}{s^{1+\varepsilon}M(x/s)} \geq \frac{1 - s^{-1-\varepsilon}}{C(\varepsilon)(1+\varepsilon)} \geq \frac{1 - 2^{-1-\varepsilon}}{C(\varepsilon)(1+\varepsilon)} = b > 0.$$

Put $y = x/s$. Then for $s > 2$ and $y > 1$,

$$\frac{M(sy)}{s^{1+\varepsilon}M(y)} \geq b,$$

which implies 3).

The equivalence 4)⇔5) follows from the relation

$$\int\limits_1^x M\left(\frac{x}{t}\right) dt = x \int\limits_1^x \frac{M'(t)}{t}\, dt + xM(1) - M(x),$$

which can be derived by changing variables and integrating by parts. ■

Proof of Proposition 1.1. Implication $f \in L_{M_1} \Rightarrow f^{**} \in L_M$. For any function g,

$$\int_0^1 M(|g(x)|)\,dx = \int_0^\infty M'(t)\mu\{|g| > t\}\,dt \tag{2.11}$$
$$\le M(1) + \int_1^\infty M'(t)\mu\{|g| > t\}\,dt.$$

We may suppose, without loss of generality, that $f = f^*$. Using (2.11) and (2.5) we get

$$\int_0^1 M(f^{**}(x))\,dx \le M(1) + 2\int_1^\infty \frac{M'(t)}{t}\Big(\int_{\{f>t/2\}} f\,d\mu\Big)\,dt.$$

Changing the order of integration shows that the last term is

$$\frac{1}{2}\int_0^{x_0} 2f(x)\Big(\int_1^{2f(x)} \frac{M'(t)}{t}\,dt\Big)\,dx = \frac{1}{2}\int_0^{x_0} M_1(2f(x))\,dx, \tag{2.12}$$

where $x_0 = \mu\{f > 1/2\}$. We may suppose that the last integral is finite by changing f to f/λ, where λ is large enough. Then

$$\int_0^1 M(f^{**}(x))\,dx < \infty \tag{2.13}$$

and hence $f^{**} \in L_M$.

Implication $f^{**} \in L_M \Rightarrow f \in L_{M_1}$. Without loss of generality we may again suppose that $f = f^*$ and that (2.13) holds. Using (2.11) and (2.5) we get, by changing the order of integration,

$$\infty > \int_0^1 M(f^{**}(x))\,dx \ge \int_1^\infty \frac{M'(t)}{t}\Big(\int_{\{f>t\}} f\,d\mu\Big)\,dt$$
$$\ge \int_0^{x_1} M_1(f(x))\,dx - M(1),$$

where $x_1 = \mu\{f > 1\}$. Hence $f \in L_{M_1}$. ∎

Proof of Proposition 1.2. For $n = 1$ we have to prove that $H(L_1) = L\log L$. One can easily verify that

$$\|f^{**}\|_{L_1} = \int_0^1 f^*(x)\log(1/x)\,dx.$$

Hence $H(L_1)$ is the Lorentz space $L_{1,\psi}$, where $\psi(x) = \log(1/x)$. We may suppose $\|f\|_{L_1} = 1$. Since $\|f\|_{L_1} \ge x f^*(x)$ for every $0 < x < 1$, we have

$$\int_0^1 |f(x)|\log^+|f(x)|\,dx \le \int_0^1 f^*(x)\log(1/x)\,dx.$$

On the other hand, it is known (see, for example, [G86]) that

$$\int_0^1 f^*(x)\log(1/x)\,dx \le c\Big(1 + \int_0^1 |f(x)|\log^+|f(x)|\,dx\Big), \tag{2.14}$$

where $c \le e/(e-1)$ is a constant. From this and the previous inequality, $H(L_1) = L_{1,\psi} = L\log L$.

For $n > 1$ the class $L\log^n L$ is an Orlicz space L_M, whose Orlicz function $M(x)$ is equivalent to the function $l_n(x) = x(\log^+ x)^n$ as $x \to \infty$. An easy computation shows that the corresponding Orlicz function M_1 is equivalent to $l_{n+1}(x)$. Thus the desired statement follows from Proposition 1.1. ∎

The best constant in (2.14) was found in [G86].

One can derive the proof of Corollary 1.3 directly from Proposition 1.1. Indeed, the condition $L_M \in$ HLP means that $L_M = L_{M_1}$ as sets. Using the well known necessary and sufficient condition for the coincidence of two Orlicz spaces (see [KR61], Ch. 2) one can easily show that 2)⇔4).

3. Remarks. 1. We will say that a r.i. space $\mathbf{E}$ satisfies the *Statistic Ergodic Theorem* ($\mathbf{E} \in$ SET), if for every $f \in \mathbf{E}$ and $A \in \mathcal{PC}$ the sequence $A_n f = \frac{1}{n}\sum_{k=0}^{n-1} A^k f$ converges in the norm of $\mathbf{E}$. According to [V85], $\mathbf{E} \in$ SET if and only if $\mathbf{E}$ is separable. It is known ([KPS82], Ch. 2) that the last condition holds if and only if $\mathbf{E} \ne L_\infty$ and L_∞ is dense in $\mathbf{E}$.

Thus we can show that there is no connection between the DET-property and SET-property of $\mathbf{E}$. Indeed, we have

$$L_p \in \text{DET} \quad\text{and}\quad L_p \in \text{SET}, \quad 1 < p < \infty,$$
$$L_1 \notin \text{DET} \quad\text{and}\quad L_1 \in \text{SET}.$$

On the other hand, the Lorentz spaces $L_{p,\infty}$, $1 < p < \infty$, are nonseparable and they have the Hardy–Littlewood property. Thus

$$L_{p,\infty} \in \text{DET} \quad\text{and}\quad L_{p,\infty} \notin \text{SET}.$$

Finally, there exist nonseparable r.i. spaces $\mathbf{E}$ without the Hardy–Littlewood property (see [BM77] and [KPS82], Ch. 2). For such a space $\mathbf{E}$ we have

$$\mathbf{E} \notin \text{DET} \quad\text{and}\quad \mathbf{E} \notin \text{SET}.$$

2. From $\mathbf{E}_1 \subset \mathbf{E}_2$ and $\mathbf{E}_2 \in$ HLP it does not follow that $\mathbf{E}_1 \in$ HLP. We describe briefly an appropriate construction (see [LT79], p. 140) for the case $\mathbf{E}_2 = L_q$, $1 < q < \infty$, and $\mathbf{E}_1 = L_M$. More exactly, we show that for every

$q > 1$ there exists an Orlicz space L_M such that $L_r \subset L_M \subset L_q$ for every $r > q$ and $L_M \notin$ HLP.

Let $q > 1, a_0 = 0$ and $a_n = 2^{n^2}$. Define $M(a_k) = a_k^q$, and

$$M(x) = \frac{a_{k-1}^q - a_k^q}{a_{k-1} - a_k}(x - a_k) + a_k^q$$

for $x \in (a_{k-1}, a_k)$. Obviously, M is an Orlicz function. Choosing $x_n = 2^{n^2+1}$ and $t_n = 2^{2n}$, one can verify that

$$\lim_{n\to\infty} \frac{M(t_n x_n)}{t_n^p M(x_n)} = 0$$

for every $p > 1$. Hence, according to Corollary 1.3, $L_M \notin$ HLP.

It follows from the definition that $M(x) \geq x^q$ for all $x > 0$ and, therefore, $L_M \subset L_q$. On the other hand, elementary calculations show that $M(x)/x^r \to \infty$ as $x \to \infty$ for each $r > q$, which implies $L_r \subset L_M$.

3. The same method which was used in the proof of Theorem 1.1 allows us to obtain corresponding results for martingales. For example, we get the following

THEOREM 3.1. *Let $0 \leq f \in L_1$ and let $\{\mathcal{F}_n\}_{n=1}^\infty$ be an increasing sequence of σ-algebras such that $\mathcal{F}_1$ is trivial and $\bigcup_{n=1}^\infty \mathcal{F}_n$ coincides with the algebra of all measurable subsets of $(0,1)$. Consider the martingale $f_n = E^{\mathcal{F}_n} f$ and put*

$$\widetilde{B} f = \sup_{n\geq 1} f_n.$$

Then for every r.i. space $\mathbf{E}$,

1) $f \in H(\mathbf{E}) \Rightarrow \widetilde{B}f \in \mathbf{E}$ *and* $\|\widetilde{B}f\|_{\mathbf{E}} \leq \|f\|_{H(\mathbf{E})}$;

2) *if there exists a constant C such that*

$$f_{n+1} \leq C f_n \quad \text{for every } n = 1, 2, \ldots, \tag{3.1}$$

then $\widetilde{B}f \in \mathbf{E} \Rightarrow f \in H(\mathbf{E})$.

To prove 1) we use Doob's maximal inequality for martingales [Do53] and the same arguments as in the proof of Theorem 1.1. The proof of 2) is based on the inequality

$$\int_{\{f_n > t\}} f_n \, d\mu \leq \int_{\{f_1 > t\}} f_1 \, d\mu + C\mu\{\sup_{1\leq k\leq n} f_k > t\} \quad (t > 0),$$

which follows from (3.1) (see [ES92], p. 98), where a detailed explanation of the topic is given for Orlicz spaces.

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