Choosing $\varphi_n \in e[2^n, \infty)(H)$ with $\varphi_n \to 0$ a.s. and $\{\varphi_n\}$ orthonormal, one can obtain (i) $\Rightarrow$ (iv) $\wedge$ (v). Moreover, $\neg$(i) (negation of (i)) $\equiv$ (j) $\Rightarrow$ (jj) $\Rightarrow$ $\neg$(iv) $\wedge$ $\neg$(v). ∎

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Minimal self-joinings and positive topological entropy II

by

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Abstract. An effective construction of positive-entropy almost one-to-one topological extensions of the Chacón flow is given. These extensions have the property of almost minimal power joinings. For each possible value of entropy there are uncountably many pairwise non-conjugate such extensions.

1. Introduction. In 1979 D. Newton [New] asked whether there exist coalescent dynamical systems with positive metric entropy (a metric dynamical system is said to be coalescent if all its endomorphisms are invertible). This problem has not been solved so far. The analogous problem in topological dynamics had been solved by P. Walters [Wal] in 1974: he gave an example of a topologically coalescent flow with positive topological entropy. His example is not minimal; strictly ergodic topological Toeplitz flows with positive entropy and trivial centralizers were constructed in [BuKw]. Of course, they are topologically coalescent.

It turns out that there exist topological flows with positive entropy and satisfying stronger conditions than having trivial centralizers. In the metric setting D. Rudolph [Rud] introduced the notion of minimal self-joinings, which is much stronger than coalescence and implies zero entropy. In topological dynamics the situation is more complicated. There exist several corresponding notions; the oldest is *graphic* flows [Mar]; later A. del Junco [delJ] introduced a set of possible definitions for topological minimal self-joinings using the orbit closures as an analogue of ergodic measures, among them *almost minimal self-joinings* (AMSJ) and *almost minimal power joinings* (AMPJ); the rather complex definitions are given in Section 2. The classes of flows mentioned above are known to satisfy the following inclusions:

$$\text{coalescent} \supset \text{trivial centralizer} \supset \text{graphic} \supset \text{AMSJ} \supseteq \text{AMPJ}.$$

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In 1990 J. King [King] proved that there exist flows with the strong property of *minimal self-joinings* (MSJ) suggested by del Junco; MSJ implies zero entropy. In [BlGlKw] E. Glasner and the present authors proved abstractly the existence of positive-entropy almost one-to-one extensions of the Chacón flow (X,T). One result in [delJ] shows that such a flow has AMPJ. But we had no explicit example.

The purpose of this article is to give an effective construction of such a flow. In Section 3 we describe a class of almost one-to-one extensions $(\Omega(\omega),T)$ of the Chacón flow. In Section 4 we give sufficient conditions for $(\Omega(\omega),T)$ to be strictly ergodic and to have positive entropy. Section 5 is devoted to constructing uncountably many pairwise non-conjugate extensions $(\Omega(\omega),T)$ having the same entropy; it is possible to do this for an unbounded set of values of entropy. An arresting feature of the flow $(O(\omega),T)$ is that the Chacón flow is a topological canonical factor of each of them.

2. Definitions and background. We adopt here the topological view: by a *dynamical system* we mean a compact metric set X endowed with a homeomorphism T. The pair (X,T) is also called a (topological) *flow*. A dynamical system (Y,S) is a *factor* of the system (X,T) if there exists a continuous onto map $\pi : X \to Y$ such that $\pi T = S\pi$; (X,T) is called an *extension* of (Y,S). Topological entropy plays an essential part in this article; readers who want to be reminded of its rather long definition can find it e.g. in [DeGrSi]. π is called a *homomorphism* of the flow (X,T) to (Y,S). (X,T) and (Y,S) are *topologically conjugate* if π is one-to-one. The sets $\pi^{-1}(y)$, $y\in Y$, form a partition of X into closed pairwise disjoint sets.

We say that the factor (Y,S) of (X,T) is *topologically canonical* if for every homomorphism $\pi' : (X,T)\to(Z,U)$ such that (Z,U) and (Y,S) are topologically conjugate we have $\{\pi'^{-1}(z): z\in Z\}=\{\pi^{-1}(y): y\in Y\}$. By the *topological centralizer* of T we mean the set $C_{\rm top}(X,T)=\{S : X\to X : S \text{ is continuous and } TS=ST\}$. A flow (X,T) is called *minimal* if X has no proper closed T-invariant subset. We say that (X,T) is *uniquely ergodic* if there is a unique borelian normalized T-invariant measure μ on X. (X,T) is said to be *strictly ergodic* if it is minimal and uniquely ergodic.

Let $O(x,T)=\{T^i(x): i\in\mathbb{Z}\}$ be the orbit of $x\in X$. For a finite set E with $\#E=k$ and $l:E\to\mathbb{Z}$ let T^{*l} denote the self-map $\bigotimes_{i\in E}T^{l(i)}$ of X^k. An *off-diagonal* on X^k is a set of the form $T^{*l}\Delta$, where Δ is the diagonal in X^k, i.e. $\Delta=\{(x,\ldots,x)\in X^k : x\in X\}$. A POOD (product of off-diagonals) is a product of such sets; it is a closed T^{*l}-invariant subset of X^k and its projection onto each coordinate is equal to X. (X,T) is called *coalescent* if any topological endomorphism is an automorphism, and *graphic* if it is totally minimal and X^2 contains no minimal T^2-invariant subset except the off-diagonals [Mar].

A flow (X,T) has *almost minimal power-joinings* (AMPJ) if it is totally minimal and there is a dense T-invariant G_δ set X^* in X such that whenever E is finite with cardinality k, l is a map from E to $\mathbb{Z}\setminus\{0\}$ and $x=(x_i)_{i\in k}\in(X^*)^k$, then $\overline{O}(x,T^{*l})$ is a POOD [delJ]. An equivalent condition is that for each $x=(x_i)_{i\in k}\in(X^*)^k$ such that x_i and x_j are on different T-orbits whenever $l(i)=l(j)$, one has $\overline{O}(x,T^{*l})=X^k$.

The flow (X,T) is said to have *almost minimal self-joinings* (AMSJ) if it satisfies the condition defining AMPJ for $l=1$ only (i.e. $l(i)=1$ for each $i\in k$). A natural strengthening of AMPJ and AMSJ would be the requirement that $X^*=X$. J. King has proved that there is no topological flow having AMPJ with $X^*=X$ [King]. But he constructed a flow (X,T) having *minimal self-joinings* (MSJ), in the sense that

(i) every non-zero power of T is minimal,

(ii) for any pair of points $x,y\in X$ not in the same orbit, $\overline{O}(x,y)=X\times X$.

Some Toeplitz flows are coalescent without being graphic and there are graphic flows without SAMPJ (a property stronger than AMPJ, see [delJ, Prop. 11]). MSJ flows have zero entropy, while there exist AMPJ flows with positive entropy [BlGlKw]. A map $\pi:(X,T)\to(Y,S)$ is said to be an *almost one-to-one extension* if there are dense G_δ sets $Y'\subset Y$ such that π is one-to-one on $\pi^{-1}(Y)$. The formally weaker definition, when only Y' is assumed to be dense, is equivalent when X is minimal [BlGlKw, Lemma 1], which is the only case we consider in this article.

All examples in this paper are symbolic, so a few definitions are in order. Given a finite alphabet A, let A^* be the set of finite sequences of letters of A. The elements of A^* are called *words* or *blocks* over A. The set $A^{\mathbb{Z}}$ of doubly infinite sequences is endowed with the usual product topology together with the shift homeomorphism T. A *subshift* X is a closed T-invariant subset of $A^{\mathbb{Z}}$; it is completely determined by the set $L(X)\subset A^*$ of all words that appear as blocks of coordinates in its points. Let $B=(B[0],\ldots,B[k-1])$ be a block over A. The number k is called the *length* of B and denoted by $|B|$. If $x\in A^{\mathbb{Z}}$ and B is a block then $x[i,s]$, $B[i,s]$, $0\le i\le s\le k-1$, denote the blocks $(x[i],\ldots,x[s])$ and $(B[i],\ldots,B[s])$ respectively. The *concatenation* of B and $C=(C[0],\ldots,C[m-1])$ is the block

$$BC=(B[0],\ldots,B[k-1],C[0],\ldots,C[m-1]).$$

Let $\Theta_x(n)$ be the number of different blocks of length n of X. The *entropy* of (X,T) is given by the formula

$$h(X,T)=\lim_{n\to\infty}\frac{1}{n}\log\Theta_x(n),$$

and since this limit exists one can take it also along a subsequence.

3. A class of almost one-to-one extensions of the Chacón flow

3.1. *The Chacón flow.* We start with the construction of Chacón's example [Cha, delJKe]. Define a family of words $(\overline{B}_t,\ t \in \mathbb{N})$ on the alphabet $\{0, s\}$ by $\overline{B}_0 = 0$ and the induction formula

$$\overline{B}_{t+1} = \overline{B}_t \overline{B}_t s \overline{B}_t .$$

$\overline{B}_t$ is called the *t-block*. Putting $d_t = |\overline{B}_t|$ one has

$$d_t = \frac{1}{2}(3^{t+1} - 1), \qquad t \geq 0.$$

The formulas

$$x[-d_t, d_t - 1] = \overline{B}_t \overline{B}_t \quad \text{and} \quad x'[-d_t, d_t] = \overline{B}_t s \overline{B}_t$$

define two two-sided infinite sequences x and x' on $\{0, s\}$. The *Chacón subshift* is defined as $X = \overline{O(x)} \subset \{0, s\}^{\mathbb{Z}}$, endowed with the shift T. The points belonging to $O(x) \cup O(x')$ are called *exceptional*. The remaining points of X can be constructed by the nesting block procedure [delJ]. We now describe a non-standard version of this procedure which is adapted to our requirements.

Assume that $1 \leq k_0 < k_1 < \ldots$ are positive integers and let

$$p_n = k_0 + \ldots + k_n, \quad B_n = \overline{B}_{p_n}, \quad l_n = |B_n| = \frac{1}{2}(3^{p_n + 1} - 1), \quad n \geq 0.$$

Then B_{n+1} is a concatenation of $3^{k_{n+1}}$ copies of B_n and of $\frac{1}{2}(3^{k_{n+1}} - 1)$ times the letter s in suitable order.

Suppose we are given two integers $n_0 \geq 0$ and λ_{n_0} such that $0 \leq \lambda_{n_0} \leq l_{n_0} - 1$ and $B_{n_0}[\lambda_{n_0}] = s$ if $n_0 \geq 1$, and a sequence $a = (a_n)_{n \geq n_0}$ with $0 \leq a_n \leq 3^{k_{n+1}} - 1$.

We define another sequence $\lambda(a)$ of integers by letting

$$\text{(1)} \qquad \lambda_{n+1}(a) = \lambda_n(a) + l_n a_n + q, \qquad n \geq n_0,$$

where q is the number of occurrences of s in $B_{p_{n+1}}$ between p_n-blocks and to the left of the a_nth occurrence of B_n (see Fig. 1). Then $0 \leq \lambda_{n+1} \leq l_{n+1} - 1$ and we can define

$$\text{(2)} \qquad y[-\lambda_n, l_n - \lambda_n - 1] = B_n \quad \text{for } n \geq n_0.$$

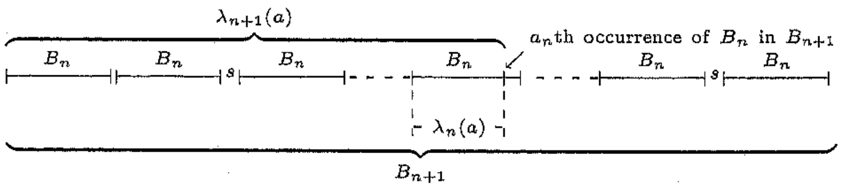

Fig. 1

If $a_n > 0$ infinitely many times and $a_n \leq 3^{k_{n+1}} - 2$ also infinitely many times, then $\lambda_n(a) \to \infty$ and so does $(l_n - \lambda_n(a) - 1)$. Then $y = y(a, \lambda_{n_0})$ is completely defined and $y \in X$. The set X consists of all $y(a, \lambda_{n_0})$ and the exceptional sequences in $O(x) \cup O(x')$.

3.2. *Some extensions of the Chacón flow.* Now we are in a position to construct almost one-to-one extensions of the Chacón flow (X, T) on the alphabet $A = \{1, 2, \ldots, 3^{k_0}, s\}$. We define a sequence $\mathcal{A}_n$ of families of blocks on A such that all blocks of $\mathcal{A}_n$ have the same length l_n; put $\#\mathcal{A}_n = m_n$.

Let σ belong to the set $\Pi(3^{k_0})$ of permutations of $\{1, 2, \ldots, 3^{k_0}\}$ and define $B_\sigma^{(1)}$ by substituting successively $\sigma(1), \sigma(2), \ldots, \sigma(3^{k_0})$ for the occurrences of 0 in B_{k_0}. The letter s stays unchanged.

Set

$$\mathcal{A}_1 = \{B_\sigma^{(1)} : \sigma \in \Pi(3^{k_0})\}.$$

We have

$$l_1 = \frac{1}{2}(3^{k_0+1} - 1) \quad \text{and} \quad m_1 = (3^{k_0})!.$$

Now assume that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are constructed and that the numbers $1 \leq k_0 < \ldots < k_n$ satisfy

$$\text{(3)} \qquad 3^{k_i} - 2 > m_{i-1}, \qquad i = 1, \ldots, n; \qquad m_0 = 0.$$

Let

$$\mathcal{A}_n = \{B_1^{(n)}, \ldots, B_{m_n}^{(n)}\}.$$

Fix two different blocks $L_n, F_n \in \mathcal{A}_n$ and choose $k_{n+1} = k \in \mathbb{N}$ satisfying (3) with $i = n + 1$. Let $t = t_{n+1} \geq 1$ be such that

$$\text{(4)} \qquad t m_n \leq 3^k - 2 < (t+1) m_n.$$

Let σ belong to the set Φ_n of maps from $\{1, \ldots, t m_n\}$ to $\{1, \ldots, m_n\}$ such that $\#\sigma^{-1}(i) = t$ for each $i = 1, \ldots, m_n$. Put $u = u_{n+1} = 3^k - 2 - t m_n \geq 0$. Define a block $B_\sigma^{(n+1)}$ by substituting L_n for the first $u + 1$ p_n-blocks of B_{n+1} and F_n for the last one, then replacing the jth of the $t m_n$ remaining p_n-blocks of B_{n+1} by $B_{\sigma(j)}^{(n)}$, $j = 1, \ldots, t m_n$ (see Fig. 2). We do not change the occurrences of s between p_n-blocks of B_{n+1}.

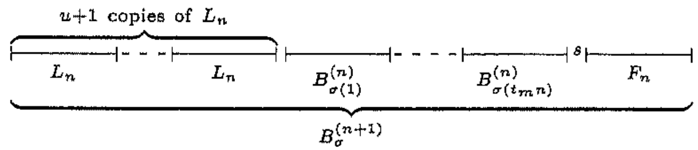

Fig. 2

Define

$$\mathcal{A}_{n+1} = \{B_\sigma^{(n+1)} : \sigma \in \Phi_n\}.$$

One has

(5) $$m_{n+1} = \#\Phi_n = \frac{(m_n t_{n+1})!}{(t_{n+1}!)^{m_n}}.$$

The blocks $B_i^{(n)}$, $i = 1, \ldots, m_n$, are called *n-symbols*. Note that each $(n+1)$-symbol $B^{(n+1)}$ is a concatenation of $3^{k_{n+1}}$ n-symbols and $l_{n+1} - l_n 3^{k_{n+1}}$ occurrences of the letter s between some of the n-symbols.

Recall that

(6) $$l_{n+1} = |B_{n+1}| = |B_j^{(n+1)}| = \frac{1}{2}(3^{p_{n+1}+1} - 1)$$

for each $j = 1, \ldots, m_{n+1}$.

Finally, define a two-sided infinite sequence ω over A by putting

(7) $$\omega[-l_n, l_n - 1] = F_n L_n$$

and the subshift $\Omega(\omega)$ to be $\overline{O(\omega)}$ endowed with T. Observe that ω and therefore $\Omega(\omega)$ are well defined once (k_n), (L_n) and (F_n) have been chosen.

4. Properties of the flows $(\Omega(\omega), T)$. In this section we are concerned with common properties of these flows; we can drop the argument ω in $\Omega(\omega)$. Denote by $\pi : (\Omega, T) \to (X, T)$ the natural homomorphism collapsing all letters $1, \ldots, 3^{k_0}$ to 0. We now describe the fibers of π. First observe that the sequence ω' defined by

$$\omega'[-l_n, l_n] = F_n s L_n, \quad n \geq 0,$$

is a point of $\Omega = \overline{O(\omega)}$. Since $\pi^{-1}(x) = \omega$ and $\pi^{-1}(x') = \omega'$, we also call the points of $O(\omega) \cup O(\omega') \subset \Omega$ *exceptional*.

The remaining part of Ω is obtained by a nesting procedure analogous to the one in Subsection 3.1: fix a sequence $a = (a_n)_{n \geq n_0}$, a non-negative $\lambda_{n_0} \leq l_{n_0} - 1$ and define a sequence (λ_n) by (2). Additionally assume that blocks $B_{i_n}^{(n)} \in \mathcal{A}_n$ are given in such a way that $B_{i_n}^{(n)}$ is the a_nth n-symbol of $B_{i_{n+1}}^{(n+1)}$. Then define a point $z = z(a, \lambda_{n_0}, \{B_{i_n}^{(n)}\})$ of Ω by putting

(8) $$z[-\lambda_n, l_n - \lambda_n - 1] = B_{i_n}^{(n)}$$

for each $n \geq 0$. Then Ω is the union of $O(\omega) \cup O(\omega')$ and the set of all z's of the form (8). Finally, one has $\pi(z(a, \lambda_{n_0}, \{B_{i_n}^{(n)}\})) = y(a, \lambda_{n_0})$ as defined in 3.1.

PROPOSITION 1. *(Ω, T) is an almost one-to-one extension of (X, T); if $t_n \to \infty$ it is strictly ergodic.*

Proof. Let C be a block over A and let $\mathrm{fr}(C, B_i^{(n)})$ be the average frequency of C in $B_i^{(n)}$. The letters of an $(n+1)$-symbol $B_j^{(n+1)}$ can be divided into 2 families:

(i) the subset I consists of all letters belonging to the first $u+1$ occurrences of L_n and to the last occurrence of F_n, and of the letters s between the n-symbols; put $K = \#I$;

(ii) the subset II consisting of the remaining letters of $B_j^{(n+1)}$; they correspond to tm_n occurrences of n-symbols.

We have

$$K \leq (u+2)l_n + u + 2 \overset{(4)}{\leq} (m_n + 2)(l_n + 1).$$

If C occurs in II then C can be contained in $B_i^{(n)}$ for some i, or be at the junction of blocks $B_i^{(n)} B_{i'}^{(n)}$ or $B_i^{(n)} s B_{i'}^{(n)}$. In the second case C can occupy at most $(|C| + 1)tm_n$ positions. Thus we have

(9) $$\mathrm{fr}(C, B_j^{(n+1)}) = \frac{tl_n}{l_{n+1}} \sum_{i=1}^{m_n} \mathrm{fr}(C, B_i^{(n)}) + \frac{R(j)}{l_{n+1}},$$

where

$$\frac{R(j)}{l_{n+1}} \leq \frac{(|C|+1)tm_n}{l_{n+1}} + \frac{(m_n+2)(l_n+1)}{l_{n+1}}, \qquad j = 1, \ldots, m_{n+1}.$$

Since

$$t_{n+1} m_n l_n < l_{n+1}$$

one gets

$$\frac{R(j)}{l_{n+1}} \leq \frac{|C|+1}{l_n} + \left(1 + \frac{2}{m_n}\right)\left(1 + \frac{1}{l_n}\right) \cdot \frac{1}{t_{n+1}} < \frac{|C|+1}{l_n} + \frac{4}{t_{n+1}} =: \varepsilon_n.$$

Then by (9),

(10) $$|\mathrm{fr}(C, B_j^{(n+1)}) - \mathrm{fr}(C, B_l^{(n+1)})| \leq 2\varepsilon_n \to 0$$

for every $j, l = 1, \ldots, m_{n+1}$. The unique ergodicity of Ω results from (10) and the construction of ω.

To prove the minimality of Ω note that if C appears in ω then it appears in some $B_j^{(n)}$, and hence in any $B_l^{(n+1)}$. Thus every fragment of ω of length $3l_{n+1}$ contains C.

To prove that π is an almost one-to-one extension note that

$$\#\pi^{-1}(y) = \begin{cases} 1 & \text{if } y \in X' = \{y \in X : a_n \leq u_{n+1} \text{ infinitely many times or } \\ & \quad a_n = 3^{k_n+1} - 1 \text{ infinitely many times}\}, \\ \text{continuum} & \text{in the remaining cases.} \end{cases}$$

and X' is a dense G_δ in X. ■

COROLLARY 1. *The flow (Ω, T) has AMPJ.*

Proof. By [delJ, Prop. 3.2], (X, T) has AMPJ and by [BlGlKw, Theorem 2.3] every almost one-to-one extension of an AMPJ flow has AMPJ. ■

Put $\tau_n = (1/l_n)\log m_n$, $n \ge 1$.

PROPOSITION 2. *If*

$$2\sum_{n=1}^{\infty}\frac{1}{3^{k_n}} + \frac{3}{2}\sum_{n=1}^{\infty}\frac{2+m_n}{3^{k_{n+1}}} < 1 \tag{11}$$

then

$$h(\Omega,T) > \tau_1\Big[1 - 2\sum_{n=1}^{\infty}\frac{1}{3^{k_n}} - \frac{3}{2}\sum_{n=1}^{\infty}\frac{2+m_n}{3^{k_{n+1}}}\Big] =: H > 0. \tag{12}$$

Proof. It follows from (5) and (6) that

$$\begin{aligned} &l_{n+1} > 3^{k_0+\ldots+k_{n+1}}, && l_{n+1} < 3^{k_{n+1}}(l_n+1),\\ &\frac{1}{l_n+1} > \frac{1}{l_n} - \frac{1}{l_n^2}, && 3^{k_0+\ldots+k_n} = \frac{2}{3}l_n + \frac{1}{3} > \frac{2}{3}l_n. \end{aligned} \tag{13}$$

A direct consequence of Stirling's formula is the inequality

$$n\log(n) - n \le \log(n!) \le n\log(n), \qquad n \ge 1. \tag{14}$$

As before let

$$k = k_{n+1}, \qquad t = t_{n+1}.$$

Using (3)–(5) and (14) we get

$$\begin{aligned} \log(m_{n+1}) &= \log((m_n t)!) - m_n\log(t!)\\ &\ge m_n t\log(m_n t) - m_n t - m_n t\log(t)\\ &= (m_n t)\log(m_n) - m_n t\\ &> (3^k - 2 - m_n)[\log(m_n) - 1]\\ &> 3^k\log(m_n) - 3^k - (2+m_n)\log(m_n). \end{aligned} \tag{15}$$

It is easy to check that $\tau_{n+1} \le \tau_n$, hence $\tau_n \le \tau_1$ for $n \ge 1$. This, together with (13) and (15), yields

$$\begin{aligned} \tau_{n+1} &= \frac{1}{l_{n+1}}\log(m_{n+1})\\ &> \frac{1}{l_n+1}\log(m_n) - \frac{3^k}{3^{k_0+\ldots+k_n+k}} - \frac{2+m_n}{3^{k_0+\ldots+k_n+k}}\log(m_n)\\ &> \frac{1}{l_n}\log(m_n) - \frac{1}{l_n}\cdot\frac{1}{l_n}\log(m_n) - \frac{1}{3^{k_n}} - \frac{2+m_n}{3^k}\cdot\frac{3}{2}\cdot\frac{1}{l_n}\log(m_n)\\ &> \tau_n - \frac{1}{l_n}\tau_1 - \frac{1}{3^{k_n}} - \frac{2+m_n}{3^{k_{n+1}}}\cdot\frac{3}{2}\tau_1. \end{aligned}$$

Summing up the above inequalities, passing to the limit and noticing that

$$\frac{1}{l_n} < \frac{3}{2}\cdot\frac{1}{3^{k_n}}$$

one gets

$$\begin{aligned} \lim_n \tau_n &\ge \tau_1\Big[1 - \sum_{n=1}^{\infty}\frac{1}{l_n} - \frac{3}{2}\sum_{n=1}^{\infty}\frac{2+m_n}{3^{k_{n+1}}} - \frac{1}{\tau_1}\sum_{n=1}^{\infty}\frac{1}{3^{k_n}}\Big]\\ &> \tau_1\Big[1 - 2\sum_{n=1}^{\infty}\frac{1}{3^{k_n}} - \frac{3}{2}\sum_{n=1}^{\infty}\frac{2+m_n}{3^{k_{n+1}}}\Big] = H, \end{aligned}$$

which, by (11) and the easily checked fact that τ_1 increases with k_0, starting from $\tau_1 = (\log 6)/4$ for $k_0 = 1$, implies

$$\lim_n \tau_n > H > 0.$$

To finish the proof note that $h(\Omega,T) \ge \lim_n \tau_n$; in fact, by standard arguments as in [Gri] one obtains $h(\Omega,T) = \lim_n \tau_n$. ∎

REMARK 1. It follows from (14) that

$$\tau_1 = \frac{1}{l_1}\log(m_1) \ge \frac{3^{k_0}}{l_1}(k_0-1) > \frac{2(k_0-1)}{3}. \tag{16}$$

We can select the numbers $k_0, k_1, \ldots$ in such a way that $h(\Omega,T)$ is arbitrarily close to τ_1, therefore arbitrarily large.

5. Non-conjugate elements in this class of examples. Recall that ω and the flow $\Omega(\omega)$ are well defined once suitable integers $(k_n)_{n\ge 0}$ and sequences (L_n) and (F_n) of words have been chosen.

Given (k_n) and two sequences ω, $\overline{\omega}$ corresponding to (k_n) but with different choices of L_n, F_n, we give a necessary and sufficient condition for $(\Omega(\omega),T)$ and $(\Omega(\overline{\omega}),T)$ to be conjugate. Suppose that ω is already constructed and $\overline{L}_i, \overline{F}_i, \overline{\mathcal{A}}_i$ are given (for $\overline{\omega}$) for $i \le n$. Let $\gamma : \mathcal{A}_n \to \overline{\mathcal{A}}_n$ be one-to-one. We construct $\overline{\mathcal{A}}_{n+1}$. Then we extend γ to a one-to-one map from $\mathcal{A}_{n+1}$ to $\overline{\mathcal{A}}_{n+1}$ as follows: $\gamma(B_j^{(n+1)})$ is obtained from $B_j^{(n+1)}$ by substituting $\overline{L}_n$ for the first $u+1$ occurrences of L_n, $\overline{F}_n$ for the last occurrence of F_n and $\gamma(B_j^{(n)})$ for any other occurrence of $B_j^{(n)}$. Then put $\overline{L}_{n+1} = \gamma(L_{n+1})$ and $\overline{F}_{n+1} = \gamma(F_{n+1})$. It is evident that $\gamma : \mathcal{A}_{n+1} \to \overline{\mathcal{A}}_{n+1}$ is one-to-one. By induction we can extend γ to a one-to-one map from $\mathcal{A}_p$ to $\overline{\mathcal{A}}_p$ for every $p \ge n$; for $p \ge n+1$ we have $\overline{L}_p = \gamma(L_p)$ and $\overline{F}_p = \gamma(F_p)$.

Now we can define a map $f_\gamma : \Omega(\omega) \to \Omega(\overline{\omega})$ in the following way:

$$f_\gamma(z) = \overline{z}(q, \lambda_{n_0}, \{\overline{B_{i_n}^{(n)}}\}),$$

where $\overline{z}$ is defined as in (8) with q, λ_{n_0}, and $\overline{B_{i_m}^{(n)}} = \gamma(B_{i_m}^{(n)})$, $m \ge n$, and

$$f_\gamma(T^i\omega) = T^i\overline{\omega}, \qquad f_\gamma(T^i\omega') = T^i\overline{\omega}', \qquad i \in \mathbb{Z}.$$

It is easy to see that f_γ is a conjugacy between $(\Omega(\omega),T)$ and $(\Omega(\overline{\omega}),T)$ such that the diagram

$$\begin{array}{ccc} \Omega(\omega) & \xrightarrow{f_\gamma} & \Omega(\overline{\omega}) \\ \pi\downarrow & & \downarrow\overline{\pi} \\ X & \xrightarrow{\text{id}} & X \end{array}$$

commutes, where $\overline{\pi} : (\Omega(\overline{\omega}),T) \to (X,T)$ is defined in the same way as π in Section 4.

THEOREM 1. *Assume ω and $\overline{\omega}$ are determined by the same integers $(k_n)_{n\geq 0}$ and by (L_n), (F_n) and $(\overline{L}_n)$, $(\overline{F}_n)$ respectively and $\widetilde{f} : \Omega(\omega) \to \Omega(\overline{\omega})$ is a conjugacy map; then there exist $n > 0$ and $n_1 \in \mathbb{Z}$ and a one-to-one map $\gamma : \mathcal{A}_n \to \overline{\mathcal{A}}_n$ such that $\widetilde{f} = T^{n_1} \circ f_\gamma$.*

Proof. By the same arguments as in [BlGlKw, Prop. 13] we have the following property for $(\Omega(\omega),T)$:

(17) for any ω as in (8), if $U : (\Omega(\omega),T) \to (Z,V)$ is a non-trivial factor then there is a homomorphism $\sigma : (Z,V) \to (X,T)$ such that $\pi = \sigma \circ U$.

Using (17) with $U = \widetilde{f}$ and $(Z,V) = (\Omega(\overline{\omega}),T)$ we get $\pi = \sigma \circ \widetilde{f}$ for some $\sigma : (\Omega(\overline{\omega}),T) \to (X,T)$. Applying (17) again with $U = \sigma$ and $(Z,V) = (X,T)$ we obtain $\overline{\pi} = S \circ \sigma$ for some S belonging to the centralizer $C_{\text{top}}(X,T)$. Then it is easy to obtain $\overline{\pi} \circ \widetilde{f} = S \circ \pi$. It is well known [delJRaSw] that $C_{\text{top}}(X,T) = \{T^n : n \in \mathbb{Z}\}$, hence $S = T^{n_1}$, $n_1 \in \mathbb{Z}$. Let $\widetilde{f}_1 = \widetilde{f} \circ T^{-n_1}$ (here $T : \Omega(\omega) \to \Omega(\omega)$). Then $\overline{\pi} \circ \widetilde{f}_1 = \pi$, which means that

$$\widetilde{f}_1(z) = \overline{z}(q, \lambda_{n_0}, \{\overline{B_{j_m}^{(m)}}\}),$$

where $z = z(q, \lambda_{n_0}, \{B_{j_m}^{(m)}\})$ (see again (8)).

Thus m-symbols in $\overline{z}$ are precisely at the same places as m-symbols in z for each $m \geq 1$. However, $\widetilde{f}_1$ is determined by a topological code ϕ. Let d be the length of ϕ. Choose n such that $l_n > 2d + 1$.

Because every $(n+1)$-symbol of $\overline{\omega}$ has $\overline{L}_n$ and $\overline{F}_n$ as initial and final n-symbols it is evident that ϕ determines a map $\gamma : \mathcal{A}_{n+1} \to \overline{\mathcal{A}}_{n+1}$. This map is one-to-one because $\widetilde{f}_1$ is. Moreover, $\widetilde{f}_1 = f_\gamma$. ■

REMARK 2. The property (17) implies that (X,T) is a topological canonical factor of $(\Omega(\omega),T)$. In fact, if $\pi' : (\Omega(\omega),T) \to (X,T)$ is a homomorphism then $\pi = S \circ \pi'$ for some $S \in C_{\text{top}}(X,T)$. Thus $S = T^n$ for some n so S is a homeomorphism. This implies $\{\pi^{-1}(x) : x \in X\} = \{\pi'^{-1} : x \in X\}$.

Now we can construct an uncountable family of pairwise non-conjugate flows $(\overline{O(\omega)},T)$ with the same positive entropy.

Fix a sequence $\{k_n\}_{n\geq 0}$ satisfying assumption (11). We will construct $L_1^{(0)}, L_1^{(1)}, F_1^{(0)}, F_1^{(1)}$ and collections $\mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1},1)}$, $\varepsilon_i = 0,1$, of blocks over $\{1,\ldots,3^{k_0},s\}$ and blocks

$$L_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1},0)}, F_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1},0)}, L_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1},1)}, F_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1},1)} \in \mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$$

for $n \geq 2$ such that

$$\#\mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})} = m_n$$

and each block of $\mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$ has length l_n for every $n \geq 1$ and every $(\varepsilon_1,\ldots,\varepsilon_{n-1})$.

Recall that in order to define $\mathcal{A}_{n+1}^{(\varepsilon_1,\ldots,\varepsilon_n)}$ we need the family $\mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$ and $L_n^{(\varepsilon_1,\ldots,\varepsilon_n)}$, $F_n^{(\varepsilon_1,\ldots,\varepsilon_n)} \in \mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$.

We illustrate this situation in Figure 3.

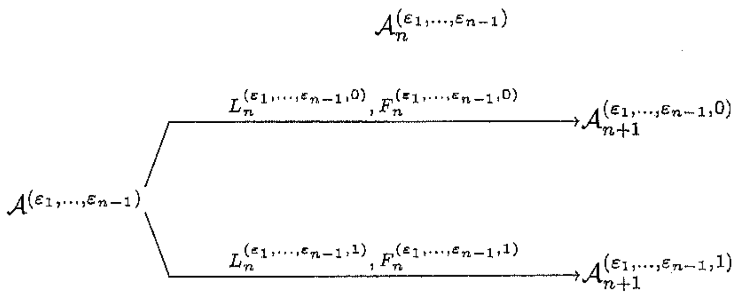

Fig. 3

Before passing to the construction of $\mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$ notice the obvious inequality

(18) $$m_{n+1} \geq m_n! > 2^{n+1} \quad \text{for } n = 0,1,\ldots$$

Start. Select $L_1^{(0)}, L_1^{(1)}, F_1^{(0)} \in \mathcal{A}_1$ in an arbitrary way. In this case there is a permutation γ of $\{1,\ldots,3^{k_0}\}$ such that $L_1^{(1)} = \gamma(L_1^{(0)})$. Then select $F_1^{(1)} \in \mathcal{A}_1$ such that $F_1^{(1)} \neq \gamma(L_1^{(1)})$; by (18) such a block $F_1^{(1)}$ exists.

Induction step. Suppose that $\mathcal{A}_i^{(\varepsilon_1,\ldots,\varepsilon_{i-1})}$ is constructed for each $i \leq n$ and each $(\varepsilon_1,\ldots,\varepsilon_{i-1})$, $\varepsilon_j = 0,1$. Select $L_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1},0)}, L_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1},1)}, F_n^{(0,\ldots,0)} \in \mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$ in an arbitrary way. Use the symbol $\prec$ to denote the lexicographic relation in the set of all n-tuples $(\varepsilon_1,\ldots,\varepsilon_n)$, $\varepsilon_j = 0,1$.

Assume that $F_n^{(\eta_1,\ldots,\eta_n)}$ is selected for each $(\eta_1,\ldots,\eta_n) \prec (\varepsilon_1,\ldots,\varepsilon_n)$. Choose $F_n^{(\varepsilon_1,\ldots,\varepsilon_n)} \in \mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$ in such a way that

(19) $$F_n^{(\varepsilon_1,\ldots,\varepsilon_n)} \neq \gamma(F_n^{(\eta_1,\ldots,\eta_n)}) \quad \text{for any } (\eta_1,\ldots,\eta_n) \prec (\varepsilon_1,\ldots,\varepsilon_n)$$

if there exists a one-to-one map $\gamma : \mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_n)} \to \mathcal{A}_n^{(\eta_1,\ldots,\eta_n)}$ such that $L_n^{(\eta_1,\ldots,\eta_n)} = \gamma(L_n^{(\varepsilon_1,\ldots,\varepsilon_n)})$. The choice of $F_n^{(\varepsilon_1,\ldots,\varepsilon_n)}$ satisfying (19) is possible by (18) because the number of blocks of $\mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_n)}$ which we must eliminate is at most 2^n.

The families $\mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$, $n \ge 2$, and the blocks $F_n^{(\varepsilon_1,\ldots,\varepsilon_n)}$, $L_n^{(\varepsilon_1,\ldots,\varepsilon_n)}$, $n \ge 1$, have the following property:

(20) For each n there is no one-to-one correspondence γ between $\mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_n)}$ and $\mathcal{A}_n^{(\eta_1,\ldots,\eta_n)}$ such that

$$L_{n+1}^{(\eta_1,\ldots,\eta_n,\eta_{n+1})} = \gamma(L_{n+1}^{(\varepsilon_1,\ldots,\varepsilon_n,\varepsilon_{n+1})}),$$
$$F_{n+1}^{(\eta_1,\ldots,\eta_n,\eta_{n+1})} = \gamma(F_{n+1}^{(\varepsilon_1,\ldots,\varepsilon_n,\varepsilon_{n+1})})$$

whenever $(\varepsilon_1,\ldots,\varepsilon_n,\varepsilon_{n+1}) \ne (\eta_1,\ldots\eta_n,\eta_{n+1})$.

Now we can sum up our theoretical results in the following

THEOREM 2. *There exist uncountably many pairwise non-conjugate strictly ergodic topological flows $(\Omega(\omega),T)$ having the same positive entropy and the AMPJ property.*

Proof. Take a sequence $\{k_n\}$ satisfying (11) and construct the families $\mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$ and $L_n^{(\varepsilon_1,\ldots,\varepsilon_n)}, F_n^{(\varepsilon_1,\ldots,\varepsilon_n)} \in \mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$ as above.

Let $\varepsilon = (\varepsilon_n)_{n\ge1}$ be an infinite sequence of 0's and 1's. Denote by $\omega(\varepsilon)$ the sequence defined as in (7) with the help of the families $\mathcal{A}_n^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$, $n \ge 2$, and the blocks $F_n^{(\varepsilon_1,\ldots,\varepsilon_n)}, L_n^{(\varepsilon_1,\ldots,\varepsilon_n)}$, $n \ge 1$. We obtain an uncountable family $(\Omega(\omega(\varepsilon)),T)$ of topological flows. It follows from Proposition 1 that each of them is strictly ergodic; their topological entropies have the same value, since the numbers of their n-symbols are identical; this value is positive by (15). Finally, Theorem 1 and (20) imply that $(\Omega(\omega(\varepsilon)),T)$ and $(\Omega(\omega(\varepsilon')),T)$ are not conjugate whenever ε and $\varepsilon' = (\varepsilon'_n)_{n\ge1}$ differ infinitely many times. ■

REMARK 3. Any two topological flows $(\Omega(\omega),T)$ ond $(\Omega(\overline{\omega}),T)$ constructed in this section (by using the same $k_0,k_1,\ldots$) are metrically isomorphic. To see this it suffices to construct codes between blocks. We have $\mathcal{A}_1 = \overline{\mathcal{A}}_1$ and define $f_1 : \mathcal{A} \to \overline{\mathcal{A}}_1$ by $f_1 = \mathrm{id}$. Having $f_n : \mathcal{A}_n \to \overline{\mathcal{A}}_n$ we can number the blocks of $\overline{\mathcal{A}}_n$ as follows: $\overline{B}_i^{(n)} = f_n(B_i^{n)})$, $i = 1,\ldots,m_n$, and then define $f_{n+1}(B_\sigma^{(n+1)}) := \overline{B}_\sigma^{(n+1)}$, $\sigma \in \Phi_n$. Next we define

$$U : \Omega(\omega) \setminus \pi^{-1}(X') \to \Omega(\overline{\omega}) \setminus \overline{\pi}^{-1}(X').$$

It is not hard to note that U is an isomorphism of the metric dynamical systems $(\Omega(\omega),T,\mu)$ and $(\Omega(\overline{\omega}),T,\overline{\mu})$ where $\mu,\overline{\mu}$ are the unique T-invariant measures.

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