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Almost sure approximation of unbounded operators in $L_2(X,\mathcal{A},\mu)$

by

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Abstract. The possibilities of almost sure approximation of unbounded operators in $L_2(X,\mathcal{A},\mu)$ by multiples of projections or unitary operators are examined.

1. Introduction. The purpose of this paper is to discuss a kind of approximation of unbounded operators in $L_2(X,\mathcal{A},\mu)$.

Throughout the paper, we assume that $(\mathcal{A},\mu)$ has the following properties:

(1.1) μ is separable and σ-finite

and

(1.2) there exists a sequence $\{Y_n\}\subset\mathcal{A}$ with $\mu(Y_n)>0$ and $\mu(Y_n)\to 0$.

The above assumptions imply, in particular, that the Hilbert space $H=L_2(X,\mathcal{A},\mu)$ is infinite-dimensional, separable and (by (1.1) and the martingale convergence theorem)

(1.3) there exists an increasing sequence $\{P_n\}$ of finite-dimensional orthogonal projections in H such that $P_n\to I$ strongly and $P_nf\to f$ μ-almost surely, for each $f\in H$.

Moreover, by (1.2),

(1.4) there exists a sequence $\{Z_n\}\subset\mathcal{A}$ of mutually disjoint sets such that $\sum_n\mu(Z_n)<\infty$ and the projections $\mathbf{1}_{Z_n}$ are infinite-dimensional.

Here and in the sequel we adopt the following notation. For a set $Z\in\mathcal{A}$, χ_Z will stand for the characteristic function (indicator) of Z, and $\mathbf{1}_Z$ denotes

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the orthogonal projection onto the subspace of functions with supports in Z. Always,

$$V(\varepsilon) = \{f \in H : \mu\{x : |f(x)| > \varepsilon\} < \varepsilon\}. \tag{1.5}$$

In the whole paper, we analyse the following kind of convergence. Let A be a linear operator, unbounded in general, acting in H, and let $\{A_n\}$ be a sequence of bounded linear operators in H. We say that A_n converges almost surely to A ($A_n \to A$ a.s.) iff

(*) $A_n f \to Af$ μ-almost surely, for each $f \in D(A)$ = the domain of A.

This paper is a continuation of [1; 2] where almost sure approximation of contractions in H has been considered. It should be stressed here that all the results in [1; 2] are valid for $H = L_2(X, \mathcal{A}, \mu)$ satisfying conditions (1.1) and (1.2) though they are formulated for $H = L_2(a, b)$.

This paper is devoted to almost sure approximation of an unbounded operator A by operators of the form $\lambda_n P_n$ or $\lambda_n U_n$, where P_n are orthogonal projections and U_n are unitary operators (with positive coefficients $\lambda_n \nearrow \infty$). In particular, we show that an approximation $\lambda_n U_n \to A$ a.s. ($\lambda_n \nearrow \infty$) is possible for every unbounded closed and densely defined operator A. The operator A admitting an approximation $\lambda_n P_n \to A$ a.s. can be characterized as "essentially unbounded" (see Definition 4.1). It is worth noting here that weak approximation in the sense that $\lambda_n U_n f \to Af$ weakly for each $f \in D(A)$ or $\lambda_n P_n f \to Af$ weakly for each $f \in D(A)$ is not possible for any unbounded operator A (see Section 5). Thus almost sure approximation seems to be important not only from the probabilistic but also from the functional-analytic point of view.

A sketch of some results presented here can be found in [4].

2. Convergence of subsequences. Let us start with the formulation of a classical result of Marcinkiewicz.

THEOREM A (Marcinkiewicz [6]). *If $\{\varphi_k\}$ is an orthonormal sequence in $L_2(0,1)$, then there exists an increasing sequence $\{n(k)\}$ of positive integers such that the subsequence of partial sums*

$$S_{n(k)}(x) = \sum_{i=1}^{n(k)} a_i \varphi_i(x) \qquad (k = 1, 2, \ldots)$$

converges almost surely on $(0,1)$ for each sequence $\{a_k\}$ with $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.

In [1; 2] the generalization of Theorem A presented below has been proved. A short and elementary proof of that generalization was given in [4].

THEOREM B. *Let $\{A_n\}$ be a sequence of finite-dimensional operators acting in $H = L_2(X, \mathcal{A}, \mu)$. Suppose that $A_n \to A$ strongly. Then there exists an increasing sequence $\{n(i)\}$ of positive integers such that $A_{n(i)} \to A$ a.s.*

The proof of Theorem B heavily depends on condition (1.3). In the above theorem, the assumption that the operators A_n are finite-dimensional cannot be omitted (in Theorem A of Marcinkiewicz, this assumption for $A_n = \sum_{k=1}^{n} \langle \cdot, \varphi_k \rangle \varphi_k$ is satisfied automatically). Namely, as shown in [2], there exists a sequence $\{P_n\}$ of orthogonal projections in $L_2(0,1)$, increasing to the identity and such that, for any increasing sequence $\{n(k)\}$ of indices, one can find a vector $f \in H$ such that $P_{n(k)} f$ does not converge a.s.

As a consequence of Theorem B we prove the following result which, together with Theorem B, will be an important tool in our further considerations.

2.1. THEOREM. *Let A be closed densely defined operator in H. If $\|A_n f - Af\| \to 0$ for some finite-dimensional A_n and all $f \in D(A)$, then $A_{n(k)} \to A$ a.s. for some increasing sequence $\{n(k)\}$.*

Proof. Let $|A| = \int_{[0,\infty)} \lambda\, e(d\lambda)$ be the spectral representation of $|A|$ and let $B = \int_{[0,\infty)} \min(1, 1/\lambda)\, e(d\lambda)$. Then $f \in D(A)$ iff $f = Bg$ for some $g \in H$. Consequently, $A_n B \to AB$ strongly, $A_n B$ being finite-dimensional. By Theorem B, for some increasing sequence $\{n(k)\}$, $A_{n(k)} B \to AB$ a.s., so $A_{n(k)} f \to Af$ for any $f \in D(A)$.

3. Approximation by a sequence $\lambda_n U_n$

3.1. THEOREM. *For any unbounded closed densely defined operator A and any increasing sequence $\{\lambda_n\}$, there exist unitary operators U_n such that $\lambda_n U_n \to A$ a.s.*

The proof of the above theorem is based on the following two results.

3.2. PROPOSITION. *For any closed densely defined operator A (bounded or not) and any sequence $\lambda_n \nearrow \infty$, there exist finite-dimensional partial isometries V_n such that $\lambda_n V_n \to A$ a.s.*

Proof. Observe that

$$A_n \to A \quad \text{a.s. for some finite-dimensional } A_n. \tag{3.1}$$

Indeed, this is obvious for $A \in B(H)$ by Theorem 2.1. For unbounded A, let $|A| = \int_{[0,\infty)} \lambda\, e(d\lambda)$ be the spectral decomposition and let $P_{n,m}$ be finite-dimensional projections satisfying $P_{n,m} \nearrow e([n-1, n))$ as $m \to \infty$, for $n = 1, 2, \ldots$ Then the operators

$$A_m = A \sum_{n=1}^{m} P_{n,m}$$

satisfy $\|A_m f - Af\| \to 0$ for $f \in D(A)$ and consequently, by Theorem 2.1, we have $A_{m(n)} \to A$ a.s. for some increasing sequence $\{m(n)\}$. Indeed, for $f \in D(A)$ and $\varepsilon > 0$ one can find m_0 and then m_1 such that

$$\sum_{n=m_0}^{\infty} n^2 \|e[n-1,n)f\|^2 < \varepsilon$$

and

$$\|A(e[n-1,n) - P_{n,m})f\|^2 < \varepsilon/m_0 \quad \text{for } n = 1, \ldots, m_0;\ m > m_1.$$

Then, for $m > \max(m_0, m_1)$, we have

$$\begin{aligned}\|(A - A_m)f\|^2 &= \sum_{m=1}^{m_0} \|A(e[n-1,n) - P_{n,m})f\|^2 \\ &\quad + \sum_{n=m_0+1}^{m} \|A(e[n-1,n) - P_{n,m})f\|^2 \\ &\quad + \sum_{n>m} \|Ae[n-1,n)f\|^2 \\ &< \frac{\varepsilon}{m_0} m_0 + \sum_{n=m_0+1}^{\infty} 4n^2 \|e[n-1,n)f\|^2 < 5\varepsilon.\end{aligned}$$

Moreover, taking the sequence

$$\underbrace{A_1, \ldots, A_1}_{n_1 \text{ terms}}, \underbrace{A_2, \ldots, A_2}_{n_2 \text{ terms}}, \ldots$$

instead of $\{A_n\}$, we can assume that, for the operators A_n in (3.1), we have $\|A_n\| < \lambda_n$ for $n > n_1$.

Let Q_n be a finite-dimensional projection satisfying $Q_n A_n Q_n = A_n$ and let $S_n^* S_n = Q_n$, $Q_n \perp S_n S_n^* \le \mathbf{1}_{Z_n}$ for Z_n satisfying (1.4) and for partial isometries S_n, $n = 1, 2, \ldots$ Then

$$V_n = \lambda_n^{-1} A_n + S_n \sqrt{Q_n - \lambda_n^{-2}|A_n|^2}, \qquad n > n_1,$$

are also partial isometries and $S_n g_n \to 0$ a.s. for any $g_n \in H$. Thus $\lambda_n V_n \to A$ a.s. ■

3.3. Lemma. *Let A be a closed densely defined unbounded operator with domain $D(A)$. Let P_n and Q_n be orthogonal projections in H such that $P_n^{\perp}$ and $Q_n^{\perp}$ are finite-dimensional. Then, for any sequence $\lambda_n \nearrow \infty$, there exist partial isometries W_n such that $W_n^* W_n = P_n$, $W_n W_n^* = Q_n$ and $\lambda_n W_n f \to 0$ a.s. for each $f \in D(A)$.*

Proof. Obviously, one can find infinite-dimensional mutually orthogonal projections $E_n = e(Y_n)$ with Y_n mutually disjoint and $Y_n \subset [\lambda_n^2, \infty)$, where

$e(\cdot)$ is the spectral measure of $|A|$. Then, for each $f \in D(A)$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n^2 \|E_n f\|^2 &= \sum_{n=1}^{\infty} \lambda_n^2 \Big\| \int_{Y_n} e(d\lambda) f \Big\|^2 \\ &\le \int_{[0,\infty)} \lambda^2 \|e(d\lambda)f\|^2 < \infty.\end{aligned}$$

The assumption that $\dim P_n^{\perp} < \infty$ and $\dim Q_n^{\perp} < \infty$ implies the existence of projections $F_n \le E_n$ and $G_n \le \mathbf{1}_{Z_n}$ (with Z_n satisfying condition (1.4)) such that $F_n \vee G_n \le P_n \wedge Q_n$ and

$$\dim(P_n - F_n) = \dim G_n = \dim(Q_n - G_n) = \dim F_n.$$

It is enough to put $W_n = S_n + T_n$ for any partial isometries S_n and T_n satisfying

$$S_n^* S_n = P_n - F_n, \quad S_n S_n^* = G_n, \quad T_n^* T_n = F_n, \quad T_n T_n^* = Q_n - G_n.$$

Indeed, $\text{support}(\lambda_n S_n f) \subset Z_n$ with $\mu(Z_n) \to 0$ and, for each $f \in D(A)$, we have

$$\sum_{n=1}^{\infty} \|\lambda_n^2 T_n f\|^2 \le \sum_{n=1}^{\infty} \lambda_n^2 \|E_n f\|^2 < \infty. \ ■$$

3.4. *Proof of Theorem 3.1.* For partial isometries V_n as in Proposition 3.2, let $P_n^{\perp} = V_n^* V_n$ and $Q_n^{\perp} = V_n V_n^*$ and let W_n be taken as in Lemma 3.3. Then the operators $U_n = V_n + W_n$ have the required properties. ■

Let us adopt the following definition.

3.5. Definition. For $A_n \in B(H)$ and a closed densely defined A, we say that A_n *converges in measure* to A iff

$$(**) \qquad A_n f \to Af \quad \text{in measure } \mu \text{ for each } f \in D(A).$$

3.6. Theorem. *Let $A \in B(H)$ and let $\lambda_n \nearrow \infty$. Then there is no sequence $\{U_n\}$ of unitary operators such that $\lambda_n U_n \to A$ in measure.*

Proof. Assume that, on the contrary, such a sequence exists. Fix an increasing sequence $\{n(k)\}$ of positive integers such that

$$(3.2) \qquad \lambda_{n(k+1)} > 2^{k+1} \lambda_{n(k)}.$$

Without loss of generality we may assume that $\mu(X) \ge 1$. Denote by h any function satisfying $h \notin V(1)$ and put $g_k = U_{n(k)}^{-1} h$. As always, $V(\varepsilon) = \{f \in H : \mu\{x : |f(x)| > \varepsilon\} < \varepsilon\}$.

Define by induction a sequence $\{f_k\}$ by putting $f_1 = g_1$ and

$$f_{k+1} = \begin{cases} f_k & \text{if } (\lambda_{n(k+1)}U_{n(k+1)} - \lambda_{n(k)}U_{n(k)})f_k \notin V(1/2), \\ f_k + \lambda^{-1}_{n(k+1)}g_{k+1} & \text{otherwise.} \end{cases}$$

For $f = \lim f_n$, we have, by (3.2),

$$\|\lambda_{n(k)}U_{n(k)}(f - f_k)\| \le \|h\|\lambda_{n(k)} \sum_{l=k+1}^{\infty} \lambda^{-1}_{n(l)} \le 2^{-k}\|h\|.$$

Since $\lambda_{n(k+1)}U_{n(k+1)}f_{k+1} - \lambda_{n(k)}U_{n(k)}f_k \notin V(1/2)$ for any $k = 1, 2, \ldots$, the sequence $\lambda_n U_n f$ does not converge in measure. ■

4. Approximation by a sequence $\lambda_n P_n$. In this section we discuss almost sure approximation of an unbounded positive selfadjoint operator A by a sequence $\lambda_n P_n$ where $0 < \lambda_n \nearrow \infty$ and P_n are orthogonal projections in H. In contrast to the case analysed in the previous section, it is not always possible to construct sequences $\{\lambda_n\}$ and $\{P_n\}$ so that $\lambda_n P_n \to A$ a.s. Such a possibility heavily depends on the structure of the spectral measure of A. To clarify the situation, consider an arbitrary unbounded selfadjoint positive operator A with spectral representation $A = \int_{[0,\infty)} \lambda\, e(d\lambda)$.

4.1. Definition. The operator A is said to be *essentially unbounded* iff

(4.1) for any $\varepsilon > 0$ and $\lambda > 0$, there exists a normalized vector $f \in H$ such that $f \in e(\lambda, \infty)(H)$ and $\mu\{x : |f(x)| > \varepsilon\} < \varepsilon$.

Condition (4.1) gives a complete characterization of those unbounded operators for which an approximation $\lambda_n P_n \to A$ a.s. exists. Let us remark that most of the well-known selfadjoint operators are bounded or essentially unbounded (see examples in Section 6). It should be stressed here that a selfadjoint positive unbounded, but not essentially unbounded, operator can be constructed (see Theorem 6.5).

The main result of this section is the following

4.2. Theorem. *Let A be any positive selfadjoint operator in H (bounded or not) and $A \ne 0$. Then the following conditions are equivalent:*

(i) *A is essentially unbounded;*

(ii) *there exists a sequence $\{P_n\}$ of orthogonal projections and positive coefficients $\lambda_n \nearrow \infty$ such that $\lambda_n P_n \to A$ a.s.;*

(iii) *there exists a sequence $\{P_n\}$ of orthogonal projections and positive coefficients $\lambda_n \nearrow \infty$ such that $\lambda_n P_n \to A$ in measure.*

In the condition (ii) (*or* (iii)) *the projections P_n can be taken finite-dimensional.*

It is enough to show that (i) implies (ii), and (iii) implies (i) (see Section 4.9).

4.3. (i)⇒(ii). Let $A = \int_{[0,\infty)} \lambda\, e(d\lambda)$ and let $R_{n,k}$ be finite-dimensional projections such that $R_{n,k} \nearrow e[n-1, n)$ as $k \to \infty$ for $n = 1, 2, \ldots$, and let $R_n = \sum_{k=1}^{n} R_{n,k}$. Then the operators $A_n = R_n A R_n$ are selfadjoint, finite-dimensional and $\|A_n f - Af\| \to 0$ for $f \in D(A)$. By Theorem 2.1, there is an increasing sequence $\{n(k)\}$ such that $A_{n(k)} \to A$ a.s. Moreover, $0 \le A_{n(k)} \le n(k)e[0, n(k))$.

Putting $B_k = A_{n(k)}$, we can write

$$B_k f = \sum_{s=1}^{m(k)} \lambda_s^{(k)} \langle f, f_s^{(k)}\rangle f_s^{(k)}$$

where $(f_1^{(k)}, \ldots, f_{m(k)}^{(k)})$ is an orthonormal system, $0 < \lambda_s^{(k)} \le n(k)$ and $f_s^{(k)} \in e[0, n(k))(H)$ for $s = 1, \ldots, m(k)$. Take $\nu_s^k = \lambda_s^k / n(k)$ and let $(\varphi_1^{(k)}, \ldots, \varphi_{m(k)}^{(k)})$ be an orthonormal system orthogonal to $(f_1^{(k)}, \ldots, f_{m(k)}^{(k)})$. Then we can define a projection P_k by putting

$$\begin{aligned} P_k f = &\sum_{s=1}^{m(k)} \nu_s^{(k)} \langle f, f_s^{(k)}\rangle f_s^{(k)} \\ &+ \sum_{s=1}^{m(k)} (1 - \nu_s^{(k)}) \langle f, \varphi_s^{(k)}\rangle \varphi_s^{(k)} \\ &+ \sum_{s=1}^{m(k)} \sqrt{\nu_s^{(k)}(1 - \nu_s^{(k)})} \langle f, \varphi_s^{(k)}\rangle f_s^{(k)} \\ &+ \sum_{s=1}^{m(k)} \sqrt{\nu_s^{(k)}(1 - \nu_s^{(k)})} \langle f, f_s^{(k)}\rangle \varphi_s^{(k)} \\ = &\ \pi_1^{(k)} + \pi_2^{(k)} + \pi_3^{(k)} + \pi_4^{(k)}. \end{aligned} \tag{4.2}$$

By (4.1), we can fix $(\varphi_1^{(k)}, \ldots, \varphi_{m(k)}^{(k)})$ in such a way that, for any k and $s = 1, \ldots, m(k)$,

$$n(k)\varphi_s^{(k)} \in V(2^{-k} m(k)^{-1}) \tag{4.3}$$

according to (1.5), and

$$\varphi_s^{(k)} \in e[\Lambda_k, \infty)(H) \tag{4.4}$$

where Λ_k is taken large enough to satisfy

$$\sum_{k=1}^{\infty} n(k)^2 m(k) \Lambda_k^{-2} < \infty. \tag{4.5}$$

Since

$$|\langle f,\varphi_s^{(k)}\rangle|^2 \le \langle e[\Lambda_k,\infty)f,f\rangle \le \Lambda_k^{-2}\int_{[\Lambda_k,\infty)}\lambda^2\,\langle e(d\lambda)f,f\rangle \le \Lambda_k^{-2}\|Af\|^2,$$

we get, by (4.5),

$$\sum_{k=1}^{\infty}\|n(k)\pi_3^{(k)}\|^2<\infty. \tag{4.6}$$

Moreover, since $f'\in V(\varepsilon')$, $f''\in V(\varepsilon'')$ imply $f'+f''\in V(\varepsilon'+\varepsilon'')$, condition (4.3) implies $n(k)\pi_2^{(k)}\in V(2^{-k})$ for $k=1,2,\ldots$ Consequently,

$$\sum_{k=1}^{\infty}\mu(n(k)\pi_2^{(k)}>2^{-k})<\sum_{k=1}^{\infty}2^{-k}<\infty$$

and we get

$$n(k)\pi_2^{(k)}\to 0\quad\text{a.s.} \tag{4.7}$$

For the same reason

$$n(k)\pi_4^{(k)}\to 0\quad\text{a.s.} \tag{4.8}$$

By (4.2), (4.6), (4.7) and (4.8), we obtain $n(k)P_k\to A$ a.s. ■

4.4. LEMMA. *For any projections P, Q and $\varepsilon>0$, there exists a partition $P=P'+P''$, $P',P''\in\operatorname{Proj}H$, satisfying the conditions:*

(4.9) *for any $f\in P'(H)$ with $\|f\|=1$, there exists $g\in Q(H)$ with $\|g\|=1$ and $\|f-g\|<2\varepsilon$,*

(4.10) *for any $f\in P''(H)$ with $\|f\|=1$, there exists $g\in Q^{\perp}(H)$ with $\|g\|<1/\varepsilon^2$ and $Pg=f$.*

P r o o f. For some decomposition $H=H_1\oplus H_2\oplus H_3\oplus H_4\oplus H_5$, one can write

$$P=1\oplus1\oplus0\oplus0\oplus P_5,\qquad Q=0\oplus1\oplus1\oplus0\oplus Q_5,$$

and P_5, Q_5 have a generic position in H_5 [3; 9, V, 1*; 7]. This means that $H_5=K\oplus K$,

$$P_5=\begin{bmatrix}c^2 & sc\\ sc & s^2\end{bmatrix},\qquad Q_5=\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

for some operators $c,s\ge0$ in K with $\ker c=\ker s=0$ and $c^2+s^2=1$. It suffices to define

$$P'=0\oplus1\oplus0\oplus0\oplus\begin{bmatrix}Ec^2 & Esc\\ Esc & Es^2\end{bmatrix},$$

$$P''=1\oplus0\oplus0\oplus0\oplus\begin{bmatrix}E^{\perp}c^2 & E^{\perp}sc\\ E^{\perp}sc & E^{\perp}s^2\end{bmatrix}$$

for $E=e((0,\varepsilon))$ defined by the spectral measure of $s=\int_{(0,1)}\lambda\,e(d\lambda)$ in K. Indeed, every vector $f\in P'(H)$ with $\|f\|=1$ can be written in the form

$$f=0\oplus f_2\oplus0\oplus0\oplus\begin{bmatrix}k_1\\k_2\end{bmatrix}$$

for some $k_1,k_2\in K$ satisfying

$$k_1=E(c^2k_1+sck_2)=ck,\qquad k_2=E(sck_1+s^2k_2)=sk,$$

with $k=E(ck_1+sk_2)\in E(K)$. Observe that

$$\|k_1\|^2+\|k_2\|^2=\int_{(0,1)}\|e(d\lambda)k_1\|^2+\int_{(0,1)}\|e(d\lambda)k_2\|^2 =\int_{(0,\varepsilon)}[(1-\lambda^2)+\lambda^2]\,\|e(d\lambda)k\|^2=\|k\|^2.$$

Thus $\|k\|^2=\|f\|^2-\|f_2\|^2\le1$ and

$$\|k_2\|\le\Big\|\int_{(0,\varepsilon)}\lambda\,e(d\lambda)k\Big\|\le\varepsilon,$$

$$\|k-k_1\|=\Big\|\int_{(0,\varepsilon)}(1-\sqrt{1-\lambda^2})\,e(d\lambda)k\Big\|\le\varepsilon,$$

and we have (4.9) for

$$g=0\oplus f_2\oplus0\oplus0\oplus\begin{bmatrix}k\\0\end{bmatrix}.$$

Analogously, every vector $f\in P''(H)$ with $\|f\|=1$ can be written in the form

$$f=f_1\oplus0\oplus0\oplus0\oplus\begin{bmatrix}ck\\sk\end{bmatrix}$$

with $k\in E^{\perp}(K)$. Define $\widetilde{k}=\int_{[\varepsilon,1)}(1/\lambda)\,e(d\lambda)k$. Then $\|\widetilde{k}\|\le\|k\|/\varepsilon$ and, for

$$g=f_1\oplus0\oplus0\oplus0\oplus\begin{bmatrix}0\\\widetilde{k}\end{bmatrix}\in Q^{\perp}(H),$$

we have $f=Pg$ and $\|g\|^2\le\|f_1\|^2+\|k\|^2/\varepsilon^2\le\|f\|^2/\varepsilon^2=1/\varepsilon^2$. Thus (4.10) holds. ■

The following lemma generalizes the idea used in 3.5.

4.5. LEMMA. *For sequences $\{A_n\}$ of bounded operators and $\{\psi_n\}$ of normalized vectors, and for a number $\delta>0$, suppose that*

$$A_n\psi_n\notin V(\delta)$$

for $n=1,2,\ldots$, and let $\nu_n\nearrow\infty$. Then $\nu_nA_nf_0$ does not converge in measure for some f_0 in the closed linear span of $(\psi_1,\psi_2,\ldots)$.

Proof. Obviously, $\nu_{n(k+1)} > 2^{k+1}\nu_{n(k)}\|A_{n(k)}\|$ for some increasing sequence $\{n(k)\}$. Define by induction $f_1 = \psi_1$, and

$$f_{k+1} = \begin{cases} f_k & \text{if } (\nu_{n(k+1)}A_{n(k+1)} - \nu_{n(k)}A_{n(k)})f_k \notin V(\delta/2), \\ f_k + \psi_{n(k+1)}/\nu_{n(k+1)} & \text{otherwise.} \end{cases}$$

For $f_0 = \lim_{k\to\infty} f_k$, we have

$$\|\nu_{n(k)}A_{n(k)}(f_0 - f_k)\| \le \nu_{n(k)}\|A_{n(k)}\| \sum_{l=k+1}^{\infty} 1/\nu_{n(l)} \le 2^{-k},$$
$$\nu_{n(k+1)}A_{n(k+1)}f_{k+1} - \nu_{n(k)}A_{n(k)}f_{n(k)} \notin V(\delta/2)$$

for $k = 1, 2, \ldots$ In consequence, $\{\nu_n A_n f_0\}$ cannot satisfy the Cauchy condition in measure. ■

4.6. Lemma. *Let $\alpha_i\varphi_i \to h \ne 0$ a.s. for weakly converging vectors $\varphi_i \to g$, $\varphi_i, h, g \in H$, $\|\varphi_i\| = 1$, $\alpha_i \in \mathbb{C}$, $i = 1, 2, \ldots$ Then $g = \beta h$ and $1/\alpha_i \to \beta$.*

Proof. Take sets $Z_s \in \mathcal{A}$ such that $\mu(X\backslash Z_s) \to 0$ and $\alpha_i\varphi_i\chi_{Z_s}$ tends to $h\chi_{Z_s}$ uniformly as $i \to \infty$, for $s = 1, 2, \ldots$ For any $\psi \in H$ with $\|\psi\| = 1$, we have

$$\langle \alpha_i\varphi_i, \psi\chi_{Z_s}\rangle \to \langle h, \psi\chi_{Z_s}\rangle \quad \text{as } i \to \infty.$$

Thus $\alpha_i\varphi_i\chi_{Z_s} \to h\chi_{Z_s}$ weakly and $\varphi_i\chi_{Z_s} \to g\chi_{Z_s}$ weakly as $i \to \infty$ and

$$g \ne 0 \quad \text{implies} \quad \alpha_i \to \alpha,\ h = \alpha g \text{ for some } 0 \ne \alpha \in \mathbb{C},$$
$$g = 0 \quad \text{implies} \quad \alpha_i \to \infty.$$

The lemma is proved. ■

4.7. Lemma. *If $\langle f, h\rangle \ne 0$ and $\lambda_k\langle f, \varphi_k\rangle\varphi_k \to h$ in measure for some $\lambda_k \nearrow \infty$, $f, h, \varphi_k \in H$ with $\|\varphi_k\| = 1$ for $k = 1, 2, \ldots$, then $\varphi_k \to 0$ weakly.*

Proof. In any subsequence $\{\varphi_{k(i)}\} \subset \{\varphi_k\}$ we choose a subsequence $\{\varphi_{n(i)}\} \subset \{\varphi_{k(i)}\}$ weakly converging to some $g \in H$ and satisfying $\lambda_{n(i)}\langle f, \varphi_{n(i)}\rangle\varphi_{n(i)} \to h$ a.s. By Lemma 4.6, $g = \beta h$, $(\lambda_{n(i)}\langle f, \varphi_{n(i)}\rangle)^{-1} \to \beta$. If $g \ne 0$, then we would have $\lambda_{n(i)}\langle f, \varphi_{n(i)}\rangle \to 1/\beta \in \mathbb{C}$. On the other hand, $\langle f, \varphi_{n(i)}\rangle \to \langle f, g\rangle = \overline{\beta}\langle f, h\rangle$ and this contradicts $\lambda_k \nearrow \infty$. We have thus proved that $g = 0$, and that $\varphi_n \to 0$ weakly by the arbitrariness of the subsequence $\{\varphi_{k(i)}\}$. ■

4.8. Lemma. *If $\varphi_n \to 0$ weakly, $\varphi_n \in H$, $\varphi_n \notin V(\varepsilon)$ for some $\varepsilon > 0$, $n = 1, 2, \ldots$, then for any sequence $\{\nu_n\} \subset \mathbb{C}$, $\nu_n\varphi_n$ cannot converge in measure to a vector $h \in H$, $h \ne 0$.*

Proof. Assume that $\nu_n\varphi_n \to h$ in measure, $h \ne 0$. For $c = (\varepsilon/(2\|h\|))^{3/2}$, we have $c\nu_n\varphi_n \to ch$ in measure and $ch \in V(\varepsilon/2)$. Thus $c\nu_n\varphi_n \in V(\varepsilon)$ and $|\nu_n| \le 1/c$ for n large enough, by the assumption $\varphi_n \notin V(\varepsilon)$. By the elementary properties of integration, the relations $\nu_n\varphi_n - h \to 0$ in measure and $\nu_n\varphi_n - h \not\to 0$ weakly imply $\|\nu_n\varphi_n - h\| \to \infty$ in H and $|\nu_n| \nearrow \infty$. The contradiction obtained ends the proof. ■

4.9. (iii)⇒(i). Assume that, on the contrary, $0 \ne A \ge 0$, A is not essentially unbounded, $P_n \in \operatorname{Proj} H$, $\lambda_n \nearrow \infty$ and $\lambda_n P_n \to A$ a.s. According to Definition 4.1, there exist $\varepsilon, \lambda > 0$ such that

$$\psi \in Q(H),\ \|\psi\| = 1 \quad \text{imply} \quad \psi \notin V(\varepsilon) \tag{4.11}$$

for

$$Q = e((\lambda, \infty)), \qquad A = \int_{[0,\infty)} \lambda\, e(d\lambda), \tag{4.12}$$

and λ can be chosen so large that $\langle f, h\rangle \ne 0$ where $h = Af$ for some $f \in Q^{\perp}(H)$ with $\|f\| = 1$.

By Lemma 4.4 (with $\varepsilon^{3/2}/2^{5/2}$ instead of ε), for some decomposition $P_n = P'_n + P''_n$, $\varphi_n \in P'_n(H)$, $\|\varphi_n\| = 1$ imply $\|\varphi_n - \psi_n\| < (\varepsilon/2)^{3/2}$ for some $\psi_n \in Q(H)$ with $\|\psi_n\| = 1$; then $\varphi_n - \psi_n \in V(\varepsilon/2)$ and, by (4.11),

$$\varphi_n \in P'_n(H),\ \|\varphi_n\| = 1 \quad \text{imply} \quad \varphi_n \notin V(\varepsilon/2).$$

Moreover, by (4.10),

(4.13) $\varphi_n \in P''_n(H)$, $\|\varphi_n\| = 1$ imply $\varphi_n = P_n g_n$ for some $g_n \in Q^{\perp}(H)$ with $\|g_n\| < 2^5/\varepsilon^3$.

By Lemmas 4.7 and 4.8, $\lambda_n P'_n f$ cannot converge in measure to h such that $\langle f, h\rangle \ne 0$. Thus $\lambda_n P''_n f$ cannot converge to 0 almost surely. Choose $\delta > 0$ and an increasing sequence $\{n(k)\}$ in such a way that

$$\lambda_{n(k)} P''_{n(k)} f \notin V(\delta). \tag{4.14}$$

Writing, for the moment, $\varphi_k = P_{n(k)}f/\|P_{n(k)}f\|$, we have

$$\lambda_{n(k)}\langle f, \varphi_k\rangle\varphi_k = \lambda_{n(k)}P_{n(k)}f \to h$$

in measure, $h = Af$. By Lemma 4.7, $\varphi_k \to 0$ weakly, which means that $P_{n(k)}f \to 0$ weakly. This implies $\|P_{n(k)}f\| \to 0$ and

$$\|P''_{n(k)}f\| \to 0. \tag{4.15}$$

Set now $\varphi_k = P''_{n(k)}f/\|P''_{n(k)}f\|$ and $\psi_k = g_k/\|g_k\|$ where $\varphi_k = P_{n(k)}g_k$ according to (4.13). Condition (4.14) can be written in the form $A_k\psi_k \notin V(\delta)$ for

$$A_k = \lambda_{n(k)}\|P''_{n(k)}f\| \cdot \|g_k\| P_{n(k)}.$$

Moreover, (4.15) and $\|g_k\| < 2^5/\varepsilon^3$ imply $\nu_k = (\|P''_{n(k)}f\| \cdot \|g_k\|)^{-1} \nearrow \infty$. The assumptions of Lemma 4.5 are satisfied, and $\lambda_{n(k)}P_{n(k)}f_0 = \nu_k A_k f_0$ does not converge in measure for some $f_0 \in$ (closed linear span of $\psi_1, \psi_2, \ldots$) $\subset Q^{\perp}(H) \subset D(A)$ by (4.12). ■

Modifying the arguments in Section 4.3 and applying the Naimark dilation method one can obtain almost sure approximation for a sequence of (in general) unbounded, positive selfadjoint operators. We start with the following simple

4.10. Remark. Let $0 \le A \le \mathbf{1}$ be an operator acting in some Hilbert space H_1, and let $P : H_1 \oplus H_2 \to H_1 \oplus H_2$ be a dilation of A, i.e. P is a projection such that $Q_{H_1} P(h_1 \oplus h_2) = Ah_1$ where $Q_{H_1} : H_1 \oplus H_2 \to H_1$ is the canonical projection. If H_3 is isomorphic to H_2 (with isomorphism $\tau : H_2 \to H_3$), then P_0 defined by

$$P_0(h_1 \oplus h_3) = Q_{H_1} P(h_1 \oplus \tau^{-1}(h_3)) + \tau Q_{H_2} P(h_1 \oplus \tau^{-1}(h_3))$$

is also a dilation of A.

Proof. A simple check. ∎

4.11. Theorem. *Let $A^{(1)}, A^{(2)}, \ldots$ be positive selfadjoint unbounded operators in H. Assume that one of them, say $A^{(1)}$, is essentially unbounded. Then there exist a sequence $0 < \lambda_n \nearrow \infty$ and a matrix $\{P_n^{(i)}\}_{i,n=1,2,\ldots}$ of finite-dimensional projections such that, for every n, $P_n^{(1)}, P_n^{(2)}, \ldots$ are mutually orthogonal and $\lambda_n P_n^{(i)} f \to A^{(i)} f$ a.s. for each $f \in D(A^{(i)}) \cap D(A^{(1)})$, $i = 1, \ldots, n$.*

Proof. By Theorem 2.1, there exists a matrix $\{A_n^{(i)}\}_{i,n=1,2,\ldots}$ of positive finite-dimensional operators such that $A_n^{(i)} \to A^{(i)}$ a.s., $i = 1, 2, \ldots$ Let $\lambda_n = \max_{1 \le i \le n} n \|A_n^{(i)}\|$ and put $B_n^{(i)} = \lambda_n^{-1} A_n^{(i)}$, $i = 1, \ldots, n$. Then $B_n^{(i)} \ge 0$ and $\sum_{i=1}^n B_n^{(i)} \le \mathbf{1}$. Moreover, $B_n^{(i)}$ are finite-dimensional. Let $\operatorname{ran} B_n^{(i)} \subset H_n$ for $i = 1, \ldots, n$, where H_n is finite-dimensional. For every n, there exists a dilation of $B_n^{(i)}$, $i = 1, \ldots, n$, that is, there exist mutually orthogonal projections $P_n^{(i)}$, $i = 1, \ldots, n$, acting in $H_n \oplus \widetilde{H}_n$ for some finite-dimensional $\widetilde{H}_n$, such that $Q_n P_n^{(i)} Q_n^* = B_n^{(i)}$, $i = 1, \ldots, n$, where $Q_n : H_n \oplus \widetilde{H}_n \to H_n$ is the canonical projection.

Define $\nu(n) = \dim H_n + \dim \widetilde{H}_n$. By the essential unboundedness of $A^{(1)}$, one can find a subspace $K_n \subset H$ with $\dim K_n = \nu(n)$ satisfying

$$\varphi \in V(2^{-n}) \quad \text{for } \varphi \in K_n,\ \|\varphi\| = 1, \quad K_n \subset e[\lambda_n 2^n, \infty)(H), \tag{4.16}$$

e being the spectral measure of $A^{(1)}$. For the orthogonal projections S, T defined by

$$S(H) = H_n, \quad T(H) = K_n,$$

we have (see [8, 4.4]) $T - T \wedge S^\perp \sim S - S \wedge T^\perp$ where $\sim$ denotes unitary equivalence. In consequence, $\dim(T \wedge S^\perp)(H) \ge \dim T(H) - \dim S(H) = \dim \widetilde{H}_n$. By Remark 4.10, we can assume that $\widetilde{H}_n \subset K_n$. Moreover, $\lambda_n P_n^{(i)} =$

$(Q_n + \widetilde{Q}_n) \lambda_n P_n^{(i)} (Q_n + \widetilde{Q}_n)$ where $\widetilde{Q}_n : H_n \oplus \widetilde{H}_n \to \widetilde{H}_n$ is the canonical projection. Thus we have

$$\lambda_n P_n^{(i)} f = A_n^{(i)} f + \widetilde{Q}_n \lambda_n P_n^{(i)} Q_n f + Q_n \lambda_n P_n^{(i)} \widetilde{Q}_n f + \widetilde{Q}_n \lambda_n P_n^{(i)} \widetilde{Q}_n f, \quad i = 1, \ldots, n.$$

Put additionally $P_n^{(i)} = 0$ for $n < i$. By (4.16), we easily see that, for $f \in H$,

$$\widetilde{Q}_n \lambda_n P_n^{(i)} Q_n f \to 0 \quad \text{a.s., } i = 1, 2, \ldots,$$

and

$$\widetilde{Q}_n \lambda_n P_n^{(i)} \widetilde{Q}_n f \to 0 \quad \text{a.s., } i = 1, 2, \ldots$$

(compare (4.7), (4.8) in Section 4.3). Since $A_n^{(i)} f \to A^{(i)} f$ a.s. for $f \in D(A^{(i)})$, it is enough to show that $Q_n \lambda_n P_n^{(i)} \widetilde{Q}_n f \to 0$ a.s. for $f \in D(A^{(1)})$. Since $\widetilde{Q}_n \le e[\lambda_n 2^n, \infty)$, we have

$$\sum_{n=1}^\infty \|Q_n \lambda_n P_n^{(i)} \widetilde{Q}_n f\|^2 \le \sum_{n=1}^\infty \|\lambda_n \widetilde{Q}_n f\|^2 \le \sum_{n=1}^\infty \lambda_n^2 \int_{[\lambda_n 2^n, \infty)} \|e(d\sigma) f\|^2$$
$$\le \sum_{n=1}^\infty 4^{-n} \int_0^\infty \sigma^2 \|e(d\sigma) f\|^2 = \frac{1}{3} \|A^{(1)} f\|^2 < \infty. \ ∎$$

As an immediate application of the previous theorem we can get the following

4.12. Theorem. *Let T be an unbounded normal operator in H with $|T|$ essentially unbounded. Then there exist $0 < \lambda_n \nearrow \infty$ and four sequences $\{P_n^{(k)}\}$, $k = 1, 2, 3, 4$, of orthogonal projections, mutually orthogonal for any fixed n, such that $\lambda_n (P_n^{(1)} - P_n^{(2)} + i P_n^{(3)} - i P_n^{(4)}) \to T$ a.s.*

5. Weak approximation of unbounded operators. It is worth while completing our reasonings by some comments concerning the following weak convergence for operators acting in any separable Hilbert space H. We say that A_n tends to A weakly, $A_n \to A$ weakly ($A_n \in B(H)$, A unbounded in general), iff $A_n f \to Af$ weakly for any $f \in D(A)$.

5.1. Remark. For $0 < \lambda_n \nearrow \infty$, U_n unitary, P_n orthogonal projections, A densely defined, neither the approximation $\lambda_n U_n \to A$ weakly nor $\lambda_n P_n \to A$ weakly is possible.

Proof. Let $\langle f, g \rangle = \alpha \ne 0$ for some $f \in D(A)$, $g = Af$. Assume that, on the contrary, $\lambda_n P_n f \to g$ weakly. Then $|\langle \lambda_n P_n f, f \rangle| = \lambda_n |\langle f, \varphi_n \rangle|^2 \to |\alpha|$ for $\varphi_n = P_n f / \|P_n f\|$, and $\|\lambda_n P_n f\| = \lambda_n |\langle f, \varphi_n \rangle| \to \lim_{n\to\infty} \sqrt{\lambda_n} \sqrt{\alpha} = \infty$.

Each of the assumptions $\lambda_n U_n \to A$ weakly, $\lambda_n P_n \to A$ weakly ($\lambda_n \nearrow \infty$) contradicts the Banach–Steinhaus principle.

5.2. Remark. Approximations

$$\lambda_n\langle P_n f, g\rangle \to \langle |A| f, g\rangle \quad \text{for } f, g \in D(A), \tag{5.1}$$

$$\lambda_n\langle U_n f, g\rangle \to \langle A f, g\rangle \quad \text{for } f, g \in D(A), \tag{5.2}$$

where $0 < \lambda_n \nearrow \infty$, U_n are unitary, and P_n are orthogonal projections, are always possible for any A densely defined, unbounded and closed.

Proof. We only sketch the proof of (5.1). Let $\sum_{s=1}^{m(k)} \lambda_s^{(k)} \langle \cdot, f_s^{(k)}\rangle f_s^{(k)} \to |A|$ weakly where $(f_1^{(k)}, \ldots, f_{m(k)}^{(k)})$ are orthonormal systems. Putting $\nu_s^{(k)} = \lambda_s^{(k)}/\lambda_k$, $\lambda_k = \max_{1\le s\le m(k)} \lambda_s^{(k)}$ and writing down formula (4.2), it is enough to take $\varphi_s^{(k)} \in e[2^k\lambda_k, \infty)(H)$ where e is the spectral measure of $|A|$.

5.3. Proposition. *For any densely defined, closed and unbounded operator A and $0 < \lambda_n \nearrow \infty$, there exists a sequence $\{V_n\}$ of partial isometries such that $\lambda_n V_n \to A$ weakly.*

Proof. Let $A_n \to A$ weakly for some finite-dimensional operators A_n with polar decomposition

$$A_n = t_n|A_n|, \qquad |A_n| = \sum_{i=1}^{k(n)} \mu_i^{(n)} \langle \cdot, f_i^{(n)}\rangle f_i^{(n)} \tag{5.3}$$

for some orthonormal systems $(f_1^{(n)}, \ldots, f_{k(n)}^{(n)})$. Choose the A_n so that $\|A_n\| < \lambda_n$. Let $(\varphi_1^{(n)}, \ldots, \varphi_{k(n)}^{(n)})$ be an orthonormal system in $e[2^n\lambda_n, \infty)(H)$, orthogonal to $(f_1^{(n)}, \ldots, f_{k(n)}^{(n)})$, where e is the spectral measure of $|A|$. It is enough to put

$$V_n = t_n \sum_{i=1}^{k(n)} \langle \cdot, \psi_i^{(n)}\rangle f_i^{(n)}$$

for $\psi_i^{(n)} = (\mu_i^{(n)}/\lambda_n) f_i^{(n)} + \sqrt{1 - (\mu_i^{(n)}/\lambda_n)^2}\, \varphi_i^{(n)}$. In fact, for $p_n = \sum_{i=1}^{k(n)} \langle \cdot, f_i^{(n)}\rangle f_i^{(n)}$ and $q_n = \sum_{i=1}^{k(n)} \langle \cdot, \varphi_i^{(n)}\rangle \varphi_i^{(n)}$, it is clear that $V_n(p_n + q_n) = V_n$, $V_n p_n = A_n$, and $\sum \|V_n q_n f\| \le \sum \|e[2^n\lambda_n, \infty) f\| < \infty$ for any $f \in D(A)$.

6. Comments and examples. The leitmotive of this section is a clarification of our concept of essentially unbounded operators. It turns out that most of the classical unbounded operators in $L_2(-\infty, \infty)$ are essentially unbounded (Examples 6.1–6.3). What is important, the property of essential unboundedness depends on a given *realization in* L_2 of an operator and is not inherent in the pure geometric structure of the spectral measure. The existence of such realizations can be described precisely (Theorem 6.5). We also give several conditions equivalent to essential unboundedness (Theorem 6.6).

6.1. Example. Let $A = \sum_{n=1}^{\infty} \lambda_n \langle \cdot, \varphi_n\rangle \varphi_n$ with $0 < \lambda_n \nearrow \infty$ and an orthonormal basis $\{\varphi_n\}$. For any $\lambda > 0$,

$$(e[\lambda, \infty))^{\perp} = \sum_{\substack{n=1,2,\ldots \\ \lambda_n < \lambda}} \langle \cdot, \varphi_n\rangle \varphi_n$$

is a finite-dimensional projection and condition (4.1) is a rather immediate consequence of the infinite-dimensionality of the projections $\mathbf{1}_{Z_n}$ with $\mu(Z_n) \to 0$ given by (1.4). Thus A is essentially unbounded.

6.2. Example. The position operator $(Qf)(x) = xf(x)$ acting in $L_2(0, \infty)$ is essentially unbounded because of the form of its spectral measure $e(Z) = \mathbf{1}_Z$.

In this case an approximating sequence $\lambda_n P_n$ can easily be constructed in a direct way. Let γ_n be a linear function satisfying $\gamma_n([0, n]) = [n^2, n^2 + 1/n]$. Then $U_n : f(x) \to nf(\gamma_n^{-1}(x))\chi_{[0,n]}(\gamma_n^{-1}(x))$ is a partial isometry with $U_n^* U_n = \mathbf{1}_{[0,n]}$, $U_n U_n^* = \mathbf{1}_{[n^2, n^2+1/n]}$, $n = 1, 2, \ldots$ Then the operators

$$P_n = \int_{[0,n]} \frac{\lambda}{n} \mathbf{1}_{d\lambda} + \int_{[0,n]} \left(1 - \frac{\lambda}{n}\right) \mathbf{1}_{\gamma(d\lambda)} + \int_{[0,n]} \sqrt{\frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)}\, (U_n \mathbf{1}_{d\lambda} + (U_n \mathbf{1}_{d\lambda})^*)$$

are projections and $nP_n \to 0$ a.s.

Proof. Let $f \in D(Q)$, i.e. $\int_{[0,\infty)} \lambda^2 |f(\lambda)|^2\, d\lambda < \infty$. Then

$$\sum_{n=1}^{\infty} n^2 \Big\| \int_{[0,n]} \sqrt{\frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)}\, (U_n \mathbf{1}_{d\lambda})^* f \Big\|^2 \le \sum_{n=1}^{\infty} n^2 \Big\| \int_{[0,n]} (U_n \mathbf{1}_{d\lambda})^* f \Big\|^2 = \sum_{n=1}^{\infty} n^2 \|\mathbf{1}_{[n^2, n^2+1/n]} f\|^2 < \infty.$$

The rest is obvious. ∎

6.3. Example. Let $P = \int_{(-\infty,\infty)} \lambda\, e(d\lambda)$ be the momentum operator, $P = id/dx$, $i = \sqrt{-1}$, in $L_2(-\infty, \infty)$. Then $|P|$ as well as its positive part $P^+ = \int_{[0,\infty)} \lambda\, e(d\lambda)$ are essentially unbounded.

Proof. Take $\lambda, \varepsilon > 0$ and $f \in V(\varepsilon/2)$ with $\|f\| = 1$. The spectral measure $e(\cdot)$ of the momentum operator is given by the Fourier transform

$$e(Z) = \mathcal{F}\mathbf{1}_Z\mathcal{F}^{-1}, \quad Z \subset (-\infty, \infty),$$

and thus $\|e[\lambda,\infty)(f(x)e^{i\nu x})\| = \|e[\lambda-\nu,\infty)f(x)\|$. For $\nu > 0$ large enough, we have $\|f(x)e^{i\nu x} - f_1(x)\| < (\varepsilon/2)^{3/2}$ for some $f_1 \in e[\lambda,\infty)(H)$ with $\|f_1\| = 1$. Obviously, $f(x)e^{i\nu x} - f_1(x) \in V(\varepsilon/2)$ and condition (4.1) is proved. ■

6.4. EXAMPLE. The operator $A = \sum_{n=1}^{\infty} n\langle\cdot, r_n\rangle r_n$ given by the Rademacher functions $r_n(x)$, $x \in (0,1)$, is not essentially unbounded in $L_2(0,1)$.

Proof. It is enough to prove the existence of a constant $c > 0$ such that

$$\sum_{n=1}^{\infty} \alpha_n r_n \notin V(c) \quad \text{for any } \alpha_n \in \mathbb{C} \text{ with } \sum_{n=1}^{\infty} |\alpha_n|^2 = 1. \tag{6.1}$$

Assume that, on the contrary,

$$s^{(k)} = \sum_{n=1}^{\infty} \alpha_n^{(k)} r_n \to 0 \quad \text{in measure} \tag{6.2}$$

for some coefficients satisfying $\sum_{n=1}^{\infty} |\alpha_n^{(k)}|^2 = 1$, $k = 1,2,\ldots$ Let $|\alpha_{n(k)}^{(k)}| = \max_{n\ge1} |\alpha_n^{(k)}|$ for each $k = 1,2,\ldots$ The functions $\widetilde{s}_k = s_k - \alpha_{n(k)}^{(k)} r_{n(k)}$ are symmetrically distributed. In particular,

$$\mu(\mathrm{Re}(\widetilde{s}_k/\alpha_{n(k)}^{(k)}) \ge 0) = \mu(\mathrm{Re}(\widetilde{s}_k/\alpha_{n(k)}^{(k)}) \le 0)$$

and both sides of the equality are at least $1/2$. Here μ is the Lebesgue measure on $(0,1)$. The random variables $\widetilde{s}_k$ and $r_{n(k)}$ on $((0,1),\mu)$ are independent and

$$(|s_k| > (|\alpha_{n(k)}^{(k)}|) \supset ((r_{n(k)} = 1) \cap (\mathrm{Re}\,\widetilde{s}_k/\alpha_{n(k)}^{(k)} \ge 0))$$
$$\cup ((r_{n(k)} = -1) \cap (\mathrm{Re}\,\widetilde{s}_k/\alpha_{n(k)}^{(k)} \le 0)).$$

This means that $\mu(|s_k| > |\alpha_{n(k)}^{(k)}|) \ge 1/2$. By (6.2), $|\alpha_{n(k)}^{(k)}| = \max_{n\ge1} |\alpha_n^{(k)}| \to 0$ as $n \to \infty$, and the matrix

$$\begin{matrix} \alpha_1^{(1)} r_1, & \alpha_2^{(1)} r_2, & \ldots \\ \alpha_1^{(2)} r_1, & \alpha_2^{(2)} r_2, & \ldots \\ \ldots & \ldots & \ldots \end{matrix}$$

of random variables satisfies the Lindeberg condition [4]. By the Lindeberg–Feller central limit theorem, we have convergence to the standard normal distribution:

$$\mu(s_n < y) \to \Phi(y) \quad \text{as } n \to \infty$$

for any $y \in (-\infty,\infty)$. This contradicts $s_n \to 0$ in measure. ■

In any Hilbert space H, each positive unbounded operator A satisfies one of the following conditions:

(6.3) $A = \sum_{n=1}^{\infty} \lambda_n \langle\cdot,\varphi_n\rangle\varphi_n$ for some $\lambda_n \nearrow \infty$ and an orthonormal basis $\{\varphi_n\}$ in H;

(6.4) for some $\lambda > 0$ the spectral projection $e[0,\lambda)$ of A is infinite-dimensional.

For any unitary operation $U : H \to L_2(X,\mathcal{A},\mu)$, the operator UAU^{-1} will be called a *realization* of A. Conditions (1.1), (1.2) for $(X,\mathcal{A},\mu)$ are always assumed.

Our Example 6.4 leads to the following immediate generalization.

6.5. THEOREM. *Let A be a positive unbounded operator in a separable infinite-dimensional Hilbert space H. If A satisfies* (6.3), *then each realization of A in $L_2(X,\mathcal{A},\mu)$ is essentially unbounded. If A satisfies* (6.4), *then some realizations of A in $L_2(0,1)$ are not essentially unbounded, but essentially unbounded realizations obviously exist.*

Proof. Let Q be an orthogonal projection on the closed linear span of the Rademacher functions. If $A = \int_{[0,\infty)} \lambda\, e(d\lambda)$ satisfies (6.4), then $Q = Ue[\lambda,\infty)U^{-1}$ for some unitary operator $U : H \to L_2(0,1)$ and some $\lambda > 0$. By (6.1), the proof is clear. ■

6.6. THEOREM. *For an operator $A = \int_{[0,\infty)} \lambda\, e(d\lambda)$ with domain $D(A)$, acting in L_2, the following conditions are equivalent*:

(i) *A is essentially unbounded*;

(ii) *for any numbers $\lambda_n \nearrow \infty$, there exist orthogonal projections P_n such that $\lambda_n P_n \to A$ a.s.*;

(iii) *for any numbers $\lambda_n \nearrow \infty$, there exist orthogonal projections P_n such that $\lambda_n P_n \to A$ in measure*;

(iv) *$D(A) \subset D(B)$ for some positive essentially unbounded operator B*;

(v) *for some orthonormal system $\{\varphi_n\}$, we have $\varphi_n \to 0$ a.s. and*

$$\sum_{n=1}^{\infty} |\langle f,\varphi_n\rangle| < \infty \quad \textit{for } f \in D(A).$$

Moreover, the following conditions are also equivalent:

(j) *A is not essentially unbounded*;

(jj) *$Q^\perp(H) \subset D(A)$ and $f \notin V(\varepsilon)$ if only f is normalized in $Q(H)$, for some orthogonal projection Q and $\varepsilon > 0$.*

Proof. Operators $A_n \ge 0$ with $\|A_n\| \le 1$ satisfying $\lambda_n A_n \to A$ a.s. can be found for any $\lambda_n \nearrow \infty$. Thus (i)⇒(ii) (cf. Section 4.3), and (i)⇔(ii)⇔(iii).

Choosing $\varphi_n \in e[2^n,\infty)(H)$ with $\varphi_n \to 0$ a.s. and $\{\varphi_n\}$ orthonormal, one can obtain (i) $\Rightarrow$ (iv) $\wedge$ (v). Moreover, $\neg$(i) (negation of (i)) $\equiv$ (j) $\Rightarrow$ (jj) $\Rightarrow$ $\neg$(iv) $\wedge$ $\neg$(v). ■

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Minimal self-joinings and positive topological entropy II

by

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Abstract. An effective construction of positive-entropy almost one-to-one topological extensions of the Chacón flow is given. These extensions have the property of almost minimal power joinings. For each possible value of entropy there are uncountably many pairwise non-conjugate such extensions.

1. Introduction. In 1979 D. Newton [New] asked whether there exist coalescent dynamical systems with positive metric entropy (a metric dynamical system is said to be coalescent if all its endomorphisms are invertible). This problem has not been solved so far. The analogous problem in topological dynamics had been solved by P. Walters [Wal] in 1974: he gave an example of a topologically coalescent flow with positive topological entropy. His example is not minimal; strictly ergodic topological Toeplitz flows with positive entropy and trivial centralizers were constructed in [BuKw]. Of course, they are topologically coalescent.

It turns out that there exist topological flows with positive entropy and satisfying stronger conditions than having trivial centralizers. In the metric setting D. Rudolph [Rud] introduced the notion of minimal self-joinings, which is much stronger than coalescence and implies zero entropy. In topological dynamics the situation is more complicated. There exist several corresponding notions; the oldest is *graphic* flows [Mar]; later A. del Junco [delJ] introduced a set of possible definitions for topological minimal self-joinings using the orbit closures as an analogue of ergodic measures, among them *almost minimal self-joinings* (AMSJ) and *almost minimal power joinings* (AMPJ); the rather complex definitions are given in Section 2. The classes of flows mentioned above are known to satisfy the following inclusions:

$$\text{coalescent} \supset \text{trivial centralizer} \supset \text{graphic} \supset \text{AMSJ} \supseteq \text{AMPJ}.$$

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