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Entropy numbers of embeddings of Sobolev spaces in Zygmund spaces

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Abstract. Let id be the natural embedding of the Sobolev space $W_p^l(\Omega)$ in the Zygmund space $L_q(\log L)_a(\Omega)$, where $\Omega = (0,1)^n$, $1 < p < \infty$, $l \in \mathbb{N}$, $1/p = 1/q + l/n$ and $a < 0$, $a \neq -l/n$. We consider the entropy numbers $e_k(id)$ of this embedding and show that

$$e_k(id) \asymp k^{-\eta},$$

where $\eta = \min(-a, l/n)$. Extensions to more general spaces are given. The results are applied to give information about the behaviour of the eigenvalues of certain operators of elliptic type.

1. Introduction. Over the years, the idea of the entropy of a set has attracted a great deal of attention, as has the related notion of the entropy numbers of embeddings between function spaces. An early significant result was that of Kolmogorov and Tikhomirov [KT] (see also Vitushkin [V]) concerning the embedding id_1 of $C^l([0,1]^n)$ in $C([0,1]^n)$, where $l \in \mathbb{N}$. In terms of entropy numbers this stated that

$$e_k(id_1) \asymp k^{-l/n},$$

where $e_k(id_1)$ is the kth entropy number of id_1. Vitushkin and Henkin used ideas like this in their work on the superposition of functions related to Hilbert's thirteenth problem: see [VH], and also the interesting article by Lorentz [L], where further references will be found. Then there was the pioneering work of Birman and Solomyak [BiS1], in which they considered the embedding

$$id_2 : W_p^l(\Omega) \to L_q(\Omega),$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $l \in \mathbb{N}$, $1 < p < \infty$, $1 < q < \infty$, $l > n(1/p - 1/q)_+$ and $W_p^l(\Omega)$ is the usual Sobolev space of order l, based on $L_p(\Omega)$. They introduced the (by now) standard method of

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piecewise polynomial approximation and showed that the entropy numbers $e_k(id_2)$ satisfy

$$e_k(id_2) \asymp k^{-l/n}.$$

A remarkable feature of this estimate is that the exponent of k which appears is independent of p and q, although l is constrained by the inequality $l > n(1/p-1/q)_+$ in which p and q do appear. Sharp two-sided estimates of the entropy numbers of embeddings involving a wider range of function spaces (including Besov and Lizorkin–Triebel spaces) are established in [ET], where further historical remarks will be found.

When $l = n(1/p - 1/q)$ the embedding id_2 is continuous but not compact, and it is natural to enquire into its nature by approaching this limiting situation by means of more finely tuned scales of spaces. This was investigated in [ET], using the Zygmund space $L_p(\log L)_a(\Omega)$: we recall that this is simply the set of all those functions f such that

$$\int_\Omega |f(x)|^p \log^{ap}(2+|f(x)|)\,dx < \infty.$$

It is shown in [ET] that if $a < 0$, then the embedding

$$id_3 : W_p^l(\Omega) \to L_q(\log L)_a(\Omega)$$

is compact when $1/p = 1/q + l/n$, and that

$$e_k(id_3) \asymp k^{-l/n} \tag{1}$$

if $a < -2l/n$. Two-sided estimates were also obtained when $-2l/n \le a < 0$, but in this case the upper and lower bounds involved different powers of k and the results could not be claimed to be optimal.

The principal object of the present paper is to find the largest d such that (1) holds when $a < d$. Throughout it is assumed that $\Omega = (0,1)^n$, but this is for convenience and simplicity rather than necessity. We show that this largest d is $-l/n$, and that

$$e_k(id_3) \asymp k^a$$

if $-l/n < a < 0$. These results are obtained by a different method from that of [ET], which largely depended upon accurate estimates of the constants generated by various intermediate embeddings; here we rely more on the construction of ε-nets different from those used earlier. Our technique works equally well if the Sobolev spaces $W_p^l(\Omega)$ are replaced by fractional Sobolev spaces $H_p^s(\Omega)$ or logarithmic Sobolev spaces $H_p^s(\log H)_a(\Omega)$ (see [ET]): when $s = l \in \mathbb{N}$, $H_p^l(\log H)_a(\Omega)$ is just $\{u : D^\alpha u \in L_p(\log L)_a(\Omega) \text{ if } |\alpha| \le l\}$, the Sobolev space of order l based on $L_p(\log L)_a(\Omega)$. It turns out that for the embedding

$$id : H_p^l(\log H)_a(\Omega) \to L_q(\log L)_b(\Omega),$$

where $1 < p < q < \infty$, $-\infty < b < a < \infty$, $l \in \mathbb{N}$ and $a - b \ne l/n = 1/p - 1/q$, we have

$$e_k(id) \asymp k^{-\eta},$$

where $\eta = \min(a-b, l/n)$. This extends the work of [ET] concerned with such embeddings. We formulate and prove analogues of the results described above for embeddings between spaces of Lizorkin–Triebel type.

One of the steps in the proof of these estimates is the derivation of bounds for the entropy numbers of embeddings between finite-dimensional spaces with symmetric bases; see Theorem 4.2. The technique used to do this is well known to specialists, but the result is new, so far as we are aware.

It may also be of interest to remark that the celebrated Cwikel–Lieb–Rozenblum result about the negative spectrum of the Schrödinger operator and its generalisations (see, for example, [BiS2]) may be interpreted in the following way:

Let $s \in \mathbb{N}$, $s < n/2$ and let $V \in L_q(\mathbb{R}^n)$, $V \ge 0$, where $1/q = 2s/n$. Then for some constant $c > 0$,

$$e_m(id : W_2^s(\mathbb{R}^n) \to L_2(V dx)) \le cm^{-s/n}, \quad m \in \mathbb{N}. \tag{2}$$

In this situation our Theorem 3.1, together with an appropriate form of Hölder's inequality, gives a weaker result, namely:

Let $s \in \mathbb{N}$, $s < n/2$ and let $V \in L_q(\log L)_u(\mathbb{R}^n)$, where $1/q = 2s/n$, $u > 2s/n$ and $\operatorname{supp} V \subset (0,1)^n$. Then (2) holds.

This loss of sharpness, in our opinion, is connected with the fact that to prove Theorem 3.1 we construct ε-nets independent of the particular V satisfying

(a) $\operatorname{supp} V \subset (0,1)^n$;
(b) $V \in L_q(\log L)_u(\mathbb{R}^n)$, where $1/q = 2s/n$, $u > 2s/n$;
(c) $\|V|L_q(\log L)_u(\mathbb{R}^n)\| \le c_1$.

These ε-nets depend only on the constants q, c_1, s and n. It thus appears that the loss of sharpness is related to some additional information which we have in our result. This is the possibility to construct ε-nets which work uniformly well for all weights V satisfying conditions (a)–(c). Moreover, it is also an easy consequence of our result that if $u < 2s/n$, then such a uniform estimate is impossible.

Finally, we indicate how these results may be applied to give information about the behaviour of the eigenvalues of certain elliptic operators.

2. Definitions and preliminary results. First, some notation. Given two (quasi-) Banach spaces X and Y, we shall write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. For non-negative expressions (that is, functions or functionals) F_1, F_2 the notation $F_1 \asymp F_2$ is used to mean that $C_1F_1 \le F_2 \le C_2F_1$ for some positive constants C_1, C_2 independent of the variables in the expressions F_1, F_2. By Q we shall always mean the unit cube $(0,1)^n$ in $\mathbb{R}^n$, and we shall consistently write $Q_{i,k} = 2^{-i}(k+Q)$, $k \in \mathbb{Z}$, $i \in \mathbb{Z}$, and $\chi_{i,k} = \chi_{Q_{i,k}}$ (the characteristic function of $Q_{i,k}$). Moreover, given a function ϕ defined on $\mathbb{R}^n$, we shall write

$$\phi_{i,k} = \phi\left(\frac{\cdot - k2^{-i}}{2^{-i}}\right).$$

For any p, $1 \le p < \infty$, $L_p(\mathbb{R}^n)$ will stand for the usual Lebesgue space with respect to Lebesgue measure μ_n and equipped with the norm

$$\|f|L_p(\mathbb{R}^n)\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p\,dx\Big)^{1/p}.$$

Definition 1. Let f be a function defined on $\mathbb{R}^n$. The *non-increasing rearrangement* of f is the function f^* defined on $(0,\infty)$ by

$$f^*(t) = \inf\{\tau > 0 : \mu_n(\{x \in \mathbb{R}^n : |f(x)| > \tau\}) \le t\}, \quad t > 0.$$

The function f^{**} is defined on $(0,\infty)$ by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \quad t > 0.$$

The idea of the rearrangement of a function is very important in connection with the Zygmund space $L_p(\log L)_a(\mathbb{R}^n)$ which we now define.

Definition 2. Let $1 < p < \infty$ and $a \in \mathbb{R}$. Then $L_p(\log L)_a(\mathbb{R}^n)$ is the set of all measurable functions defined on $\mathbb{R}^n$ such that

$$\text{(1)} \qquad \|f|L_p(\log L)_a(\mathbb{R}^n)\| := \Big(\int_0^\infty (f^*(t)(\log(t^{-1}+2))^a)^p\,dt\Big)^{1/p} < \infty.$$

Moreover, $L_\infty(\log L)_a(\mathbb{R}^n)$ is the set of all measurable functions on $\mathbb{R}^n$ such that

$$\|f|L_\infty(\log L)_a(\mathbb{R}^n)\| := \sup_{t>0} f^*(t)(\log(t^{-1}+2))^a < \infty.$$

We shall also need the logarithmic Sobolev space $W^l_p(\log W)_a$ introduced in [ET] and denoted there by $H^l_p(\log H)_a$.

Definition 3. Let $1 < p < \infty$, $l \in \mathbb{N}$, $a \in \mathbb{R}$. Then $W^l_p(\log W)_a(\mathbb{R}^n)$ is the set of all measurable functions defined on $\mathbb{R}^n$ such that for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\{0,1,\ldots,\})^n$ with $|\alpha| = \alpha_1 + \ldots + \alpha_n \le l$, the weak derivative $D^\alpha f$ belongs to the space $L_p(\log L)_a(\mathbb{R}^n)$. This is a Banach space with the norm $\sum_{|\alpha|\le l} \|D^\alpha f|L_p(\log L)_a(\mathbb{R}^n)\|$.

All the spaces introduced so far may be defined on open subsets of $\mathbb{R}^n$ rather than $\mathbb{R}^n$ itself by the standard process of restriction. We shall mainly be concerned with spaces defined on Q.

Remark 1. It can be shown (see for instance [BeS]) that when $1 < p < \infty$, then $f \in L_p(\log L)_a(Q)$ if, and only if,

$$\int_Q |f(x)|^p(\log(|f(x)|+2))^{ap}\,dx < \infty.$$

The expression in (1) is, in general, a quasi-norm but use of a convenient form of the Hardy inequality shows that the analogue of (1) with f^* replaced by f^{**} is a norm on $L_p(\log L)_a(Q)$ equivalent to the quasi-norm (1); with this norm the space is complete. We shall suppose that $L_p(\log L)_a(Q)$ is equipped with the quasi-norm corresponding to (1) and shall regard it as a Banach space.

Remark 2. Other characterisations of $L_p(\log L)_a(Q)$ when $1 < p < \infty$ are given in [ET]. For example, it is shown that if $a < 0$ then the space is just the set of all measurable functions defined on Q such that

$$\text{(2)} \qquad \Big(\sum_{j=J}^\infty 2^{jap}\|f|L_{p(j)}(Q)\|^p\Big)^{1/p} < \infty,$$

where $J \in \mathbb{N}$, $J > p$, $1/p(j) = 1/p + 2^{-j}/n$. Moreover, the quantity in (2) is an equivalent norm on the space. If instead $a > 0$, the space can be characterised as the set of all those measurable functions g defined on Q such that there is a representation

$$\text{(3)} \qquad g = \sum_{j=J}^\infty g_j,$$

$J \in \mathbb{N}$, $J > p$, such that $g_j \in L_{q(j)}(Q)$ with

$$\text{(4)} \qquad \Big(\sum_{j=J}^\infty (2^{ja}\|g_j|L_{q(j)}(Q)\|)^p\Big)^{1/p} < \infty$$

and $1/q(j) = 1/p - 2^{-j}/n$. The infimum of the quantity (4), taken over all admissible representations (3), is an equivalent norm on $L_p(\log L)_a(Q)$.

Next we introduce spaces of Marcinkiewicz and Lorentz type.

Definition 4. Let ψ be a non-negative, increasing, continuous function defined on $[0,\infty)$ and such that

(i) $\psi(0) = 0$,
(ii) $\psi(t) > 0$ if $t > 0$,
(iii) the function ψ_1, $\psi_1(t) = t/\psi(t)$, is increasing.

Denote by $M_\psi(\mathbb{R}^n)$ the set of functions f defined on $\mathbb{R}^n$ such that

$$\|f|M_\psi(\mathbb{R}^n)\| := \sup_{0<t<\infty} f^{**}(t)\psi(t) < \infty,$$

and let $M_\psi(\mathbb{Z})$ stand for the space of all sequences $\{a_j\}_{j\in\mathbb{Z}}$ such that

$$\sum_{j=-\infty}^{\infty} a_j\chi_{[j,j+1]} \in M_\psi(\mathbb{R}).$$

Equipped with the norm $\|\cdot|M_\psi(\mathbb{R}^n)\|$, $M_\psi(\mathbb{R}^n)$ is a Banach space. If we identify $\{a_j\}_{j\in\mathbb{Z}}$ with the function $\sum_{-\infty}^{\infty} a_j\chi_{[j,j+1]}$ we may regard $M_\psi(\mathbb{Z})$ as a subspace of $M_\psi(\mathbb{R})$ and give it the induced norm. We may define $M_\psi(\Gamma)$, when Γ is any finite set, in the same way by identifying Γ with a subset of $\mathbb{Z}$.

DEFINITION 5. Let ψ be a non-negative, continuous, concave function defined on $[0,\infty)$ such that conditions (i) and (ii) from the previous definition hold. Denote by $\Lambda_\psi(\mathbb{R}^n)$ the space of all measurable functions defined on $\mathbb{R}^n$ such that

$$\|f|\Lambda_\psi(\mathbb{R}^n)\| := \int_0^\infty f^*(t)\,d\psi(t) < \infty.$$

Using the formula

$$\|f|\Lambda_\psi(\mathbb{R}^n)\| = -\int_0^\infty f^{**}(t)t\psi''(t)\,dt + b\|f|L_1(\mathbb{R}^n)\|,$$

where $b = \lim_{t\to\infty}\psi'(t)$, it is not hard to check that the space is a Banach space with the norm $\|\cdot|\Lambda_\psi(\mathbb{R}^n)\|$. The spaces $\Lambda_\psi(\mathbb{Z})$ and $\Lambda_\psi(\Gamma)$ may be defined as for the spaces $M_\psi(\mathbb{Z})$ and $M_\psi(\Gamma)$. It is very well known that

$$(\Lambda_\psi(\mathbb{Z}))^* = M_{\psi_1}(\mathbb{Z}),$$

where the function ψ_1 is defined by the relation $\psi_1(t) = t/\psi(t)$. In the case of the spaces $\Lambda_\psi(\Gamma)$ and $M_\psi(\Gamma)$ we also have the relations

$$(\Lambda_\psi(\Gamma))^* = M_{\psi_1}(\Gamma), \quad (M_{\psi_1}(\Gamma))^* = \Lambda_\psi(\Gamma).$$

DEFINITION 6. Let $K>0$. We say a non-negative function ψ defined on $[0,\infty)$ is *K-increasing* if $\psi(s)\le K\psi(t)$ whenever $0\le s\le t$.

LEMMA 1. *Let ψ_1 be a non-negative function defined on $[0,\infty)$ such that*

(i) $\psi_1(0)=0$,

(ii) $\psi_1(t)>0$ *if* $t>0$,

(iii) *the functions ψ_1 and ψ_2, $\psi_2(t)=t/\psi_1(t)$, are K-increasing for some $K>0$.*

Then there are a concave, continuous function ψ defined on $[0,\infty)$ and positive constants c_1, c_2 (independent of ψ_1 and K) such that for all $t>0$,

$$c_1 \le \psi(t)/\psi_1(t) \le c_2K^2.$$

This lemma is well known in the mathematical folklore; we give a proof of it in the Appendix.

REMARK 3. Let ψ and ψ_0 be functions satisfying the conditions of Definition 4 and suppose $c_1\le\psi/\psi_0\le c_2$ for some positive constants c_1 and c_2. Then for every finite set Γ the estimates $c_1 \le \|\cdot|M_\psi(\Gamma)\|/\|\cdot|M_{\psi_0}(\Gamma)\| \le c_2$ hold.

REMARK 4. Let ψ and ψ_0 be functions such that the conditions of Definition 5 are satisfied and $c_1\le\psi/\psi_0\le c_2$ for some positive constants c_1,c_2. Then for every finite set Γ, the estimates $c_1 \le \|\cdot|\Lambda_\psi(\Gamma)\|/\|\cdot|\Lambda_{\psi_0}(\Gamma)\| \le c_2$ hold.

In view of Remarks 3 and 4 and the assertion of Lemma 1 we may and shall suppose in the definition of the spaces Λ_ψ and M_ψ that ψ_1, where $\psi_1(t)=t/\psi(t)$, is K-increasing for some $K>0$; of course, we also suppose that $\psi(0)=0$, that $\psi(t)>0$ if $t>0$ and that ψ is increasing.

LEMMA 2. *Let E_1 be a finite-dimensional Banach space with unconditional basis $\{e_j\}_{j=1}^{\dim E_1}$. Denote the set $\{1,\dots,\dim E_1\}$ by Γ.*

(i) *Let ψ be as in Definition 4. Suppose there are numbers $\varepsilon>0$ and $K>0$ such that the function $t\mapsto t^{1-\varepsilon}/\psi(t)$ is K-increasing, and assume there is a constant $c>0$ such that for every set $\Gamma_1\subset\Gamma$, the following inequality holds:*

$$\Big\|\sum_{j\in\Gamma_1} e_j\Big|E_1\Big\| \ge c\psi(\#\Gamma_1).$$

Then for every sequence $\{a_j\}_{j\in\Gamma}$ of scalars the estimate

$$\Big\|\sum_{j\in\Gamma} a_je_j\Big|E_1\Big\| \ge c_1\|\{a_j\}_{j\in\Gamma}|M_\psi(\Gamma)\|$$

holds, where c_1 (>0) depends on the parameters ε, K and c only.

(ii) *Let ψ be as in Definition 5 and suppose that there is a constant $c>0$ such that for every set $\Gamma_1\subset\Gamma$ the following inequality holds:*

$$\Big\|\sum_{j\in\Gamma_1} e_j\Big|E_1\Big\| \le c\psi(\#\Gamma_1).$$

Then for every sequence $\{a_j\}_{j\in\Gamma}$ the estimate

$$\Big\|\sum_{j\in\Gamma} a_je_j\Big|E_1\Big\| \le c_1\|\{a_j\}_{j\in\Gamma}|\Lambda_\psi(\Gamma)\|$$

holds, where c_1 (>0) depends on the parameters ε, K and c only.

When the basis of E_1 is symmetric, this lemma is well known: see, for example, [BeS]. As we prove a more general result in §4, Lemma 4.1, we shall not give a proof here.

Turning now to entropy numbers, we give the formal definition.

DEFINITION 7. Let X and Y be Banach spaces with X compactly embedded in Y, and let $id : X \to Y$ be the natural embedding. Given any $k \in \mathbb{N}$, the kth *entropy number* of id, written $e_k(id)$, is the infimum of all those $\varepsilon > 0$ such that there are 2^{k-1} balls in Y of radius ε which cover $\{id(x) \in Y : \|x|X\| \le 1\}$. More generally, if K is any compact subset of Y and $k \in \mathbb{N}$, we write

$$e_k(K, Y) = \inf\{\varepsilon > 0 : K \text{ can be covered by } 2^{k-1} \text{ balls in } Y \text{ of radius } \varepsilon\}.$$

For properties of entropy numbers in general we refer to [ET], Chapter 1.

LEMMA 3. *Let $s \in \mathbb{N}$, $m \in \mathbb{N}_0$, $s(j) \in \mathbb{N}$, for $j = 0, \ldots, m$, $s = \sum_{j=0}^{m} s(j)$, and let E, F be Banach spaces of sequences, with $F \hookrightarrow E$, which both have an unconditional basis $\{e_j\}_{j\in\mathbb{N}}$. Suppose that $T_0, T_1, \ldots, T_{m+1}$ are disjoint subsets of $\mathbb{N}$ with union $\mathbb{N}$, let $E(T_j)$, $F(T_j)$ be the subspaces of E, F respectively which are spanned by those e_i with $i \in T_j$, and let $id : F \to E$, $id_j : F_j \to E_j$ be the natural embeddings. Then*

$$e_s(id) \le \sum_{j=0}^{m} e_{s(j)}(id_j) + \|id_{m+1}\|.$$

Proof. This follows in much the same way as the subadditivity of the entropy numbers is proved (see [ET], p. 8, for example).

LEMMA 4. *Let Γ be a finite set, let $m \in \mathbb{N}$ satisfy $m \le \#\Gamma$, and let $\varepsilon > 0$, $0 < \gamma < 1$. Suppose that $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ are continuous and such that*

(i) *$\psi_1(t), \psi_2(t) > 0$ if $t > 0$;*

(ii) *ψ_1 and ψ_2 are increasing and the functions with values at $t > 0$ given by $t/\psi_1(t)$, $t/\psi_2(t)$, $t^{\gamma-\varepsilon}/\psi_2(t)$, $\psi_1(t)/t^{\gamma+\varepsilon}$ are C-increasing, for some constant $C > 0$.*

Let $id : M_{\psi_1}(\Gamma) \to \Lambda_{\psi_2}(\Gamma)$ be the natural embedding. Then there are constants C_1, C_2, depending only on C, γ and ε, such that

$$C_2\psi_2(\max(1, S))/\psi_1(\max(1, S)) \le e_m(id) \le C_1\psi_2(\max(1, S))/\psi_1(\max(1, S)) \le C^2 C_1 \psi_2(S)/\psi_1(S),$$

where

$$S = \frac{m}{\log(1 + \#\Gamma/m)}.$$

Proof. We defer this until §4, after Theorem 4.2 has been proved.

LEMMA 5. *Let $l \in \mathbb{N}$, $1 < p < \infty$, $b \in \mathbb{R}$. Then there is a function $\phi \in C_0^\infty(\mathbb{R}^n)$, with $\int_{\mathbb{R}^n} \phi\, dx = 0$, such that given any $f \in W_p^l(\log W)_b(Q)$ there are numbers $\alpha_{i,k}$, for each $i \in \mathbb{N}_0$ and all $k \in S(i)$, where*

$$S(i) = \{k = (k_1, \ldots, k_n) \in \mathbb{Z}^n : -2^i \le k_j \le 2^{i+1} - 1, \text{ for } j = 1, \ldots, n\},$$

and a function $f_0 \in C_0^\infty(\mathbb{R}^n)$, such that

$$f = \sum_{i=1}^{\infty} \sum_{k\in S(i)} \alpha_{i,k}\, \phi_{i,k} + f_0 \tag{5}$$

and

$$\Big\| \Big(\sum_{i=1}^{\infty} 2^{2il} \Big(\sum_{k\in S(i)} |\alpha_{i,k}|^2 \chi_{i,k} \Big) \Big)^{1/2} \Big| L_p(\log L)_b(Q) \Big\| + \|f_0|C^{l+1}(\overline{Q})\| \le C\|f|W_p^l(\log W)_b(Q)\|. \tag{6}$$

Here C is a constant which depends on ϕ, p and b only.

LEMMA 6. *Let $l \in \mathbb{N}$, $1 < p < \infty$, $b \in \mathbb{R}$, $\phi \in C_0^\infty(\mathbb{R}^n)$, $f_0 \in C^{l+1}(\overline{Q})$ and let $\alpha_{i,k}$, $i \in \mathbb{N}_0$, $k \in S(i)$ be numbers such that the left side of inequality (6) is finite. Then the function f defined by means of formula (5) belongs to the Sobolev space $W_p^l \log W_b(Q)$ and the following estimate holds:*

$$\|f|W_p^l \log W_b(Q)\| \le C\Big\| \Big(\sum_{i=1}^{\infty} 2^{2il} \Big(\sum_{k\in S(i)} |\alpha_{i,k}|^2 \chi_{i,k} \Big) \Big)^{1/2} \Big| L_p(\log L)_b(Q) \Big\| + C\|f_0|C^{l+1}(\overline{Q})\|.$$

Here C is a constant which depends on ϕ, p and b only.

REMARK. Lemma 6 is still valid if we replace the Sobolev space $W_p^l(\log W)_b(Q)$ by the Zygmund space $L_p(\log L)_b(Q)$ and require that $\int_{\mathbb{R}^n} \phi\, dx = 0$. Lemmas 5 and 6 are special cases of Lemma 4.2, and so will not be proved here.

3. The main result.

It is convenient to introduce some new notation at this point. Let $i(1) \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $i(2) \in \mathbb{N}_0 \cup \{\infty\}$, $\widetilde{Q} = [-1, 2]^n$ and define

$$\Gamma(i(1), i(2)) = \{(i, k) : \text{for each } i \in \mathbb{N},\ i(1) \le i \le i(2),\ k \in \mathbb{Z}^n \text{ satisfies } -2^i \le k_j \le 2^{i+1} - 1, \text{ for } j = 1, \ldots, n\}.$$

In the notation of the last section (see Lemma 2.5)

$$\Gamma(i(1), i(2)) = \bigcup_{i(1)\le i\le i(2)} S(i).$$

Definition 1. Given $1 < r < \infty$, $1 \le \theta \le \infty$, $b \in \mathbb{R}$, $i(1) \in \mathbb{N}_0$ and $i(2) \in \mathbb{N} \cup \{\infty\}$, $i(1) \le i(2)$, define $E = E_{r,b,\theta}(i(1), i(2))$ to be the space of sequences of numbers indexed by $\Gamma = \Gamma(i(1), i(2))$ and given by

$$E = \{(\alpha_{i,k})_{(i,k)\in\Gamma} : \|\{\alpha_{i,k}\}|E\| < \infty\},$$

where

$$\|\{\alpha_{i,k}\}|E_{r,b,\theta}\| = \Big\|\Big(\sum_{i=i(1)}^{i(2)}\Big(\sum_{(i,k)\in\Gamma}|\alpha_{i,k}|\chi_{i,k}2^{in/r}\Big)^{\theta}\Big)^{1/\theta}\Big|L_r(\log L)_b(\widetilde{Q})\Big\|.$$

Lemma 1. *Let $\Gamma \subset \Gamma(0,\infty)$, $1 < r < \infty$, $1 \le \theta \le \infty$ and put $e_{i,k} = 2^{in/r}\chi_{i,k}$ for each $(i,k) \in \Gamma$. Then*

(i) $$\Big|\sum_{(i,k)\in\Gamma} e_{i,k}\Big| \le (1-2^{-n/r})^{-1}\sup_{(i,k)\in\Gamma} e_{i,k},$$

(ii) $$\Big\|\Big(\sum_{(i,k)\in\Gamma}(e_{i,k})^{\theta}\Big)^{1/\theta}\Big|L_r(\widetilde{Q})\Big\| \asymp (\#\Gamma)^{1/r}.$$

Proof. (i) Let $x \in \widetilde{Q}$, $\sum_{(i,k)\in\Gamma} e_{i,k}(x) \ne 0$ and put

$$i(x) = \max\{j \in \mathbb{N}_0 : \text{there exists } k \in \mathbb{Z}^n \text{ with } x \in Q_{j,k},\ (j,k) \in \Gamma\}.$$

Then

$$\sum_{(i,k)\in\Gamma} e_{i,k}(x) \le \sum_{i=-\infty}^{i(x)} 2^{in/r} = 2^{i(x)n/r}/(1-2^{-n/r}) = (1-2^{-n/r})^{-1}\sup_{(i,k)\in\Gamma} e_{i,k}(x).$$

(ii) Note that

$$\sup_{(i,k)\in\Gamma} e_{i,k} \le \Big(\sum_{(i,k)\in\Gamma}(e_{i,k})^{\theta}\Big)^{1/\theta} \le \sum_{(i,k)\in\Gamma} e_{i,k}$$

and

$$\sup_{(i,k)\in\Gamma} e_{i,k} \le \Big(\sum_{(i,k)\in\Gamma}(e_{i,k})^{r}\Big)^{1/r} \le \sum_{(i,k)\in\Gamma} e_{i,k}.$$

Moreover,

$$\Big\|\Big(\sum_{(i,k)\in\Gamma}(e_{i,k})^{r}\Big)^{1/r}\Big|L_r(Q)\Big\| = (\#\Gamma)^{1/r}.$$

The result now follows from (i).

Lemma 2. *Let $1 < r < \infty$, $1 \le \theta \le \infty$, $b \in \mathbb{R}$, $i(1) \in \mathbb{N}_0$, $i(2) \in \mathbb{N}_0$. Let $\Gamma = \Gamma(i(1), i(2))$ and $E = E_{r,b,\theta}$. Define functions ψ_1, ψ_2 by*

$$\psi_1(t) = \begin{cases} t^{1/r} & \text{when } 3^n 2^{ni(1)} \le t < \infty,\\ t^{1/r}(\log_2(3^n 2^{ni(1)}t^{-1}+2))^b & \text{when } 0 \le t \le 3^n 2^{ni(1)},\end{cases}$$

$$\psi_2(t) = t^{1/r}(\log_2(2^{ni(2)}t^{-1}+2))^b \quad \text{when } 0 \le t < \infty.$$

Then

(i) $\Lambda_{\psi_2}(\Gamma) \hookrightarrow E \hookrightarrow M_{\psi_1}(\Gamma)$ *if* $b \ge 0$,

(ii) $\Lambda_{\psi_1}(\Gamma) \hookrightarrow E \hookrightarrow M_{\psi_2}(\Gamma)$ *if* $b \le 0$.

Moreover, the upper bounds of the norms of these embeddings may be chosen independent of the parameters $i(1)$ and $i(2)$.

Proof. It is enough to deal with (i), as (ii) then follows from duality. Due to Lemma 2.2, it is enough to prove that for all $\Gamma_0 \subset \Gamma$, the inequalities

$$\Big\|\sum_{(i,k)\in\Gamma_0} e_{i,k}\Big|L_r(\log L)_b(\widetilde{Q})\Big\| \ge c_1\psi_1(\#\Gamma_0),$$

$$\Big\|\sum_{(i,k)\in\Gamma_0} e_{i,k}\Big|L_r(\log L)_b(\widetilde{Q})\Big\| \le c_2\psi_2(\#\Gamma_0)$$

hold for some positive constants c_1 and c_2, where $e_{i,k} = 2^{in/r}\chi_{i,k}$.

Put $g = \sum_{(i,k)\in\Gamma_0} e_{i,k}$. By Lemma 1(ii),

$$\|g|L_r(\widetilde{Q})\| \ge c_3(\#\Gamma_0)^{1/r}$$

for some $c_3 > 0$. Also,

$$\mu_n(\operatorname{supp} g) \le \min(3^n, 2^{-ni(1)}\#\Gamma_0).$$

Hence

$$\begin{aligned}\|g|L_r(\widetilde{Q})\|^r &= \|g^r|L_1(\widetilde{Q})\| \le \|g^r|L_1(\log L)_{br}\|\|\chi_{(0,\mu_n(\operatorname{supp} g))}|L_\infty(\log L)_{-br}\|\\ &\le c_2\|g|L_r(\log L)_b\|\min(1,(1+|\log(2^{-ni(1)}\#\Gamma_0)|^{rb})^{-1}),\end{aligned}$$

the first of the desired inequalities. For the second, put $f = \sup_{(i,k)\in\Gamma_0} e_{i,k}$. By Lemma 1,

$$\|f|L_r(\widetilde{Q})\| \asymp (\#\Gamma_0)^{1/r};$$

also,

$$M := \|f|L_\infty(\widetilde{Q})\| \le 2^{ni(2)/r}.$$

Let d be the Luxemburg norm of f in the Orlicz space $L_r(\log L)_b(\widetilde{Q})$. This norm is equivalent to that with which we earlier endowed the space: see Remark 2.1.

Then

$$1=\int_{\widetilde{Q}}\left(\frac{f(x)}{d}\right)^r\left\{\log\left(2+\frac{|f(x)|}{d}\right)\right\}^{rb}dx$$
$$\le\int_{\widetilde{Q}}\left(\frac{f(x)}{d}\right)^r\left\{\log\left(2+\frac{M}{d}\right)\right\}^{rb}dx$$
$$\le c_4(\#\Gamma_0)d^{-r}\{\log(2+M(\#\Gamma_0)^{-1/r})\}^{rb}$$
$$\le c_5\psi_2(\#\Gamma_0)^r d^{-r}.$$

REMARK 1. If $i(1)=k+2^j$, $i(2)=k+2^{j+1}-1$, $0<t\le 2^{kn}$, then $\psi_1(t)\asymp\psi_2(t)\asymp t^{1/r}\{2^j+\log_2(m/t)\}^b$, where $m=2^{kn}$.

LEMMA 3. *Let $m,k\in\mathbb{N}$, $1<p<q<\infty$, $m=2^{kn}$, $0<\delta\le 1$ and $\delta m\in\mathbb{N}$. Then*

$$e_{m\delta}(E_{p,2,a}(k+2^j,k+2^{j+1}-1))\hookrightarrow E_{q,2,b}(k+2^j,k+2^{j+1}-1))$$
$$\le c(m\delta)^{-d}(2^j-\log_2\delta)^{b+d-a}\le c(m\delta)^{-d}2^{j(b+d-a)}$$

where $d=1/p-1/q$ and the constant c is independent of m, k, j and δ.

Proof. From Lemma 3.2, Remark 1 and Lemma 2.4 it follows that

$$e_{m\delta}(I)\le\frac{c_1}{t^d(2^j+\log_2(m/t))^{a-b}},$$

where

$$t=\frac{m\delta}{\log_2(1+2^{(k+2^j)n}(m\delta)^{-1})}$$

and where I is the natural embedding of $E_{p,2,a}(k+2^j,k+2^{j+1}-1)$ in $E_{q,2,b}(k+2^j,k+2^{j+1}-1)$. Hence

$$e_{m\delta}(I)\le\frac{c_1}{m^d\delta^d(\log_2(1+2^{(k+2^j)n}(m\delta)^{-1}))^{-d}}\cdot\frac{1}{[2^j+\log_2(m/t)]^{a-b}}.$$

But

$$\log_2(1+2^{(k+2^j)n}(m\delta)^{-1})\asymp 2^j-\log\delta$$

and

$$2^j+\log_2(m/t)=2^j+\log_2(\delta^{-1}(\log_2(1+2^{2^jn}/\delta)))$$
$$=2^j-\log_2\delta+\log_2(\log_2(1+2^{2^jn}/\delta))\asymp 2^j-\log\delta.$$

Thus we have the estimate

$$e_{m\delta}(I)\le\frac{c}{(m\delta)^d}(2^j-\log_2\delta)^{b+d-a}.$$

REMARK 2. Using the same calculations it is not hard to see that the following inequality holds for all $j,k\in\mathbb{N}_0$:

(1) $$\|E_{p,2,a}(k+2^j,k+2^{j+1}-1)\hookrightarrow E_{q,2,b}(k+2^j,k+2^{j+1}-1)\|$$
$$\le c_1(k+2^j)^{-a+b}\le c_1 2^{j(b-a)}.$$

From inequality (1) it follows that for all $j,k\in\mathbb{N}_0$, if $a>b$,

(2) $$\|E_{p,2,a}(k+2^j,\infty)\hookrightarrow E_{q,2,b}(k+2^j,\infty)\|\le c2^{j(b-a)},$$

and hence

(3) $$\|E_{p,2,a}(k+2^{kn+1},\infty)\hookrightarrow E_{q,2,b}(k+2^{kn+1},\infty)\|\le c2^{kn(b-a)}.$$

LEMMA 4. *Let $m,k\in\mathbb{N}$, $m=2^{kn}$, $1<p<q<\infty$, $a,b\in\mathbb{R}$, $b-a\ne 1/q-1/p$. Then*

$$e_{mc}(E_{p,2,a}(k+2,\infty)\hookrightarrow E_{q,2,b}(k+2,\infty))\le c_0 m^{\max(b-a,1/q-1/p)},$$

where the constants c,c_0 are independent of the parameter k.

Proof. Thanks to inequality (3) and Lemma 2.3 it is enough to prove the following statement:

There are positive constants c_1 and c_2 such that for every $k\in\mathbb{N}_0$ there are numbers $s(1),\ldots,s(k)$ with

(I) $$\sum_{j=1}^{kn}s(j)\le c_1 2^{kn},$$

(II) $$\sum_{j=1}^{kn}e_{s(j)}(E_{p,2,a}(k+2^j,k+2^{j+1}-1)\hookrightarrow E_{q,2,b}(k+2^j,k+2^{j+1}-1))$$
$$\le c_2 2^{k\max(1/q-1/p,b-a)}.$$

Let integers $s(j)$ be such that

$$\frac{m2^n}{(\min(j,kn-j+1))^2}\le s(j)\le\frac{m2^n}{(\min(j,kn-j+1))^2}+1.$$

Then condition (I) is a trivial consequence of our construction. Condition (II) follows easily from Lemma 3 and the relations

$$\sum_{j=1}^{kn}c(s(j))^{-d}2^{j(b+d-a)}\asymp m^{-d}\sum_{j=1}^{kn}(\min(j,nk-j+1))^{2d}2^{j(b+d-a)}$$
$$\asymp m^{-d}\max(1,2^{k(b+d-a)n})$$
$$=\max(m^{-d},m^{b-a})$$

when $d=1/p-1/q$ and $a-b\ne d$.

We need one more lemma.

LEMMA 5. (i) *Let K_1, K_2 be compact subsets of a Banach space X and let $m \in \mathbb{N}$. Then (see Definition 2.7)*

$$e_{2m}(K_1 + K_2, X) \le e_m(K_1, X) + e_m(K_2, X).$$

(ii) *Let $1 < p < q < \infty$, let $a, b \in (0, \infty)$ and suppose that $s \in \mathbb{N}$ satisfies $s > n(1/p - 1/q)$. Let $M > 0$ and let*

$$K = \{f \in W_p^s(\log W)_a(Q) : \|f|W_p^s(\log W)_a(Q)\| \le M\}.$$

Then there is a constant c, independent of M but depending on n, s, p, q, a and b, such that for all $m \in \mathbb{N}$,

$$e_m(K, L_q(\log L)_b(Q)) \le cMm^{-s/n}.$$

Proof. (i) This follows in the same way as the subadditivity of the entropy numbers is proved (see [ET], Chapter 2).

(ii) This is an easy consequence of the well-known result of Birman and Solomyak [BiS1] concerning the entropy numbers of the embedding of $W_{p_1}^s(Q)$ in $L_{q_1}(Q)$ when $1 < p_1 < q_1 < \infty$ and $s > n(1/p_1 - 1/q_1)$.

Now we can give the main result of the paper.

THEOREM 1. *Let $1 < p < q < \infty$ and suppose that $l = n(1/p - 1/q) \in \mathbb{N}$, let $-\infty < b < a < \infty$ and assume that $a - b \ne 1/p - 1/q$. Let*

$$id : W_p^l(\log W)_a(Q) \to L_q(\log L)_b(Q)$$

be the natural embedding. Then

$$e_m(id) \asymp m^{-\delta},$$

where $\delta = \min(a - b, 1/p - 1/q)$.

Proof. Let f belong to the unit ball of $W_p^l(\log W)_a(Q)$, let $m = 2^{un}$, $u \in \mathbb{N}$, let

$$f = f_0 + \sum_{i=1}^{\infty} \sum_{k \in S(i)} \alpha_{i,k}\phi_{i,k}$$

be the decomposition given by Lemma 2.5, and put

$$f_1 = f_0 + \sum_{i=1}^{u} \sum_{k \in S(i)} \alpha_{i,k}\phi_{i,k}, \quad f_2 = \sum_{i=u+1}^{\infty} \sum_{k \in S(i)} \alpha_{i,k}\phi_{i,k}.$$

Note that f_1 is a smooth function. Now put

$$K_1 = \{f : \|f|W_p^{l+1}(\log W)_a(Q)\| \le m\}$$

and

$$K_2 = \Big\{f \in W_p^l(\log W)_a(Q) : f = \sum_{i=u+1}^{\infty} \sum_{k \in S(i)} \alpha_{i,k}\phi_{i,k} \text{ and}$$
$$\|\{\alpha_{i,k}2^{il - in/p}\}_{i=u+1,\dots,\infty;\, k \in S(i)}|E_{p,2,a}(u+1, \infty)\| \le 1\Big\}.$$

Then f_1 and f_2 belong to suitable multiples of K_1 and K_2 respectively. The upper estimate in the theorem now follows from Lemma 5, Lemma 2.6 and Lemma 4. The lower estimates follow from the inequalities

$$(4) \qquad e_m(W_p^l(\log W)_a(Q) \hookrightarrow L_q(\log L)_b(Q)) \ge c_1 e_m(W_r^l(Q) \hookrightarrow L_r(Q)) \asymp m^{-l/n},$$

where r is any number from the interval (p, q), and

$$e_m(W_p^l(\log W)_a(Q) \hookrightarrow L_q(\log L)_b(Q)) \ge c_2 e_m(E_{p,2,a}(m,m) \hookrightarrow E_{q,2,b}(m,m))$$
$$\ge \|2^{mn/q}\chi_{m,0}|L_q(\log L)_b\|/\|2^{mn/p}\chi_{m,0}|L_p(\log L)_a\| \asymp m^{b-a}.$$

The final relation in (4) is a particular case of the results of [BiS1] (see also [KT], [V]). We refer to [ET] for a different explanation of the lower estimates. When $a = 0$, the lower estimates are contained in Theorem 3.4.1 of [ET]; when $a \ne 0$, they follow from natural adaptations of the proof of that theorem, as indicated in the proof of Theorem 3.4.2 of [ET].

4. More general results.

Let X be a linear space and let $0 < p \le 1$. A map $\|\cdot\| : X \to [0, \infty)$ is called a *p-norm* if it satisfies the usual norm axioms except for the triangle inequality, which is replaced by the following:

$$\|x + y\|^p \le \|x\|^p + \|y\|^p \quad \text{for all } x, y \in X.$$

If $p = 1$, the map is simply a norm.

Cauchy sequences are defined in the obvious way with respect to a p-norm on X, and if every Cauchy sequence in X converges (to a point of X) the pair $(X, \|\cdot\|)$ (or, more loosely, X) is called a *p-quasi-Banach space*.

DEFINITION 1. Let $0 < r \le 1$, $0 < t < \infty$. Given a measurable function f on $\mathbb{R}^n$, denote by $f_r^{**}(t)$ the quantity

$$\Big(t^{-1}\int_0^t (f^*(u))^r\,du\Big)^{1/r}.$$

We remark that when $r = 1$, then $f_r^{**}(t) = f^{**}$ (see Section 2).

DEFINITION 2. Let $0 < r \le p$, $0 < r \le 1$ and let ψ be a non-negative, continuous, increasing function defined on $[0, \infty)$ and such that

(i) $\psi(0) = 0$,

(ii) $\psi(t) > 0$ if $t > 0$,

(iii) there are constants $\varepsilon > 0$ and $C_1 > 0$ such that the function ψ_1 is C_1-increasing, where $\psi_1(t) = t^{1/r-\varepsilon}/\psi(t)$.

Denote by $\Lambda_{\psi,p}(\mathbb{R}^n)$ the space of all measurable functions defined on $\mathbb{R}^n$ such that

$$\|f|\Lambda_{\psi,p}(\mathbb{R}^n)\| = \Big(\int_0^\infty (f_r^{**}(t)\psi(t))^p\,\frac{d\psi(t)}{\psi(t)}\Big)^{1/p} < \infty. \tag{1}$$

REMARK 1. When $p = 1$, the space $\Lambda_{\psi,p}(\mathbb{R}^n)$ is just the Lorentz space $\Lambda_\psi(\mathbb{R}^n)$ introduced in §2. When $p = \infty$ it coincides with the Marcinkiewicz space $M_\psi(\mathbb{R}^n)$ of §2, provided that we assume that there is $\varepsilon > 0$ such that the function ψ_1, $\psi_1(t) = t^{1-\varepsilon}/\psi(t)$, is K-increasing for some constant K. It will be noticed that the definition of $\Lambda_{\psi,p}(\mathbb{R}^n)$ involves a number r. However, given any r_1, $0 < r_1 \le \min(1,p)$, the space $\Lambda_{\psi,p}(\mathbb{R}^n)$ is independent of the particular $r \in (0,r_1)$ used to define it. Moreover, the quantity (1) will be equivalent to the corresponding quantity obtained by replacement of f_r^{**} with $f_{r_1}^{**}$ or f^*. This follows from the inequalities

$$f^* \le f_{r_1}^{**} \le f_{r_2}^{**}, \quad 0 < r_1 \le r_2 \le 1,$$

and the Hardy inequality. Moreover, the expression in (1) is an r-norm on $\Lambda_{\psi,p}(\mathbb{R}^n)$, which becomes an r-quasi-Banach space when endowed with it. The spaces $\Lambda_{\psi,p}(\mathbb{Z}^n)$, $\Lambda_{\psi,p}(\Gamma)$ and $\Lambda_{\psi,p}(Q)$ are defined just as their counterparts in §2 were specified.

Finally, we observe that when ψ is given by

$$\psi(t) = t^{1/p}(\log(2+1/t))^a, \quad t > 0,$$

the space $\Lambda_{\psi,p}$ becomes the Zygmund space $L_p(\log L)_a$.

The next lemma is an analogue of Lemma 2.2 for the case of p-normed spaces.

LEMMA 1. *Let $0 < p \le 1$, let E_p be a finite-dimensional p-quasi-Banach space with unconditional basis $\{e(j)\}_{j=1}^{\dim E_p}$, and let ψ_1 and ψ_2 be non-negative, continuous, increasing functions such that*

(i) $\psi_1(0) = \psi_2(0)$,

(ii) $\psi_1(t) > 0$, $\psi_2(t) > 0$ *when* $t > 0$,

(iii) *there are positive constants ε and K such that the functions ψ_3 and ψ_4 are K-increasing, where*

$$\psi_3(t) = t^{1/p-\varepsilon}/\psi_1(t), \quad \psi_4(t) = t^{1/p-\varepsilon}/\psi_2(t).$$

Suppose also that for every set $\Gamma \subset \{1,2,\ldots,\dim E_p\}$ the following estimate holds:

$$\psi_1(\#\Gamma) \le \Big\|\sum_{j\in\Gamma} e(j)\Big|E_p\Big\| \le \psi_2(\#\Gamma). \tag{2}$$

Then we have the embeddings

$$\Lambda_{\psi_1,\infty} \supset E_p \supset \Lambda_{\psi_2,p}.$$

Moreover, the norms of these embeddings depend only on the constants p,ε and K.

Proof. Let us prove the right-hand embedding. Define a sequence of integers $\{i(s)\}_{s=0}^\infty$ as follows:

$$i(0) = 0, \quad i(1) = 1, \quad i(s+1) = \inf\{j : \psi_2(j) \ge 2\psi_2(i(s))\}, \quad s \in \mathbb{N}.$$

Then from the properties of our function ψ_2 it follows that

$$2 \le \frac{\psi_2(i(s+1))}{\psi_2(i(s))} \le c_1, \quad s \in \mathbb{N}. \tag{3}$$

Denote by $s(0)$ the positive integer such that $i(s(0)) < \dim E_p \le i(s(0)+1)$.

Let $\sum_{i=1}^{\dim E_p} a(i)e(i)$ be an element of our space $\Lambda_{\psi_2,p}$ such that

$$\Big\|\sum_{i=1}^{\dim E_p} a(i)e(i)\Big|\Lambda_{\psi_2,p}\Big\| \le 1.$$

Without loss of generality we can assume that

$$a(1) \ge a(2) \ge \ldots \ge a(\dim E_p) \ge 0.$$

Using inequalities (3) it is not hard to check that

$$\begin{aligned}
\|x|\Lambda_{\psi_2,p}\| &\ge \Big\|\sum_{j=1}^{s(0)}\ \sum_{u=i(j-1)+1}^{\min(\dim E_p,i(j))} a(i(j))e(u)\Big|\Lambda_{\psi_2,p}\Big\| \\
&\ge c_1\Big(\sum_{j=1}^{s(0)}(a(i(j)))^p\psi(i(j))\Big)^{1/p} \\
&\ge c_2\Big(\sum_{j=1}^{s(0)-1}(a(i(j)))^p\psi(i(j+1)) + a(i(s(0)))^p\psi(\dim E_p)\Big)^{1/p} \\
&= c_2\Big(\sum_{j=1}^{s(0)-1}\Big(a(i(j))\Big\|\sum_{u=1}^{i(j+1)} e(u)\Big|\Lambda_{\psi_2,p}\Big\|\Big)^p \\
&\quad + \Big(a(i(s(0)))\Big\|\sum_{u=1}^{\dim E_p} e(u)\Big|\Lambda_{\psi_2,p}\Big\|\Big)^p\Big)^{1/p}.
\end{aligned}$$

Now apply the estimate (2) and after that the p-triangle inequality to obtain the required estimate:

$$\|x|\Lambda_{\psi_2,p}\| \ge c_3\Big(\sum_{j=1}^{s(0)-1}\Big(a(i(j))\Big\|\sum_{u=1}^{i(j+1)} e(u)\Big|E_p\Big\|\Big)^p + \Big(a(i(s(0)))\Big\|\sum_{u=1}^{\dim E_p} e(u)\Big|E_p\Big\|\Big)^p\Big)^{1/p} \ge c_3\|x|E_p\|.$$

The proof of the left-hand embedding is easy. Let

$$x = \sum_{u=1}^{\dim E_p} a(i)e(i) \in E_p.$$

Without loss of generality we assume that $a(1) \ge \ldots \ge a(\dim E_p) \ge 0$. Then

$$\|x|\Lambda_{\psi_1,\infty}\| \le c_1 \sup \psi_1(u)a(u) \le c_2 \sup a(u)\Big\|\sum_{j=1}^{u} e(j)\Big|E_p\Big\| \le c_2\|x|E_p\|,$$

where the suprema are taken over all $u \in \mathbb{N}$ with $u \le \dim E_p$.

DEFINITION 3. Let $a, l \in \mathbb{R}$, $0 < p < \infty$, $0 < \theta \le \infty$. A function $f \in S'(\mathbb{R}^n)$ belongs to the space $F^l_{p,\theta}(\log F)_a(\mathbb{R}^n)$ if and only if there is a representation

$$f = \sum_{i=0}^{\infty} f_i \tag{4}$$

(this representation converges in S') such that

(i) $\operatorname{supp} \mathcal{F} f_0 \subset B(0,2)$,
(ii) $\operatorname{supp} \mathcal{F} f_i \subset B(0,2^{i+1})\backslash B(0,2^{i-1})$,
(iii) the quantity

$$\Big\|\Big(\sum_{i=0}^{\infty}(|f_i|2^{il})^\theta\Big)^{1/\theta}\Big|L_p(\log L)_a(\mathbb{R}^n)\Big\| \tag{5}$$

is finite.

REMARK 2. The infimum of the quantity (5) taken over all representations satisfying conditions (i), (ii) is equivalent to an r-quasi-norm, where r may be taken to have the form $\min(1,p,\theta) - \varepsilon$, where ε is any positive number. Moreover, equipped with this quasi-norm our space becomes a quasi-Banach space.

The space $F^l_{p,\theta}(\log F)_a(Q)$ is defined by restriction. If $g \in S'(\mathbb{R}^n)$, the restriction $g|_Q$ is an element of $\mathcal{D}'(Q)$. We put

$$F^l_{p,\theta}(\log F)_a(Q) = \{f \in \mathcal{D}'(Q) : f = g|_Q \text{ for some } g \in F^l_{p,\theta}(\log F)_a(\mathbb{R}^n)\}$$

and it is quasi-normed by

$$\|f|F^l_{p,\theta}(\log F)_a(Q)\| = \inf \|g|F^l_{p,\theta}(\log F)_a(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in F^l_{p,\theta}(\log F)_a(\mathbb{R}^n)$ such that $g|_Q = f$.

REMARK 3. Special cases of the spaces introduced in Definition 3 coincide with known spaces. Thus $F^l_{p,\theta}(\log F)_0$ is simply the Lizorkin–Triebel space $F^l_{p,\theta}$. Moreover, the space $F^0_{p,2}(\log L)_a$ coincides with the Zygmund space $L_p(\log L)_a$ when $1 < p < \infty$, while $F^l_{p,2}(\log F)_a$ is just $W^l_p(\log W)_a$ when $l \in \mathbb{N}$ and $1 < p < \infty$.

LEMMA 2. *Let* $0 < p < \infty$, $a \in \mathbb{R}$, $0 < \theta \le \infty$, $l \in \mathbb{R}$, $L \in \mathbb{N}_0 \cup \{-1\}$, $L > -l + n/\min(1,p,\theta) - n$, $M \in \mathbb{N}$, $M > l$. *Then the following two assertions hold:*

1) *There is a function* $\varphi \in C_0^\infty(\mathbb{R}^n)$ *such that*

(a) *We have*

$$\int \varphi x^\alpha\,dx = 0 \quad \text{for every } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \le L. \tag{6}$$

(b) *Every function* $f \in F^l_{p,\theta}(\log F)_a(Q)$ *has a representation*

$$f = f_0 + \sum_{i=1}^{\infty}\sum_{k\in S(i)} \alpha_{i,k}\varphi_{i,k} \tag{7}$$

where $\alpha_{i,k} \in \mathbb{R}$, $\varphi_{i,k} = \varphi((\cdot - k2^{-i})/2^{-i})$, $i \in \mathbb{N}$, $k \in S(i)$, $\operatorname{supp} f_0 \subset [-1,2]^n$. *Here* $S(i) = \{k = (k_1,\ldots,k_n) \in \mathbb{Z}^n : -2^i \le k_j \le 2^{i+1} - 1,\ j = 1,\ldots,n\}$. *Moreover,*

$$\|f_0|C^M([-1,2]^n)\| + \Big\|\Big(\sum_{i=1}^{\infty}\Big(\sum_{k\in S(i)}|\alpha_{i,k}|2^{il}\chi_{i,k}\Big)^\theta\Big)^{1/\theta}\Big|L_p(\log L)_a(Q)\Big\| \le c_1\|f|F^l_{p,\theta}(\log F)_a(Q)\|. \tag{8}$$

2) *Let* $\varphi \in C_0^M(\mathbb{R}^n)$ *satisfy condition* (6). *Then every function* f *which has a representation* (7) *such that the quantity on the left-hand side of* (8) *is finite belongs to the space* $F^l_{p,\theta}(\log F)_a(Q)$; *moreover, an estimate reverse to* (8) *holds.*

When $a = 0$, the space $F^l_{p,\theta}(\log F)_a$ is just the usual Lizorkin–Triebel space $F^l_{p,\theta}$ and the result is known: see [FJ], [N1, 2]. When $a \ne 0$, the proof is essentially the same, with L_p replaced by $L_p(\log L)_a$: see [FJ], [N1,2], [T]. Due to Remark 3 above, Lemmas 2.5, 2.6 are special cases of Lemma 2.

Now everything is ready for the generalization of our main result to the case of Lizorkin–Triebel spaces.

THEOREM 1. *Let* $a,b\in\mathbb{R}$, $0<\theta_1,\theta_2\le\infty$, $0<p,q<\infty$, $l_1,l_2\in\mathbb{R}$, $l_1-l_2=n/p-n/q>0$, $a-b>0$, $a-b\neq 1/p-1/q$. *Denote by* id_1 *the embedding*

$$F^{l_1}_{p,\theta_1}(\log F)_a(Q)\hookrightarrow F^{l_2}_{p,\theta_2}(\log F)_b(Q).$$

Then

$$e_m(id_1)\asymp m^{-\min(a-b,1/p-1/q)}.$$

Proof. This follows exactly the same lines as the proof of Theorem 3.1, using Lemma 1 instead of Lemma 2.2.

Before formulation of the next result, some additional notation and definitions are required.

DEFINITION 4. Let E be an m-dimensional r-quasi-Banach space. We say that a basis $\{f_i\}_{i=1}^m$ of E is *symmetric* if, and only if, for any sequence $\{a_i\}_{i=1}^m$ of scalars and any permutation π of $\{1,\dots,m\}$,

$$\Big\|\sum_{i=1}^m a_i f_i\Big|E\Big\|=\Big\|\sum_{i=1}^m a_{\pi(i)} f_i\Big|E\Big\|$$

A typical example of such a space with a symmetric basis is the Lorentz space $\Lambda_{\psi,p}(\Gamma)$ introduced in Definition 2.

From now on in this section we shall suppose that E and F are d-dimensional r-quasi-Banach spaces with a common symmetric basis $\{f_i\}_{i=1}^d$. Denote by $id_{E,F}$ the natural embedding of E in F.

DEFINITION 5. Let $s\in\mathbb{N}$. We write $u(E,F,s)$ for the quantity

$$\sup\Big\|\sum_{i=1}^d\min(\alpha_s,\alpha_i)f_i\Big|F\Big\|,$$

where the supremum is taken over all scalar sequences $\{\alpha_i\}_{i=1}^d$ such that $\alpha_1\ge\dots\ge\alpha_d\ge0$ and $\|\sum_{i=1}^d\alpha_i f_i|E\|\le1$.

THEOREM 2. *Let* $m\in\mathbb{N}$, $m<d/2$, *and let* s *be the unique integer such that*

$$\frac{m}{\log(1+d/m)}<s\le1+\frac{m}{\log(1+d/m)}.$$

Then

$$e_m(id_{E,F})\asymp u(E,F,s),$$

where the constants implicit in the symbol $\asymp$ *depend only on* r.

To help in the proof of this result we need the following lemmas.

LEMMA 3. *Let* m *and* s *be as in Theorem 2. Then:*

(i) $\binom{s}{d}\ge2^{c_1m}$; *moreover, if* $2^m\ge d$ *then* $\binom{s}{d}\le2^{c_2m}$, *where* c_1,c_2 *are positive constants independent of* d,m *and* s.

(ii) *Let* X *be an* r*-quasi-Banach space of dimension* s *with a symmetric basis* $\{f_i\}_{i=1}^s$, *and let* $id:X\to\ell_\infty$ *be the natural embedding. Then*

$$e_s(id)\le c/\Big\|\sum_{i=1}^s f_i\Big|X\Big\|,$$

where the constant c *depends only on* r.

(iii) *There are positive constants* α,γ,c,K *such that given any* $d\in\mathbb{N}$, *any set* Γ *with* $\#\Gamma=d$ *and any* $v\in\mathbb{N}$ *with* $v<d/K$, *there is a set* $S\subset P(\Gamma,v):=\{\Gamma_0\subset\Gamma:\#\Gamma_0=v\}$ *such that*

(a) *if* $U_1,U_2\in S$, $U_1\neq U_2$, *then*

$$(1+\gamma)\#(U_1\cap U_2)<\#(U_1\cup U_2);$$

(b) $\#S\ge c\binom{v}{d}^{\alpha}$.

Proof. (i) Computations using Stirling's formula give this result: see [ET], Chapter 3, §2.

(ii) First, we note that if $\#\Gamma=s$, then

$$e_s(\ell_{p,\infty}(\Gamma)\to\ell_\infty(\Gamma))\asymp s^{-1/p}.$$

This follows from the same arguments as in [ET], p. 100, where in fact only weak estimates are used. For convenience we take $\Gamma=\{1,\dots,s\}$. The result will follow if we can prove that

$$M:=\|X\hookrightarrow\ell_{p,\infty}(\Gamma)\|\le s^{1/p}/\Big\|\sum_{i\in\Gamma}f_i\Big|X\Big\|.$$

To establish this, let $x=\sum_{i=1}^s\alpha_if_i\in X$, $\|x|X\|\le1$, and suppose, without loss of generality, that $\alpha_1\ge\dots\ge\alpha_s\ge0$ and that

$$M=\sup\{\alpha_ii^{1/p}:i=1,\dots,s\}=\|\{\alpha_i\}_{i\in\Gamma}|\Lambda_{p,\infty}(\Gamma)\|.$$

Let $M=\alpha_{i(0)}i(0)^{1/p}$. Then

$$\alpha_1\ge\dots\ge\alpha_{i(0)}=M/i(0)^{1/p}.$$

Thus

$$\Big\|\sum_{i=1}^{i(0)}Mi(0)^{-1/p}f_i\Big|X\Big\|\le1.$$

Assume that $s/i(0) \in \mathbb{N}$. Then dividing Γ into $s/i(0)$ groups each of length $i(0)$, we see that

$$\Big\| \sum_{i=1+ui(0)}^{(u+1)i(0)} Mi(0)^{-1/p} f_i \Big| X \Big\| \le 1, \quad u \in \{0,1,\ldots,s/i(0)-1\},$$

in view of the properties of the symmetric norm, and

$$\Big\| \sum_{i=1}^{i(0)} f_i \Big| X \Big\| \le i(0)^{1/p}/M.$$

Hence

$$\Big\| \sum_{i=1}^{s} f_i \Big| X \Big\| \le (s/i(0))^{1/p} \Big\| \sum_{i=1}^{i(0)} f_i \Big| X \Big\| \le (s/i(0))^{1/p} i(0)^{1/p}/M = s^{1/p}/M,$$

as required. If $s/i(0) \notin \mathbb{N}$ routine modifications of this argument are all that is needed.

(iii) It is easy to see that there are positive constants c_1, c_2 such that for any $d, v \in \mathbb{N}$ with $v \le d$,

$$(c_1 d/v)^v \le \binom{d}{v} \le (c_2 d/v)^v.$$

Also

$$\# P(\Gamma, v) = \binom{d}{v}.$$

Let $\Gamma_0 \in P(\Gamma, v)$. Then if $2v/3 \in \mathbb{N}$,

$$\#\Big\{ \Gamma_1 \in P(\Gamma, v) : \#(\Gamma_1 \cap \Gamma_0) \ge \frac{2v}{3} \Big\} \le \binom{v}{2v/3}\binom{d}{v/3}.$$

From this it follows that there is an S satisfying condition (a) with $\gamma = 2$, and that

$$\# S \ge \binom{d}{v} \Big/ \Big\{ \binom{v}{2v/3}\binom{d}{v/3} \Big\} \ge c^v (d/v)^v/(d/v)^{v/3}.$$

Choosing K sufficiently large, we may take α to be any positive number less than $2/3$. If $2v/3 \notin \mathbb{N}$ we make natural changes to this argument.

LEMMA 4.

(i) *There is a positive constant c, depending only on r, such that for all $s \in \mathbb{N}$,*

$$u(E, F, 2s) \ge c\, u(E, F, s).$$

(ii) *For any positive K and r, there is a number $A \ge 4$ such that for any $s, d \in \mathbb{N}$ with $s \le d/K$, there are integers $m(i)$ and sets $B(i) \subset \{f_1, \ldots, f_d\}$, with $1 \le i \le i(1)$ and $i, i(1) \in \mathbb{N}$, such that*

(a) $\# B(i+1)/\# B(i) \ge 9$, $m(i+1) > m(i)$;
(b) $\# B(1) \ge s$;
(c) $\sum_{i=1}^{i(1)} \# B(i) \le d/K$;
(d) $B(i) \cap B(j) = \emptyset$ *when* $i \ne j$;
(e) *the estimate*

$$\|x|F\|/\|x|E\| \ge c\, u(E, F, s)$$

holds, where

$$x = \sum_{i=1}^{i(1)} A^{-m(i)} \chi_{B(i)}.$$

(iii) *Let $A \in \mathbb{N}$, $A \ge 4$, and for each $i \in \mathbb{N}$ with $1 \le i \le i(1)$ let $n(i) \in \mathbb{N}$ be such that*

$$n(i+1)/n(i) \ge 3, \quad m(i+1) > m(i), \quad \sum_{i=1}^{i(1)} n(i) \le d/K.$$

Then there are positive constants c_1 and c_2, depending upon r and A, such that whenever

$$x = \sum_{i=1}^{i(1)} A^{-m(i)} \chi_{B(i)}, \quad y = \sum_{i=1}^{i(1)} A^{-m(i)} \chi_{D(i)},$$

where $D(i)$, $B(i) \subset \{1, \ldots, d\}$ and $\# D_i = \# B(i)$, then

$$c_1 \le \|x|E\|/\|y|E\| \le c_2.$$

(iv) *Let the conditions of the last statement be satisfied and let $j \in \mathbb{N}$, $1 \le j \le i(1)$, $\varepsilon(1,i), \varepsilon(2,i) \in \{0,1\}$, $i = 1, \ldots, i(1)$. Suppose that $\varepsilon(1,j) \ne \varepsilon(2,j)$. Then*

$$\Big| \sum_{i=1}^{i(1)} A^{-m(i)} (\varepsilon(1,i) - \varepsilon(2,i)) \Big| > \frac{2}{3} A^{-m(j)}.$$

(v) *If*

$$x = \sum_{i=1}^{i(1)} A^{-m(i)} \chi_{B(i)}, \quad y = \sum_{i=1}^{i(1)} A^{-m(i)} \chi_{D(i)},$$

where $A \ge 4$ and $B(i), D(i) \subset \{f_1, \ldots, f_d\}$, $m(i) \in \mathbb{Z}$, with $\# B(i) = \# D(i) = n(i)$, where $n(i+1)/n(i) \ge 9$, $m(i+1) > m(i)$, $1 \le i \le i(1)$ and $\#(B(i) \cap D(i)) \le 2n(i)/(2+\gamma)$, $\gamma > 0$, then

$$\|y - x|F\| \ge c\|x|F\|,$$

where the positive constant c depends only on the constants γ and r.

Proof. (i) Let $\{\alpha(i)\}_{i=1}^{d}$ be non-negative numbers such that

$$\alpha(1) = \ldots = \alpha(s) \ge \alpha(s+1) \ge \ldots \ge \alpha(d) \ge 0$$

and

$$\Big\|\sum_{i=1}^{d}\alpha(i)f_i\Big|F\Big\| = u(E,F,s), \qquad \Big\|\sum_{i=1}^{d}\alpha(i)f_i\Big|E\Big\| = 1.$$

Then

$$\begin{aligned} &u(E,F,2s)\\ &\ge \Big\|\sum_{i=1}^{2s}\alpha(1)f_i + \sum_{i=2s+1}^{d}\alpha(i)f_i\Big|F\Big\| \Big/ \Big\|\sum_{i=1}^{2s}\alpha(1)f_i + \sum_{i=2s+1}^{d}\alpha(i)f_i\Big|E\Big\|\\ &\ge u(E,F,s)/2^r. \end{aligned}$$

(ii) Let $i(2)$ be an integer such that

$$10^{i(2)}s \le d/K, \qquad 10^{i(2)+1}s > d/K,$$

and let $x(i)$, $1 \le i \le d$, $i \in \mathbb{Z}$, be numbers such that

$$x(1) = x(2) = \ldots = x(s) \ge x(s+1) \ge \ldots \ge x(d) \ge 0,$$

$$\Big\|\sum_{i=1}^{d}x_if_i\Big|F\Big\| \ge u(E,F,s)\Big\|\sum_{i=1}^{d}x_if_i\Big|E\Big\|.$$

Denote by $m(1,i)$, $i = 0,\ldots,i(2)$, integers such that

$$A^{-m(1,i)} \ge x(10^is) > A^{-m(1,i)-1}.$$

Let M be the subset of $\mathbb{Z}$ defined by means of the relation $m \in M$ if, and only if, there exists i, $0 \le i \le i(2)$, such that $m = m(1,i)$. Let $M = \{m(i) : 1 \le i \le i(1)\}$, $m(1) < m(2) < \ldots < m(i(1))$, where $i(1) = \#M$.

Put $B(i) = \{i : 1 \le i \le 10^{i(2)}s,\ A^{-m(i)-1} < x(i) \le A^{-m(i)}\}$. All the properties (a)–(e) are trivial consequences of our construction.

(iii) Denote by $x^*(i)$ the non-decreasing rearrangement of the element $x \in E$. For the proof of the statement it is enough to check that the following two inequalities hold:

$$x^*(j) \le \frac{4}{3}z^*(j), \tag{9}$$

$$g\Big(\frac{2}{3}j\Big) \ge z^*(j) \tag{10}$$

for any j, $1 \le j \le d$. Here the sequence $z^*(i)$, $1 \le i \le d$, $i \in \mathbb{Z}$, is defined by means of the relation

$$z^*(j) = \begin{cases} 0 & \text{when } d \ge j > n(1) + \ldots + n(i(1)),\\ A^{-m(i)} & \text{when } n(1) + \ldots + n(i) \ge j > n(1) + \ldots + n(i-1), \end{cases}$$

and the function $g(t)$, $t \in [0,d]$ is defined by means of the relations

$$g(0) = x^*(1), \qquad g(i) = x^*(i),$$

$$\begin{aligned} &g(i+\lambda) = (1-\lambda)g^*(i) + \lambda g^*(i+1),\\ &\qquad\qquad i = 0,1,\ldots,n(1)+\ldots+n(i(1)), \qquad 0 < \lambda < 1,\\ &g(u) = 0, \qquad n(1)+\ldots+n(i(1)) < u \le d. \end{aligned}$$

Let us prove inequality (9). Suppose that i and j are such that

$$n(1) + \ldots + n(i) \ge j > n(1) + \ldots + n(i-1). \tag{11}$$

Then

$$\begin{aligned} x^*(j) &\le \sum_{u=i}^{i(1)} A^{-m(i)} \le \sum_{s=0}^{\infty} A^{-m(i)-s}\\ &\le A^{-m(i)}\big(1 + \tfrac14 + \tfrac1{4^2} + \ldots\big) \le \tfrac43 A^{-m(i)} \le \tfrac43 z^*(j). \end{aligned}$$

Now we prove inequality (10). Suppose that i and j are such that (11) holds. Then

$$\begin{aligned} \tfrac23 j &\le \tfrac23(n(1)+\ldots+n(i))\\ &\le \tfrac23(n(i) + n(i)/3 + n(i)/3^2 + \ldots) < \big(\tfrac23\cdot\tfrac32 n(i)\big) = n(i). \end{aligned}$$

The estimate (10) is a trivial consequence of this inequality and the estimate $x^*(n(i)) \ge A^{-m(i)}$, which follows immediately from the relation $|B(i)| = n(i)$.

(iv) The proof of this is obvious.

(v) Let $u^*(i)$ be the non-decreasing rearrangement connected with the vector $x - y$ and let the sequence $z^*(i)$ be defined in the same way as in the proof of statement (iii). Denote by $g_1(t)$ the function constructed from the sequence $\{u(j)\}_{j=1}^{d}$ in the same way as the function $g(t)$ (from (iii)) was defined by the sequence $\{x^*(i)\}$. Now suppose that i and j are such that inequality (11) holds. Then

$$\begin{aligned} \frac{9}{10}\cdot\frac{2\gamma}{2+\gamma}j &\le \frac{9}{10}\cdot\frac{2\gamma}{2+\gamma}(n(i)+n(i-1)+\ldots+n(1))\\ &< \frac{9}{10}\cdot\frac{2\gamma}{2+\gamma}n(i)\Big(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots\Big) = \frac{9}{10}\cdot\frac{2\gamma}{2+\gamma}\cdot n(i)\cdot\frac{10}{9}\\ &\le \#((B(i)\cup D(i))\setminus(B(i)\cap D(i))). \end{aligned}$$

From (iv) it follows that

$$g_1\Big(\frac{9}{10}\cdot\frac{2\gamma}{2+\gamma}j\Big) \ge \frac23 A^{-m(i)} = \frac23 z^*(j),$$

which leads directly to the required statement.

Proof of Theorem 2. First we prove the estimate of $e_m(id_{E,F})$ from above. By Lemma 4(i), to do this it is enough to show that for some positive constants c_1, c_2 and c_3,

$$e_{c_1 m}(id_{E,F}) \le c_2 u(E,F,c_3 s).$$

When $2^m < d$, this estimate is a consequence of the inequality

$$e_m(id_{E,F}) \le \|id_{E,F}\| = u(E,F,1).$$

We may thus assume that $2^m \ge d$.

Let X be the linear span of $f_1,\dots,f_s$. Using the definition of entropy numbers, choose elements x_l ($l \in L$, $\#L \le 2^m$) such that given any $x \in X$ with $\|x|E\| \le 1$, there exists $x_l \in L$ such that

$$\|x - x_l|\ell_\infty\| \le e_m(X \hookrightarrow \ell_\infty). \tag{12}$$

For any $J \in P(\{1,\dots,d\},s)$, let $\pi(J)$ be a bijection mapping J onto $\{1,\dots$ $\dots,s\}$. We now construct our approximating set K. Let $x_l(i)$ denote the ith coordinate of x_l and take

$$K = \Big\{y(J,l) = \sum_{j\in J} x_l(\pi(J)(j)) f_j : l \in L,\ J \in P(\{1,\dots,d\},s)\Big\}. \tag{13}$$

We check that K has the necessary approximation properties and is not too large. First, using Lemma 3(i) and the definition of L, we see that

$$\#K \le (\#L)\#P(\{1,\dots,d\},s) \le 2^{cm}.$$

As for the approximation properties, take any $z = \sum_{i=1}^n z_i f_i \in E$ with $\|z|E\| \le 1$ and let $J \subset \{1,\dots,d\}$ be any set such that for any $j(1) \in J$ and $j(2) \in \{1,\dots,d\}\setminus J$, the inequality $z_{j(1)} \ge z_{j(2)}$ holds. Put

$$x = \sum_{j\in J} z_j f_{\pi(J)(j)}.$$

Choose l so that (12) holds. We verify that the element $y(J,l)$ defined in (13) approximates z in the desired way. This follows since

$$\begin{aligned}\|z - y(J,l)|F\| &\le c\|P_J(z - y(J,l))|F\| \\ &\quad + c\|P_{\{1,\dots,d\}\setminus J}(z - y(J,l))|F\|,\end{aligned} \tag{14}$$

where we denote by P_Γ, $\Gamma \subset \{1,\dots,d\}$, the natural projection from E to the linear span of $\{f_j\}_{j\in\Gamma}$. From the construction of the set L we see that

$$\|P_J(z - y(J,l))|F\| \le \Big\|\sum_{j\in J} f_j\Big|F\Big\| e_m(X,\ell_\infty)$$

and, by Lemma 3(ii), the right-hand side of this is bounded above by

$$\Big\|\sum_{j\in J} f_j\Big|F\Big\| \Big/ \Big\|\sum_{j\in J} f_j\Big|E\Big\| \le u(E,F,s).$$

Moreover,

$$\|P_{\{1,\dots,d\}\setminus J}(z - y(J,l))|F\| = \|P_{\{1,\dots,d\}\setminus J}\, z|F\| \le u(E,F,s).$$

This establishes the desired estimate from above.

For the lower estimates of the theorem, in view of Lemma 3(i) and Lemma 4(i), it is enough to check that there exist positive constants K,α,c_1, c_2,c_3 such that for any $m \le d/K$, there exists $G \subset E$ with the properties:

(a) $\#G \ge c_1\binom{d}{s}^\alpha$;
(b) for all $x \in G$, $\|x|E\| \le c_2$;
(c) for all $x,y \in G$, $x \ne y$, we have $\|x - y|F\| \ge c_3 u(E,F,s)$.

Take K and α as in Lemma 3(iii). Apply Lemma 4(ii), let A and $B(i)$ be the same as in Lemma 4(ii) and put $\#B(i) = n(i)$, $i(1) \le i \le i(2)$. Apply Lemma 3(iii) with $v = n(i)$ and construct corresponding sets $S(i) \subset P(\{1,\dots,d\},n(i))$. Then

$$\#S(i) \ge c\binom{d}{s}^\alpha, \quad i(1) \le i \le i(2).$$

For any i with $i(1) \le i \le i(2)$ we can find sets $B(j,i) \subset S(i)$, $j = 1,\dots,c_1\binom{d}{s}^\alpha$, with $B(j_1,i) \ne B(j_2,i)$ if $j_1 \ne j_2$, and define

$$G = \Big\{x_j : j = 1,\dots,c\binom{d}{s}^\alpha\Big\}$$

where

$$x_j = b\sum_{i=i(1)}^{i(2)} A^{-m(i)}\chi_{B(j,i)},$$

and b is a constant. On account of Lemma 4(iii) we can choose b in such a way that

$$\|x_j|E\| \le c_5, \quad \|x_j|F\| \ge c_4 u(E,F,s).$$

Properties (a) and (b) follow from our construction; we simply have to check (c). But this follows directly from Lemma 4(v).

Now that Theorem 2 has been proved, we can return to Lemma 2.4.

Proof of Lemma 2.4. In view of Theorem 2, it is sufficient to prove that

$$u(M_{\psi_1}(\Gamma), \Lambda_{\psi_2}(\Gamma), s) \le C_1\psi_2(s)/\psi_1(s),$$

where $\Gamma = \{1,\dots,d\}$ and C_1 depends only on ε. Let $a = (a_i)_{i\in\Gamma}$ be such that $\|a|M_{\psi_1}(\Gamma)\| \le 1$; without loss of generality we may suppose that $a_1 \ge a_2 \ge \dots \ge a_d \ge 0$, so that $a_i \le c/\psi_1(i)$, $i \in \{1,\dots,d\}$. Let $i(0) \in \mathbb{N}$ satisfy $s/2 \le 2^{i(0)} \le s$ and let $i(1) \in \mathbb{N}$ be such that $d/2 < 2^{i(1)} \le d$. Then

$$(15)\quad \|(\min(a_i,a_s))_{i\in\Gamma}|\Lambda_{\psi_2}(\Gamma)\| \le c\|(1/\psi_1(\max(i,s)))_{i\in\Gamma}|\Lambda_{\psi_2}(\Gamma)\| \asymp \sum_{i=i(0)}^{i(1)} \psi_2(2^i)/\psi_1(2^i).$$

From the conditions on ψ_1 and ψ_2, together with the equality

$$t^{2\varepsilon}\cdot\frac{\psi_2(t)}{\psi_1(t)} = \frac{\psi_2(t)}{t^{\gamma-\varepsilon}}\cdot\frac{t^{\gamma+\varepsilon}}{\psi_1(t)},$$

it follows that $t\mapsto t^{2\varepsilon}\psi_2(t)/\psi_1(t)$ is C^2-decreasing and that

$$\frac{\psi_2(2^i)}{\psi_1(2^i)} \le C^2 2^{-2i\varepsilon}\cdot\frac{\psi_2(2^{i(0)})}{\psi_1(2^{i(0)})}.$$

We thus see that the right-hand side of (15) is bounded above by

$$\frac{C^2}{1-2^{-2\varepsilon}}\cdot\frac{\psi_2(2^{i(0)})}{\psi_1(2^{i(0)})},$$

and the required estimate follows.

5. Applications. Just as in [ET], the sharp estimates for the entropy numbers of embeddings which we have obtained can be used to obtain information about the eigenvalues of operators of elliptic type acting in a bounded domain of $\mathbb{R}^n$. To explain the procedure we first extend the notion of entropy numbers given in Definition 2.7 to a more general context. Let X,Y,Z,W be Banach spaces, let $U_X=\{x\in X:\|x|X\|\le 1\}$ and let $T:X\to Y$ be bounded and linear. Given $k\in\mathbb{N}$, the kth *entropy number* of T is

$$e_k(T)=\inf\{\varepsilon>0: T(U_X) \text{ can be covered by } 2^{k-1} \text{ balls in } Y \text{ of radius } \varepsilon\}.$$

It is easy to see that if $S:Z\to X$, $R:Y\to W$ are bounded and linear, then for all $k\in\mathbb{N}$,

$$(1)\quad e_k(R\circ T\circ S)\le\|R\|e_k(T)\|S\|.$$

If T is a compact linear map from X to itself, let $\{\lambda_k(T)\}$ be the sequence of all non-zero eigenvalues of T, repeated according to algebraic multiplicity and ordered by decreasing modulus; here we make the convention that if T has only a finite number of distinct eigenvalues and M is the sum of their algebraic multiplicities, then we put $\lambda_k(T)=0$ for all $k>M$. Carl's inequality (see [ET], Corollary 1.3.4, p. 30) gives a most useful connection between the eigenvalues and entropy numbers of T:

$$(2)\quad |\lambda_k(T)|\le\sqrt{2}\,e_k(T)\quad(k\in\mathbb{N}).$$

The operators B which we shall discuss here are, as in [ET], Chapter 5, of the form

$$(3)\quad Bf=b_2A^{-\kappa}b_1f\quad(0<\kappa\le 1),$$

where $A^{-\kappa}$ is a fractional power of a regular elliptic differential operator of order $2m$, and b_1,b_2 belong to suitable spaces.

THEOREM 1. *Let $r_1,r_2\in(1,\infty]$, $\kappa\in(0,1]$, $m\in\mathbb{N}$, with*

$$(4)\quad 1>\frac{1}{r_1}+\frac{1}{r_2}=\frac{2m\kappa}{n},$$

and let $p\in(1,\infty)$, $a_1,a_2\in\mathbb{R}$ be such that

$$(5)\quad \frac{1}{r_2}<\frac{1}{p}<\frac{1}{r_1'},\qquad a_1+a_2>\frac{2m\kappa}{n}$$

(with $a_1\le 0$ if $r_1=\infty$ and $a_2\le 0$ if $r_2=\infty$). Suppose that

$$(6)\quad b_j\in L_{r_j}(\log L)_{a_j}(Q)\quad(j=1,2).$$

Then B is a compact linear map from $L_p(Q)$ to itself with eigenvalues μ_k (arranged as explained above), and there exists $c>0$ such that

$$|\mu_k|\le c\|b_1|L_{r_1}(\log L)_{a_1}(Q)\|\,\|b_2|L_{r2}(\log L)_{a_2}(Q)\|k^{-2m\kappa/n}\quad(k\in\mathbb{N}).$$

Proof. This is the same as the proof of Theorem 5.3.3/1 of [ET]. The idea is to use the decomposition

$$B=b_2\circ id\circ A^{-\kappa}\circ b_1,$$

where

$$b_1:L_p(Q)\to L_q(\log L)_{a_1}(Q),\quad 1/q=1/p+1/r_1,$$
$$A^{-\kappa}:L_q(\log L)_{a_1}(Q)\to F^{2m\kappa}_{q,2}(\log F)_{a_1}(Q),$$
$$id:F^{2m\kappa}_{q,2}(\log F)_{a_1}(Q)\to L_t(\log L)_{-a_2}(Q),\quad 1/t=1/p-1/r_2,$$
$$b_2:L_t(\log L)_{-a_2}(Q)\to L_p(Q),\quad 1/q-1/t=2m\kappa/n.$$

We now simply use Carl's inequality (2), together with the composition property (1), to obtain

$$|\mu_k|\le\sqrt{2}\|b_1\|\,\|A^{-\kappa}\|\,\|b_2\|e_k(id).$$

From this, plus the mapping properties of $A^{-\kappa}$, b_1,b_2 and the information about $e_k(id)$ given by Theorem 4.1, the theorem follows.

REMARK. This result improves Theorem 5.3.3/1 of [ET], where it was required that $a_1+a_2>4m\kappa/n$, rather than the condition imposed in (5). This improvement arises directly from our weakening of the conditions under which the sharp estimates for $e_k(id)$ hold. In the same way, Theorem 5.3.3/2 of [ET] can be improved: condition (16) on p. 216 of [ET] can be weakened so as to require merely that $\lambda_1+\lambda_2>4m\kappa/n$. Of course, we have Q as the underlying domain rather than the bounded domain with C^∞ boundary of [ET], but as explained in the Introduction, this is not of real significance.

Appendix. As promised, we give here the proof of Lemma 2.1.

Let ψ_0 be the function defined by

$$\psi_0(u) = \sup_{0<t\le u} \psi_1(t), \quad u > 0.$$

Then it is easy to see that ψ_0 has the following properties:

(i) ψ_0 is nonnegative and increasing;

(ii) $\psi_1(t) \le \psi_0(t) \le K\psi_1(t)$ for every $t > 0$;

(iii) the function $t \mapsto t/\psi_0(t)$ is K-increasing.

Put $a_i = \psi_0(2^{i+1}) - \psi_0(2^i)$, $i \in \mathbb{Z}$. Then from conditions (i)–(iii) it follows that

(a) $a_i \ge 0$, $i \in \mathbb{Z}$;

(b) for every $i(1)$ and $i(2)$ in $\mathbb{Z}$, with $i(2) > i(1)$, the inequality

$$\sum_{j=-\infty}^{i(2)} a_j \le K2^{i(2)-i(1)} \Big(\sum_{j=-\infty}^{i(1)} a_j \Big)$$

holds.

Define a sequence $\{b_i\}$ by means of the relation

$$b_i = \sup_{j\ge i} \frac{a_j}{2^{j-i}}, \quad i \in \mathbb{Z}.$$

Then from the definition of our sequence it follows that

$$(1) \qquad b_i \ge \frac{b_{i+1}}{2} \ge 0 \quad \text{for every } i \in \mathbb{Z}.$$

Now let us prove that for every integer $i(0)$ the estimates

$$(2) \qquad \sum_{j=-\infty}^{i(0)} a_j \le \sum_{j=-\infty}^{i(0)} b_j \le 2(K+1) \sum_{j=-\infty}^{i(0)} a_j$$

hold. The left-hand estimate is trivial; let us prove the right-hand inequality. With this aim we show that the inequality

$$(3) \qquad b_i \le \max\Big(\max_{i\le j\le i(0)} \Big(\frac{a_j}{2^{j-i}}\Big), K2^{i-i(0)} \sum_{j=-\infty}^{i(0)} a_j \Big), \quad i \le i(0),\ i \in \mathbb{Z},$$

holds.

From the definition of b_i it follows that

$$b_i \le \max\Big(\max_{i\le j\le i(0)} \frac{a_j}{2^{j-i}}, \max_{i(0)<j} \frac{a_j}{2^{j-i}} \Big), \quad i \le i(0).$$

Now after applying the estimate (b) when $i(2) = j$ and $i(1) = i(0)$, we have (3). Inequality (2) is now a trivial consequence of the estimate

$$b_i \le \sum_{j=i}^{i(0)} \frac{a_j}{2^{j-i}} + \frac{K}{2^{i(0)-i}} \sum_{j=-\infty}^{i(0)} a_j,$$

which itself follows immediately from (3).

Next, define the function ψ in the following way. Put

$$\psi(t) = b_{i+1}(t2^{-i} - 1) + \sum_{j=-\infty}^{i} b_j$$

when $2^i < t \le 2^{i+1}$. Then from inequalities (1) it follows that ψ is concave. From the properties of our functions ψ_0, ψ_1 and ψ we have the following inequalities:

$$\psi(t) \ge \tfrac{1}{2}\psi(2^{-j+1}) \ge \tfrac{1}{2}\psi_0(t) \ge \tfrac{1}{2}\psi_1(t),$$

and

$$\psi(t) \le 2\psi(2^{-j}) \le 4(K+1)\psi_0(2^{-j}) \le 4(K+1)\psi_0(t) \le 4(K+1)K\psi_1(t)$$

where t is any positive real, and the integer j is defined by the relation $2^{-j} \le t \le 2^{-j+1}$.

The proof is now complete.

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