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Derivations into iterated duals of Banach algebras

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Abstract. We introduce two new notions of amenability for a Banach algebra $\mathfrak{A}$. The algebra $\mathfrak{A}$ is n-weakly amenable (for $n \in \mathbb{N}$) if the first continuous cohomology group of $\mathfrak{A}$ with coefficients in the nth dual space $\mathfrak{A}^{(n)}$ is zero; i.e., $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = \{0\}$. Further, $\mathfrak{A}$ is permanently weakly amenable if $\mathfrak{A}$ is n-weakly amenable for each $n \in \mathbb{N}$.

We begin by examining the relations between m-weak amenability and n-weak amenability for distinct $m, n \in \mathbb{N}$. We then examine when Banach algebras in various classes are n-weakly amenable; we study group algebras, C^*-algebras, Banach function algebras, and algebras of operators. Our results are summarized and some open questions are raised in the final section.

1. Introduction. In this paper, we shall be concerned with determining when continuous derivations from a Banach algebra $\mathfrak{A}$ are necessarily inner. We begin by recalling some terminology.

Let $\mathfrak{A}$ be an algebra, and let X be an $\mathfrak{A}$-bimodule. Thus there are bilinear maps $(a, x) \mapsto a \cdot x$ and $(a, x) \mapsto x \cdot a$ from $\mathfrak{A} \times X$ into X such that

$$(ab) \cdot x = a \cdot (b \cdot x), \quad x \cdot (ab) = (x \cdot a) \cdot b, \quad a \cdot (x \cdot b) = (a \cdot x) \cdot b \quad (a, b \in \mathfrak{A},\ x \in X).$$

For example, the algebra $\mathfrak{A}$ is a bimodule over itself, with bimodule operations the product in $\mathfrak{A}$. An $\mathfrak{A}$-bimodule is *symmetric* if

$$a \cdot x = x \cdot a \quad (a \in \mathfrak{A},\ x \in X);$$

a symmetric bimodule over a commutative algebra $\mathfrak{A}$ is called an $\mathfrak{A}$-*module*. In the case where $\mathfrak{A}$ has an identity $e_{\mathfrak{A}}$, the bimodule X is *unital* if

$$e_{\mathfrak{A}} \cdot x = x \cdot e_{\mathfrak{A}} = x \quad (x \in X).$$

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The space of characters on an algebra $\mathfrak{A}$ is denoted by $\Phi_{\mathfrak{A}}$.

Let $\varphi \in \Phi_{\mathfrak{A}} \cup \{0\}$. Then $\mathbb{C}$ is a symmetric $\mathfrak{A}$-bimodule for the products

$$a \cdot z = z \cdot a = \varphi(a)z \quad (a \in \mathfrak{A},\ z \in \mathbb{C});$$

in this case the bimodule is denoted by $\mathbb{C}_\varphi$.

Let $\mathfrak{A}$ be an algebra, and let X be an $\mathfrak{A}$-bimodule. A *derivation* from $\mathfrak{A}$ into X is a linear map D such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathfrak{A}).$$

For example, let $x \in X$, and define

$$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in \mathfrak{A}).$$

Then δ_x is a derivation; maps of this form are called *inner derivations*. A derivation from $\mathfrak{A}$ into the $\mathfrak{A}$-bimodule $\mathbb{C}_\varphi$ is a linear functional d on $\mathfrak{A}$ such that

$$d(ab) = \varphi(a)d(b) + d(a)\varphi(b) \quad (a, b \in \mathfrak{A});$$

such a linear functional is a *point derivation* at φ. Suppose that $\mathfrak{A}$ has an identity $e_{\mathfrak{A}}$ and that X is unital. Then $D(e_{\mathfrak{A}}) = 0$ for each derivation $D : \mathfrak{A} \to X$.

Now suppose that $\mathfrak{A}$ is a Banach algebra and that X is a Banach space which is also an $\mathfrak{A}$-bimodule. Then X is a *Banach $\mathfrak{A}$-bimodule* if the module maps are continuous; in this case, by changing to an equivalent norm on E, we may suppose that

$$\|a \cdot x\| \le \|a\|\,\|x\|, \quad \|x \cdot a\| \le \|a\|\,\|x\| \quad (a \in \mathfrak{A},\ x \in X).$$

For example, if $\mathfrak{B}$ is a Banach algebra containing $\mathfrak{A}$ as a closed subalgebra, then $\mathfrak{B}$ is a Banach $\mathfrak{A}$-bimodule with respect to the products in $\mathfrak{B}$.

Let X be a Banach space with dual space X'; the value of $\lambda \in X'$ at $x \in X$ is denoted by $\langle x, \lambda\rangle$. The second dual of X is X'', and the canonical embedding of X in X'' is denoted by ι or $\widehat{\ }$. We adopt the convention when writing a duality $\langle\cdot,\cdot\rangle$ between Banach spaces that the element on the right is regarded as the functional. In particular,

$$\langle \lambda, \iota(x)\rangle = \langle \lambda, \widehat{x}\rangle = \langle x, \lambda\rangle \quad (x \in X,\ \lambda \in X').$$

When no ambiguity seems possible, we shall omit both ι and $\widehat{\ }$.

We continue to define higher duals by $X^{(1)} = X'$ and $X^{(n+1)} = X^{(n)\prime}$ for $n \in \mathbb{N}$; we also set $X^{(0)} = X$.

The weak-$*$ topology on X' is denoted by $\sigma(X', X)$. We shall frequently use Goldstine's theorem: for each $\Lambda \in X''$, there is a net (x_γ) in X such that $\|x_\gamma\| \le \|\Lambda\|$ and $x_\gamma \to \Lambda$ in $(X'', \sigma(X'', X'))$.

Let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule. Then X' is also a Banach $\mathfrak{A}$-bimodule in a natural way: for $a \in \mathfrak{A}$ and $\lambda \in X'$, we define $a \cdot \lambda$ and $\lambda \cdot a$ in X' by

$$\langle x, a \cdot \lambda\rangle = \langle x \cdot a, \lambda\rangle, \quad \langle x, \lambda \cdot a\rangle = \langle a \cdot x, \lambda\rangle \quad (a \in \mathfrak{A},\ x \in X).$$

Similarly, the higher duals $X^{(n)}$ are Banach $\mathfrak{A}$-bimodules; the definitions are consistent in the sense that $a \cdot \widehat{x} = \widehat{a \cdot x}$, etc., so that $X^{(n)}$ is a submodule of $X^{(n+2)}$ for each $n \in \mathbb{Z}^+$. If X is symmetric, then each $X^{(n)}$ is also symmetric; if X is unital, then so is X'.

We write $\mathcal{B}(E, F)$ for the Banach space of all bounded linear maps from E to F, where E and F are Banach spaces, and we write $\mathcal{B}(E)$ for the Banach algebra $\mathcal{B}(E, E)$. Let $T \in \mathcal{B}(E, F)$. Then the adjoint of T is $T' \in \mathcal{B}(F', E')$. Note that $T'' \in \mathcal{B}(E'', F'')$ is continuous when E'' and F'' both have their weak-$*$ topologies.

Let X be a Banach $\mathfrak{A}$-bimodule, and let $n \in \mathbb{N}$. The adjoint of the injective map $\iota : X^{(n-1)} \to X^{(n+1)}$ is the projection $P : X^{(n+2)} \to X^{(n)}$, defined by $P(\Lambda) = \Lambda | \iota(X^{(n-1)})$. Then P is a morphism of $\mathfrak{A}$-bimodules, and so we may write

$$X^{(n+2)} = X^{(n)} \oplus \ker P = X^{(n)} \oplus \iota(X^{(n-1)})^\perp$$

as Banach $\mathfrak{A}$-bimodules, where, in general, for $F \subseteq Y$, we write

$$F^\perp = \{\lambda \in Y' : \langle x, \lambda\rangle = 0\ (x \in F)\}.$$

Let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule. The space of continuous derivations from $\mathfrak{A}$ into X is denoted by $\mathcal{Z}^1(\mathfrak{A}, X)$, and the space of (necessarily continuous) inner derivations from $\mathfrak{A}$ into X is $N^1(\mathfrak{A}, X)$. The *first* (*topological*) *cohomology group* of $\mathfrak{A}$ with coefficients in X is defined to be the linear space

$$\mathcal{H}^1(\mathfrak{A}, X) = \mathcal{Z}^1(\mathfrak{A}, X)/N^1(\mathfrak{A}, X).$$

Thus $\mathcal{H}^1(\mathfrak{A}, X) = \{0\}$ if and only if each continuous derivation from $\mathfrak{A}$ to X is inner.

There have been very extensive studies devoted to the calculation of the cohomology groups $\mathcal{H}^1(\mathfrak{A}, X)$ and the higher groups $\mathcal{H}^n(\mathfrak{A}, X)$ for various classes of Banach algebras $\mathfrak{A}$ and Banach $\mathfrak{A}$-bimodules X; we shall continue this study here, being particularly concerned with the cohomology groups $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)})$ for $n \in \mathbb{N}$.

Let $\mathfrak{A}$ be a Banach algebra. Then $\mathfrak{A}$ is *amenable* if $\mathcal{H}^1(\mathfrak{A}, X') = \{0\}$ for each Banach $\mathfrak{A}$-bimodule X; this definition was introduced by B. E. Johnson in [20]. The Banach algebra $\mathfrak{A}$ is *weakly amenable* if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}') = \{0\}$; this definition generalizes that introduced by Bade, Curtis, and Dales in [2], where it was noted that a commutative Banach algebra $\mathfrak{A}$ is weakly amenable if and only if $\mathcal{H}^1(\mathfrak{A}, X) = \{0\}$ for each symmetric Banach $\mathfrak{A}$-module X.

For example, it was shown in [20] that the group algebra $L^1(G)$ is amenable if and only if G is an amenable group, and in [21] that $L^1(G)$

is weakly amenable for each locally compact group G; for a shorter proof of this latter result, see [7].

The following definition describes the main new property that we shall study.

DEFINITION 1.1. Let $\mathfrak{A}$ be a Banach algebra, and let $n \in \mathbb{N}$. Then $\mathfrak{A}$ is *n-weakly amenable* if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = \{0\}$; $\mathfrak{A}$ is *permanently weakly amenable* if $\mathfrak{A}$ is n-weakly amenable for each $n \in \mathbb{N}$.

Our purpose in this paper is to determine the relations between m- and n-weak amenability for general Banach algebras $\mathfrak{A}$ and for Banach algebras in various classes, and also to determine when Banach algebras in various classes are n-weakly amenable. We are most interested in the relation between weak amenability, 2-weak amenability, and 3-weak amenability. We shall give some general results in this first section, and then consider various special classes of Banach algebras in subsequent sections. We shall also obtain some new information about the second duals of various Banach algebras. Some results and open questions are summarized at the end of the paper.

We begin with the following trivial observations: (i) an amenable Banach algebra is permanently weakly amenable; (ii) a commutative Banach algebra is permanently weakly amenable if and only if it is weakly amenable.

There is also one easy remark about the relations between m- and n-weak amenability.

PROPOSITION 1.2. *Let $\mathfrak{A}$ be a Banach algebra, and let $n \in \mathbb{N}$. Suppose that $\mathfrak{A}$ is $(n+2)$-weakly amenable. Then $\mathfrak{A}$ is n-weakly amenable.*

Proof. Let $D \in \mathcal{Z}^1(\mathfrak{A}, \mathfrak{A}^{(n)})$. Then D can be viewed as an element of $\mathcal{Z}^1(\mathfrak{A}, \mathfrak{A}^{(n+2)})$, and so there exists $\Phi \in \mathfrak{A}^{(n+2)}$ with

$$D(a) = a \cdot \Phi - \Phi \cdot a \quad (a \in \mathfrak{A}).$$

Set $\Lambda = P(\Phi)$, where the projection $P : \mathfrak{A}^{(n+2)} \to \mathfrak{A}^{(n)}$ was described earlier. Then

$$D(a) = P\widehat{D(a)} = a \cdot \Lambda - \Lambda \cdot a \quad (a \in \mathfrak{A}),$$

and so D is an inner derivation. ∎

We remark that a 2-weakly amenable Banach algebra $\mathfrak{A}$ does not necessarily have the property that $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}) = \{0\}$. For take G to be an infinite, compact, non-abelian group. Then $L^1(G)$ is amenable, and hence 2-weakly amenable. However, take an element $s \in G$ which is not in the centre of G, and define

$$D : f \mapsto f \star \delta_s - \delta_s \star f, \quad L^1(G) \to L^1(G);$$

here $(\delta_s \star f)(t) = f(s^{-1}t)$ and $(f \star \delta_s)(t) = f(ts^{-1})$ for $t \in G$, noting that G is unimodular. Then D is a continuous derivation, but it is easy to see that D is not inner, and so $\mathcal{H}^1(L^1(G), L^1(G)) \neq \{0\}$.

Let $\mathfrak{A}$ be a commutative Banach algebra, let $n \in \mathbb{Z}^+$, and suppose that $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n+2)}) = \{0\}$. Take $D \in \mathcal{Z}^1(\mathfrak{A}, \mathfrak{A}^{(n)})$. Then again D can be viewed as an element of $\mathcal{Z}^1(\mathfrak{A}, \mathfrak{A}^{(n+2)})$. Since $\mathfrak{A}^{(n+2)}$ is a symmetric $\mathfrak{A}$-bimodule and since $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n+2)}) = \{0\}$, it follows that $D = 0$. Thus, in the case where $\mathfrak{A}$ is $(2n+1)$-weakly amenable for some $n \in \mathbb{Z}^+$, $\mathfrak{A}$ is permanently weakly amenable. Also, if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(2n)}) = \{0\}$ for some $n \in \mathbb{N}$, then $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}) = \{0\}$. It follows that, if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}) \neq \{0\}$, then $\mathfrak{A}$ is not n-weakly amenable for any $n \in \mathbb{N}$.

The converse to Proposition 1.2 is open. In particular, we do not know if weak amenability always implies 3-weak amenability or if 2-weak amenability always implies 4-weak amenability; these results are true in many special cases.

A further easy remark, to be used later, is the following. For an algebra $\mathfrak{A}$ and $n \in \mathbb{N}$, we write

$$\mathfrak{A}^{[n]} = \{a_1 \ldots a_n : a_1, \ldots, a_n \in \mathfrak{A}\}, \quad \mathfrak{A}^n = \operatorname{lin} \mathfrak{A}^{[n]}.$$

PROPOSITION 1.3. *Let $\mathfrak{A}$ be a weakly amenable Banach algebra. Then:*

(i) *$\mathfrak{A}^2$ is dense in $\mathfrak{A}$;*

(ii) *there are no non-zero, continuous point derivations on $\mathfrak{A}$.*

Proof. (i) Assume towards a contradiction that $\mathfrak{A}^2$ is not dense in $\mathfrak{A}$, and take $\lambda_0 \in \mathfrak{A}'$ with $\lambda_0 | \mathfrak{A}^2 = 0$ and $\lambda_0 \neq 0$. Then it is easily checked that the map

$$D : a \mapsto \langle a, \lambda_0 \rangle \lambda_0, \quad \mathfrak{A} \to \mathfrak{A}',$$

is a continuous derivation that is not inner, a contradiction of the fact that $\mathfrak{A}$ is weakly amenable.

(ii) By (i), there are no continuous point derivations at $\varphi = 0$.

Assume towards a contradiction that d is a non-zero, continuous point derivation at $\varphi \in \Phi_{\mathfrak{A}}$. Then the map

$$D : a \mapsto d(a)\varphi, \quad \mathfrak{A} \to \mathfrak{A}',$$

is certainly a continuous linear operator, and it is immediately checked that D is a derivation.

Since $\mathfrak{A}$ is weakly amenable, there exists $\lambda \in \mathfrak{A}'$ with

$$D(a) = a \cdot \lambda - \lambda \cdot a \quad (a \in \mathfrak{A}).$$

Take $a_1 \in \mathfrak{A}$ with $\varphi(a_1) = 1$, take $a_2 \in \ker \varphi$ with $d(a_2) = 1$, and set $a_0 = a_1 + (1 - d(a_1))a_2$. Then $\varphi(a_0) = d(a_0) = 1$, and so

$$1 = \langle a_0, Da_0 \rangle = \langle a_0, a_0 \cdot \lambda \rangle - \langle a_0, \lambda \cdot a_0 \rangle = 0,$$

a contradiction. ∎

Let $\mathfrak{A}$ be a non-unital algebra. We denote by $\mathfrak{A}^{\#}$ the algebra formed by adjoining an identity to $\mathfrak{A}$, so that $\mathfrak{A}^{\#} = \mathbb{C}e \oplus \mathfrak{A}$, with the product

$$(\alpha, a)(\beta, b) = (\alpha\beta, \alpha b + \beta a + ab) \quad (\alpha, \beta \in \mathbb{C},\ a, b \in \mathfrak{A}).$$

In the case where $\mathfrak{A}$ is a Banach algebra, $\mathfrak{A}^{\#}$ is also a Banach algebra. Define $e' \in \mathfrak{A}^{\#'}$ by requiring that $\langle e, e'\rangle = 1$ and $e'|\mathfrak{A} = 0$. Then we have the identifications

$$\mathfrak{A}^{\#(2n)} = \mathbb{C}e \oplus \mathfrak{A}^{(2n)} \qquad (n \in \mathbb{N}),$$
$$\mathfrak{A}^{\#(2n+1)} = \mathbb{C}e' \oplus \mathfrak{A}^{(2n+1)} \qquad (n \in \mathbb{Z}^+).$$

The module operations of $\mathfrak{A}^{\#}$ on $\mathfrak{A}^{\#(2n+1)}$ are given by

$$(\alpha e + a) \cdot (\gamma e' + \lambda) = (\alpha\gamma + \langle a, \lambda\rangle)e' + \alpha\lambda + a \cdot \lambda,$$
$$(\gamma e' + \lambda) \cdot (\alpha e + a) = (\alpha\gamma + \langle a, \lambda\rangle)e' + \alpha\lambda + \lambda \cdot a.$$

Note that $\mathfrak{A}^{(2n+1)}$ is *not* a submodule of $\mathfrak{A}^{\#(2n+1)}$. However, $\mathfrak{A}^{(2n)}$ is a submodule of $\mathfrak{A}^{\#(2n)}$.

PROPOSITION 1.4. *Let $\mathfrak{A}$ be a non-unital Banach algebra, and let $n \in \mathbb{N}$.*

(i) *Suppose that $\mathfrak{A}^{\#}$ is $2n$-weakly amenable. Then $\mathfrak{A}$ is $2n$-weakly amenable.*

(ii) *Suppose that $\mathfrak{A}$ is $(2n-1)$-weakly amenable. Then $\mathfrak{A}^{\#}$ is $(2n-1)$-weakly amenable.*

(iii) *Suppose that $\mathfrak{A}$ is commutative. Then $\mathfrak{A}^{\#}$ is n-weakly amenable if and only if $\mathfrak{A}$ is n-weakly amenable.*

Proof. (i) This is immediate.

(ii) Let

$$D : a \mapsto \langle a, \lambda\rangle e' + \widetilde{D}(a), \quad \mathfrak{A} \to \mathbb{C}e' \oplus \mathfrak{A}^{(2n-1)},$$

be a continuous derivation. We see that $\widetilde{D} : \mathfrak{A} \to \mathfrak{A}^{(2n-1)}$ is a continuous derivation, and so there exists $\Lambda \in \mathfrak{A}^{(2n-1)}$ such that

$$\widetilde{D}(a) = a \cdot \Lambda - \Lambda \cdot a \quad (a \in \mathfrak{A}).$$

Let $a, b \in \mathfrak{A}$. Then we have

$$\langle ab, \lambda\rangle = \langle a, \widetilde{D}(b)\rangle + \langle b, \widetilde{D}(a)\rangle = \langle a, b \cdot \Lambda - \Lambda \cdot b\rangle + \langle b, a \cdot \Lambda - \Lambda \cdot a\rangle = 0,$$

and so $\lambda|\mathfrak{A}^2 = 0$. By Proposition 1.2, $\mathfrak{A}$ is weakly amenable, and hence, by Proposition 1.3(i), $\mathfrak{A}^2$ is dense in $\mathfrak{A}$. It follows that $\lambda = 0$, and so $D = \widetilde{D}$ is an inner derivation.

(iii) Suppose that $\mathfrak{A}^{\#}$ is $2k$-weakly amenable. Then $\mathfrak{A}$ is $2k$-weakly amenable by (i).

Suppose that $\mathfrak{A}^{\#}$ is $(2k-1)$-weakly amenable. Then $\mathfrak{A}^{\#}$ is weakly amenable, and so $\mathfrak{A}^2$ is dense in $\mathfrak{A}$ by Proposition 1.3(ii). By [13, Corollary 1.3], $\mathfrak{A}$ is weakly amenable, and so $\mathfrak{A}$ is $(2k-1)$-weakly amenable.

Suppose that $\mathfrak{A}$ is $(2k-1)$-weakly amenable. Then $\mathfrak{A}^{\#}$ is $(2k-1)$-weakly amenable by (ii).

Suppose that $\mathfrak{A}$ is $2k$-weakly amenable, and let

$$D : a \mapsto \langle a, \lambda\rangle e + \widetilde{D}(a), \quad \mathfrak{A} \to \mathbb{C}e \oplus \mathfrak{A}^{(2k)},$$

be a continuous derivation. Again we see immediately that $\lambda|\mathfrak{A}^2 = 0$.

First, suppose that there exists $\Psi \in \mathfrak{A}^{(2k)} \setminus \{0\}$ with

$$a \cdot \Psi = \Psi \cdot a = 0 \quad (a \in \mathfrak{A}).$$

Then $\mathfrak{A}^2$ is dense in $\mathfrak{A}$, for otherwise take $\zeta \in \mathfrak{A}'$ with $\zeta|\mathfrak{A}^2 = 0$ and $\zeta \neq 0$, and note that $a \mapsto \zeta(a)\Psi$, $\mathfrak{A} \to \mathfrak{A}^{(2k)}$, is a non-zero, continuous derivation. Thus $\lambda = 0$ and D is inner.

Second, suppose that, for each $\Psi \in \mathfrak{A}^{(2k)} \setminus \{0\}$, there exists $a \in \mathfrak{A}$ with $a \cdot \Psi \neq 0$. For each $a \in \mathfrak{A}$, the map

$$b \mapsto a \cdot D(b), \quad \mathfrak{A} \to \mathfrak{A}^{(2k)},$$

is a continuous derivation (since $\mathfrak{A}$ is commutative, and hence $\mathfrak{A}^{(2k)}$ is symmetric), and so $a \cdot D(b) = 0$ $(a, b \in \mathfrak{A})$. By our supposition, $D(b) = 0$ $(b \in \mathfrak{A})$, and so $D = 0$. Thus $\mathfrak{A}^{\#}$ is $2n$-weakly amenable. ∎

Let $\mathfrak{A}$ be a non-unital Banach algebra. We do not know if $\mathfrak{A}$ is weakly amenable whenever $\mathfrak{A}^{\#}$ is weakly amenable; this is true if $\mathfrak{A}$ has a bounded approximate identity (see [15, Corollary 2.2]), or, more generally, in the case where $\mathfrak{A}$ is H-unital. Also, we do not know if $\mathfrak{A}^{\#}$ is always 2-weakly amenable whenever $\mathfrak{A}$ is 2-weakly amenable.

We shall also consider the second dual $\mathfrak{A}''$ of a Banach algebra $\mathfrak{A}$ as a Banach algebra; indeed, two products are defined on $\mathfrak{A}''$ as follows. Let $a \in \mathfrak{A}$, $\lambda \in \mathfrak{A}'$, and $\Phi, \Psi \in \mathfrak{A}''$. Then $\Phi \cdot \lambda$ and $\lambda \cdot \Phi$ are defined in $\mathfrak{A}'$ by the formulae

$$\langle a, \Phi \cdot \lambda\rangle = \langle \lambda \cdot a, \Phi\rangle, \quad \langle a, \lambda \cdot \Phi\rangle = \langle a \cdot \lambda, \Phi\rangle. \tag{1.1}$$

Next, $\Phi \,\square\, \Psi$ and $\Phi \diamond \Psi$ are defined in $\mathfrak{A}''$ by the formulae

$$\langle \lambda, \Phi \,\square\, \Psi\rangle = \langle \Psi \cdot \lambda, \Phi\rangle, \quad \langle \lambda, \Phi \diamond \Psi\rangle = \langle \lambda \cdot \Phi, \Psi\rangle. \tag{1.2}$$

Then $\mathfrak{A}''$ is a Banach algebra with respect to each of the products $\square$ and $\diamond$; these products are called the *first* and *second Arens products* on $\mathfrak{A}''$, respectively. The algebra $\mathfrak{A}$ is defined to be *Arens regular* if the two products $\square$ and $\diamond$ coincide in $\mathfrak{A}''$. For the general theory of Arens products, see [9] and [26], for example.

The products $\square$ and $\diamond$ both extend the module operations on $\mathfrak{A}''$, in the sense that

$$\left.\begin{aligned} a \cdot \Phi &= \widehat{a} \,\square\, \Phi = \widehat{a} \diamond \Phi \\ \Phi \cdot a &= \Phi \,\square\, \widehat{a} = \Phi \diamond \widehat{a} \end{aligned}\right\} \quad (a \in \mathfrak{A},\ \Phi \in \mathfrak{A}'').$$

For a commutative algebra $\mathfrak{A}$, we have $\Phi \,\Box\, \Psi = \Psi \diamond \Phi$ $(\Phi, \Psi \in \mathfrak{A}'')$, and so $\mathfrak{A}$ is Arens regular if and only if $(\mathfrak{A}'', \Box)$ is commutative.

We shall require the following standard properties of the Arens products. Suppose that (a_γ) and (b_δ) are nets in $\mathfrak{A}$ with $a_\gamma \to \Phi$ and $b_\delta \to \Psi$ in $(\mathfrak{A}'', \sigma)$, where $\sigma = \sigma(\mathfrak{A}'', \mathfrak{A}')$ is the weak-$*$ topology on $\mathfrak{A}''$. Then

$$\Phi \,\Box\, \Psi = \lim_\gamma \lim_\delta a_\gamma b_\delta \quad \text{and} \quad \Phi \diamond \Psi = \lim_\delta \lim_\gamma a_\gamma b_\delta$$

in $(\mathfrak{A}'', \sigma)$. Let $\Phi_\nu \to \Phi$ in $(\mathfrak{A}'', \sigma)$, and let $\Psi \in \mathfrak{A}''$. Then $\Phi_\nu \,\Box\, \Psi \to \Phi \,\Box\, \Psi$ in $(\mathfrak{A}'', \sigma)$, but, in general, we cannot assert that $\Psi \Box \Phi_\nu \to \Psi \Box \Phi$. The following result is [26, 1.4.11].

PROPOSITION 1.5. *Let $\mathfrak{A}$ be a Banach algebra. Then the following are equivalent*:

(a) *$\mathfrak{A}$ is Arens regular*;
(b) *for each $\Psi \in \mathfrak{A}''$, the map $\Phi \mapsto \Psi \,\Box\, \Phi$ is continuous in $(\mathfrak{A}'', \sigma)$*;
(c) *for each $\lambda \in \mathfrak{A}'$, the map $a \mapsto \lambda \cdot a$, $\mathfrak{A} \to \mathfrak{A}'$, is weakly compact*;
(d) *for each pair $((a_m), (b_n))$ of bounded sequences in $\mathfrak{A}$ and each $\lambda \in \mathfrak{A}'$,*

$$\lim_m \lim_n \langle a_m b_n,\, \lambda\rangle = \lim_n \lim_m \langle a_m b_n,\, \lambda\rangle$$

whenever both iterated limits exist. ∎

Closed subalgebras of Arens regular algebras and quotients of Arens regular algebras by a closed ideal are also Arens regular.

Let $\mathfrak{A}$ be a Banach algebra. For $a \in \mathfrak{A}$, we denote the left and right regular representations of a by L_a and R_a, so that $L_a(b) = ab$ and $R_a(b) = ba$ for $b \in \mathfrak{A}$. The closed subalgebra $\mathfrak{A}$ of $(\mathfrak{A}'', \Box)$ is an ideal if and only if both L_a and R_a are weakly compact for each $a \in \mathfrak{A}$. For example, $L^1(G)$ is an ideal in $L^1(G)''$ if and only if G is a compact group.

We shall also require the following remark, pointed out by B. E. Johnson [22].

PROPOSITION 1.6. *Let $\mathfrak{A}$ be a Banach algebra. Suppose that $\mathfrak{A}$ has a predual X such that $\mathfrak{A} \cdot \mathfrak{A}' + \mathfrak{A}' \cdot \mathfrak{A} \subseteq \iota(X)$ in $\mathfrak{A}'$. Then $\mathfrak{A}$ is Arens regular.*

Proof. We shall verify that $\mathfrak{A}$ satisfies condition (c) of Proposition 1.5. Fix $\lambda \in \mathfrak{A}'$.

Define $P : \mathfrak{A}'' \to \mathfrak{A}$ as the adjoint of the injection of X in $\mathfrak{A}'$, so that P is an $\mathfrak{A}$-bimodule morphism. Let (a_γ) be a $\|\cdot\|$-bounded net in $\mathfrak{A}$ with $a_\gamma \to a$ in $(\mathfrak{A}, \sigma(\mathfrak{A}, X))$. For each $\Phi \in \mathfrak{A}''$, we have

$$\begin{aligned}
\langle \lambda \cdot a_\gamma, \Phi\rangle &= \langle P(\Phi),\, \lambda \cdot a_\gamma\rangle && \text{because } \lambda \cdot a_\gamma \in \mathfrak{A}' \cdot \mathfrak{A} \subseteq \iota(X)\\
&= \langle a_\gamma,\, P(\Phi) \cdot \lambda\rangle\\
&\to \langle a,\, P(\Phi) \cdot \lambda\rangle && \text{because } P(\Phi) \cdot \lambda \in \mathfrak{A} \cdot \mathfrak{A}' \subseteq \iota(X)\\
&= \langle P(\Phi),\, \lambda \cdot a\rangle\\
&= \langle \lambda \cdot a, \Phi\rangle && \text{because } \lambda \cdot a \in \mathfrak{A}' \cdot \mathfrak{A} \subseteq \iota(X),
\end{aligned}$$

and so $\lambda \cdot a_\gamma \to \lambda \cdot a$ in $(\mathfrak{A}', \sigma(\mathfrak{A}', \mathfrak{A}''))$. Thus the map $a \mapsto \lambda \cdot a$ is weakly compact, as required. ∎

Now let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule. We recall a construction that shows that X'' is a Banach $(\mathfrak{A}'', \Box)$-bimodule; we must make some successive definitions.

For $\Lambda \in X''$ and $\lambda \in X'$, define $\Lambda \cdot \lambda \in \mathfrak{A}'$ by

$$\langle a,\, \Lambda \cdot \lambda\rangle = \langle \lambda \cdot a,\, \Lambda\rangle \quad (a \in \mathfrak{A}). \tag{1.3}$$

For $\Phi \in \mathfrak{A}''$ and $\Lambda \in X''$, define $\Phi \cdot \Lambda \in X''$ by

$$\langle \lambda,\, \Phi \cdot \Lambda\rangle = \langle \Lambda \cdot \lambda,\, \Phi\rangle \quad (\lambda \in X'). \tag{1.4}$$

Then $a \cdot \Lambda$ takes its previous value in the case where $a \in \mathfrak{A} \subseteq \mathfrak{A}''$. Let (a_γ) and (x_δ) be nets in $\mathfrak{A}$ and X, respectively, such that $a_\gamma \to \Phi$ in $(\mathfrak{A}'', \sigma(\mathfrak{A}'', \mathfrak{A}'))$ and $x_\delta \to \Lambda$ in $(X'', \sigma(X'', X'))$. Then

$$\Phi \cdot \Lambda = \lim_\gamma \lim_\delta a_\gamma \cdot x_\delta \quad \text{in } (X'', \sigma(X'', X')). \tag{1.5}$$

Next, for $\lambda \in X'$ and $x \in X$, define $\lambda \cdot x \in \mathfrak{A}'$ by

$$\langle a,\, \lambda \cdot x\rangle = \langle x \cdot a,\, \lambda\rangle \quad (a \in \mathfrak{A}). \tag{1.6}$$

For $\Phi \in \mathfrak{A}''$ and $\lambda \in X'$, define $\Phi \cdot \lambda \in X'$ by

$$\langle x, \Phi \cdot \lambda\rangle = \langle \lambda \cdot x, \Phi\rangle \quad (x \in X), \tag{1.7}$$

so that $a \cdot \lambda$ agrees with its previous definition in the case where $a \in \mathfrak{A} \subseteq \mathfrak{A}''$. For $\Lambda \in X''$ and $\Phi \in \mathfrak{A}''$, define $\Lambda \cdot \Phi \in X''$ by

$$\langle \lambda, \Lambda \cdot \Phi\rangle = \langle \Phi \cdot \lambda, \Lambda\rangle \quad (\lambda \in X'), \tag{1.8}$$

so that $\Lambda \cdot a$ agrees with its previous definition. Let $a_\gamma \to \Phi$ and $x_\delta \to \Lambda$, as above. Then

$$\Lambda \cdot \Phi = \lim_\delta \lim_\gamma x_\delta \cdot a_\gamma \quad \text{in } (X'', \sigma(X'', X')). \tag{1.9}$$

We claim that X'' is a Banach $(\mathfrak{A}'', \Box)$-bimodule with respect to the maps $(\Phi, \Lambda) \mapsto \Phi \cdot \Lambda$ and $(\Phi, \Lambda) \mapsto \Lambda \cdot \Phi$ from $\mathfrak{A}'' \times X''$ to X''. To verify this, one can carefully check that the various associativity rules follow from the definitions; alternatively, these rules follow from the analogous rules that show that X is a Banach $\mathfrak{A}$-bimodule, together with a limiting process specified by (1.5) and (1.9).

Let $\mathfrak{A}$ and X be as above, and let $\mathfrak{B} = \mathfrak{A} \oplus_1 X$ as a Banach space, so that

$$\|(a, x)\| = \|a\| + \|x\| \quad (a \in \mathfrak{A},\ x \in X).$$

Then $\mathfrak{B}$ is a Banach algebra for the product

$$(a_1, x_1)(a_2, x_2) = (a_1 a_2, a_1 \cdot x_2 + x_1 \cdot a_2).$$

The second dual $\mathfrak{B}''$ of $\mathfrak{B}$ is identified with $\mathfrak{A}'' \oplus_1 X''$ as a Banach space, and the first Arens product $\Box$ on $\mathfrak{B}''$ is given by

$$(1.10)\qquad (\Phi_1, \Lambda_1) \Box (\Phi_2, \Lambda_2) = (\Phi_1 \Box \Phi_2,\ \Phi_1 \cdot \Lambda_2 + \Lambda_1 \cdot \Phi_2),$$

where the products $\Phi_1 \cdot \Lambda_2$ and $\Lambda_1 \cdot \Phi_2$ are defined by (1.4) and (1.8), respectively. The algebra $\mathfrak{B}$ is Arens regular if and only if $\mathfrak{A}$ is Arens regular, and

$$(1.11)\qquad \begin{aligned} \lim_\delta \lim_\gamma a_\gamma \cdot x_\delta &= \lim_\gamma \lim_\delta a_\gamma \cdot x_\delta, \\ \lim_\delta \lim_\gamma x_\delta \cdot a_\gamma &= \lim_\gamma \lim_\delta x_\delta \cdot a_\gamma, \end{aligned}$$

whenever (a_γ) and (x_δ) are weak-$*$ convergent nets in $\mathfrak{A}''$ and X'', respectively.

PROPOSITION 1.7. *Let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule. Suppose that $D : \mathfrak{A} \to X$ is a continuous derivation. Then*

$$D'' : (\mathfrak{A}'', \Box) \to X''$$

is a continuous derivation.

Proof. Certainly, $D'' : \mathfrak{A}'' \to X''$ is a continuous linear operator. Let $\Phi, \Psi \in \mathfrak{A}''$, say $\Phi = \lim a_\gamma$ and $\Psi = \lim b_\delta$ in $(\mathfrak{A}'', \sigma(\mathfrak{A}'', \mathfrak{A}'))$, where (a_γ) and (b_δ) are nets in $\mathfrak{A}$ with $\|a_\gamma\| \le \|\Phi\|$ and $\|b_\delta\| \le \|\Psi\|$. Then

$$\begin{aligned} D''(\Phi \Box \Psi) &= D''(\lim_\gamma \lim_\delta a_\gamma b_\delta) = \lim_\gamma \lim_\delta D(a_\gamma b_\delta) \\ &= \lim_\gamma \lim_\delta (a_\gamma \cdot D(b_\delta) + D(a_\gamma) \cdot b_\delta) \\ &= \Phi \cdot D''(\Psi) + D''(\Phi) \cdot \Psi, \end{aligned}$$

and so D'' is a derivation. ∎

Let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule. By applying the construction above to the Banach $\mathfrak{A}$-bimodules X and X'', we may equip X'' and X'''' with Banach $(\mathfrak{A}'', \Box)$-bimodule structures. (NB: The Banach $(\mathfrak{A}'', \Box)$-bimodule structure on X'''' thus obtained should not be confused with the $(\mathfrak{A}'', \Box)$-bimodule structure on X'''' obtained by viewing X'''' as a bidual bimodule over $(\mathfrak{A}'', \Box)$. For example, we may take X to be the dual module $\mathfrak{A}'$ of $\mathfrak{A}$; the $(\mathfrak{A}'', \Box)$-module structures on $\mathfrak{A}'''$ just defined and the module structures on $\mathfrak{A}'''$ as the dual module of $(\mathfrak{A}'', \Box)$ do not necessarily coincide.) Certainly, the projection $P : X'''' \to X''$ is a morphism of $\mathfrak{A}$-bimodules, but we should like it to be an morphism of $(\mathfrak{A}'', \Box)$-bimodules. This is so in certain cases.

PROPOSITION 1.8. *Let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule.*

(i) *The map $P : X'''' \to X''$ is a left $(\mathfrak{A}'', \Box)$-module morphism.*

(ii) *Suppose that the map*

$$\Phi \mapsto \lambda \cdot \Phi, \quad (\mathfrak{A}'', \sigma(\mathfrak{A}'', \mathfrak{A}')) \to (X'', \sigma(X'', X')),$$

is continuous for each fixed $\lambda \in X''$. Then P is also a right $(\mathfrak{A}'', \Box)$-module morphism.

Proof. First note that $P(\widehat{\lambda}) = \lambda$ $(\lambda \in X'')$, so that, for each $a \in \mathfrak{A}$,

$$P(\widehat{a \cdot \lambda}) = a \cdot \lambda = a \cdot P(\widehat{\lambda}) \quad \text{and} \quad P(\widehat{\lambda \cdot a}) = \lambda \cdot a = P(\widehat{\lambda}) \cdot a.$$

Now let $\Phi \in \mathfrak{A}''$ and $\Lambda \in X''''$, and choose bounded nets (a_γ) in $\mathfrak{A}$ and (λ_δ) in X'' with $a_\gamma \to \Phi$ in $\sigma(\mathfrak{A}'', \mathfrak{A}')$ and $\lambda_\delta \to \Lambda$ in $\sigma(X'''', X''')$. Since P is weak-$*$ continuous, it follows from (1.5) that

$$P(\Phi \cdot \Lambda) = \lim_\gamma \lim_\delta P(\widehat{a_\gamma \cdot \lambda_\delta})^\wedge) = \lim_\gamma \lim_\delta a_\gamma \cdot P(\widehat{\lambda}_\delta) = \Phi \cdot P(\Lambda)$$

and, with the assumption in (ii), that

$$\begin{aligned} P(\Lambda \cdot \Phi) &= \lim_\delta \lim_\gamma P(\widehat{\lambda_\delta \cdot a_\gamma}) = \lim_\delta \lim_\gamma P(\widehat{\lambda}_\delta) \cdot a_\gamma \\ &= \lim_\delta P(\widehat{\lambda}_\delta) \cdot \Phi = P(\Lambda) \cdot \Phi. \end{aligned}$$

Hence P has the specified properties. ∎

Note that, in the case where $X = \mathfrak{A}$ in the above result, the extra hypothesis in (ii) is just the hypothesis that $\mathfrak{A}$ be Arens regular.

THEOREM 1.9. *Let $\mathfrak{A}$ be a Banach algebra, let $n \in \mathbb{N}$, and let $D : \mathfrak{A} \to \mathfrak{A}^{(2n)}$ be a continuous derivation. Suppose that $\mathfrak{A}^{(2n-2)}$ is Arens regular. Then there is a continuous derivation $\widetilde{D} : (\mathfrak{A}^{(2n)}, \Box) \to (\mathfrak{A}^{(2n)}, \Box)$ such that*

$$\widetilde{D}(\widetilde{a}) = D(a) \quad (a \in \mathfrak{A}),$$

where $\widetilde{a}$ is the canonical image in $\mathfrak{A}^{(2n)}$ of $a \in \mathfrak{A}$.

Proof. By Proposition 1.7, $D'' : (\mathfrak{A}'', \Box) \to \mathfrak{A}^{(2n+2)}$ is a continuous derivation. Set $\mathfrak{B} = \mathfrak{A}^{(2n-2)}$. By assumption, $\mathfrak{B}$ is Arens regular, and so, by Proposition 1.8, the canonical projection $P : \mathfrak{B}'''' \to \mathfrak{B}''$ is a $(\mathfrak{B}'', \Box)$-bimodule morphism. Let $\Phi \in \mathfrak{A}''$, and let (a_γ) be a bounded net in $\mathfrak{A}$ such that $a_\gamma \to \Phi$ in $\sigma(\mathfrak{A}'', \mathfrak{A}')$. We have $\widetilde{a}_\gamma \to \widetilde{\Phi}$ in $\sigma(\mathfrak{B}'', \mathfrak{B}')$, where $\widetilde{\Phi}$ is the canonical image of Φ in $\mathfrak{A}^{(2n)} = \mathfrak{B}''$. Thus, with the identifications $\mathfrak{B}'' = \mathfrak{A}^{(2n)}$ and $\mathfrak{B}'''' = \mathfrak{A}^{(2n+2)}$, we see that P is an $(\mathfrak{A}'', \Box)$-bimodule morphism.

First note that, for $n = 1$, we may take $\widetilde{D} = P \circ D''$, as in the note after Proposition 1.8. In general the tuple $((\mathfrak{A}'', \Box),\ P \circ D'')$ consists of a Banach algebra and a derivation satisfying the hypotheses of the theorem with n replaced by $n-1$ since the Arens products on iterated duals successively extend previously defined Arens products. An easy induction argument now finishes the proof. ∎

COROLLARY 1.10. *Let $\mathfrak{A}$ be a Banach algebra which is Arens regular, and suppose that $\mathcal{H}^1(\mathfrak{A}'', \mathfrak{A}'') = \{0\}$. Then $\mathfrak{A}$ is 2-weakly amenable.*

Proof. Let $D \in \mathcal{Z}^1(\mathfrak{A}, \mathfrak{A}'')$. By the theorem, there exists $\widetilde{D} \in \mathcal{Z}^1(\mathfrak{A}'', \mathfrak{A}'')$ with $\widetilde{D}(\widehat{a}) = D(a)$ $(a \in \mathfrak{A})$. By hypothesis, there exists $\Psi \in \mathfrak{A}''$ such that

$$\widetilde{D}(\Phi) = \Phi \Box \Psi - \Psi \Box \Phi \quad (\Phi \in \mathfrak{A}'').$$

In particular, $D(a) = a \cdot \Psi - \Psi \cdot a$ $(a \in \mathfrak{A})$, and so $D \in N^1(\mathfrak{A}, \mathfrak{A}'')$. Thus $\mathfrak{A}$ is 2-weakly amenable. ∎

COROLLARY 1.11. *Let $\mathfrak{A}$ be a commutative Banach algebra which is Arens regular and such that $\mathfrak{A}''$ is semisimple. Then $\mathfrak{A}$ is 2-weakly amenable.*

Proof. The Banach algebra $\mathfrak{A}''$ is commutative and semisimple, and so, by the commutative Singer–Wermer theorem (see [3, 18.16], for example), we have $\mathcal{Z}^1(\mathfrak{A}'', \mathfrak{A}'') = \{0\}$. ∎

COROLLARY 1.12. *Let $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A}^{(2n)}$ is Arens regular and $\mathcal{H}^1(\mathfrak{A}^{(2n+2)}, \mathfrak{A}^{(2n+2)}) = \{0\}$ for each $n \in \mathbb{Z}^+$. Then $\mathfrak{A}$ is $2n$-weakly amenable for each $n \in \mathbb{N}$.*

Proof. By Corollary 1.10, $\mathfrak{A}$ is 2-weakly amenable.

We now show that $\mathfrak{A}$ is $(2k+2)$-weakly amenable under the assumption that every algebra with the stated properties is $2k$-weakly amenable.

Let $D \in \mathcal{Z}^1(\mathfrak{A}, \mathfrak{A}^{(2k+2)})$. By Proposition 1.6, $D'' \in \mathcal{Z}^1(\mathfrak{A}'', \mathfrak{A}^{(2k+4)})$. Let $P : \mathfrak{A}^{(2k+4)} \to \mathfrak{A}^{(2k+2)}$ be the natural projection; by Proposition 1.7, P is an $\mathfrak{A}''$-bimodule morphism. Set $\widehat{D} = P \circ D''$, so that

$$\widehat{D} \in \mathcal{Z}^1(\mathfrak{A}'', \mathfrak{A}^{(2k+2)}) = \mathcal{Z}^1(\mathfrak{A}'', (\mathfrak{A}'')^{(2k)}).$$

Since $\mathfrak{A}''$ satisfies the stated properties on $\mathfrak{A}$, $\mathfrak{A}''$ is $2k$-weakly amenable, and so there exists $\Lambda \in \mathfrak{A}^{(2k+2)}$ with

$$\widehat{D}(\Phi) = \Phi \cdot \Lambda - \Lambda \cdot \Phi \quad (\Phi \in \mathfrak{A}'').$$

Clearly, $\widehat{D}(a) = a \cdot \Lambda - \Lambda \cdot a$ $(a \in \mathfrak{A})$, and so $D \in N^1(\mathfrak{A}, \mathfrak{A}^{(2k+2)})$. Hence $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(2k+2)}) = \{0\}$ and $\mathfrak{A}$ is $(2k+2)$-weakly amenable.

The result follows by induction. ∎

We note that there is a commutative, semisimple, Arens regular Banach algebra $\mathfrak{A}$ such that $\mathfrak{A}''$ is not Arens regular ([28]); however, it may be that the above corollary holds under the hypotheses that $\mathfrak{A}$ is Arens regular and that $\mathcal{H}^1(\mathfrak{A}^{(2n+2)}, \mathfrak{A}^{(2n+2)}) = \{0\}$ for each $n \in \mathbb{Z}^+$.

We have remarked that a commutative, weakly amenable Banach algebra is permanently weakly amenable; for general (non-commutative) Banach algebras, we have the following partial result under a special hypothesis.

PROPOSITION 1.13. *Let $\mathfrak{A}$ be a weakly amenable Banach algebra such that $\mathfrak{A}$ is an ideal in $(\mathfrak{A}'', \Box)$. Then $\mathfrak{A}$ is $(2n+1)$-weakly amenable for each $n \in \mathbb{Z}^+$.*

Proof. For $n \in \mathbb{Z}^+$, we regard $\mathfrak{A}^{(2n+2)}$ as the second dual of $(\mathfrak{A}^{(2n)}, \Box)$, taken with the first Arens product $\Box$, and, for $m \le n$, we regard $(\mathfrak{A}^{(2m)}, \Box)$ as a subalgebra of $(\mathfrak{A}^{(2n)}, \Box)$; the definitions of $a \cdot \Phi$ and $\Phi \cdot a$ are consistent.

Fix $n \in \mathbb{N}$. For each $a \in \mathfrak{A}$, the operators L_a and R_a on $\mathfrak{A}$ are weakly compact, and so the operators $L_a^{(2n)}$ and $R_a^{(2n)}$ are weakly compact on $\mathfrak{A}^{(2n)}$. Thus $a \cdot \Phi, \Phi \cdot a \in \mathfrak{A}^{(2n-2)}$ for $a \in \mathfrak{A}$ and $\Phi \in \mathfrak{A}^{(2n)}$. Further, $a_1 \ldots a_n \cdot \Phi$ and $\Phi \cdot a_1 \ldots a_n$ belong to $\mathfrak{A}$ for $a_1, \ldots, a_n \in \mathfrak{A}$ and $\Phi \in \mathfrak{A}^{(2n)}$. Let $\mathfrak{A}^\perp$ be the space of functionals in $\mathfrak{A}^{(2n+1)}$ which annihilate $\iota(\mathfrak{A})$. Then

$$\mathfrak{A}^{(2n+1)} = \iota(\mathfrak{A}') \oplus \mathfrak{A}^\perp$$

as Banach $\mathfrak{A}$-bimodules, and so

$$\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(2n+1)}) = \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}') \oplus \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^\perp).$$

By hypothesis, $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}') = \{0\}$, and so it suffices for the result to show that $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^\perp) = \{0\}$.

Let $D \in \mathcal{Z}^1(\mathfrak{A}, \mathfrak{A}^\perp)$, and let $a, b \in \mathfrak{A}^{[n]}$. For each $\Phi \in \mathfrak{A}^{(2n)}$,

$$\begin{aligned}\langle \Phi, D(ab)\rangle &= \langle \Phi, D(a) \cdot b\rangle + \langle a, \Phi \cdot D(b)\rangle \\ &= \langle b \cdot \Phi, D(a)\rangle + \langle \Phi \cdot a, D(b)\rangle = 0,\end{aligned}$$

and so $D(ab) = 0$. It follows that $D | \mathfrak{A}^{2n} = 0$. By Proposition 1.3(i), $\mathfrak{A}^{2n}$ is dense in $\mathfrak{A}$, and so $D = 0$. Hence $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^\perp) = \{0\}$, and the result follows. ∎

COROLLARY 1.14. *Let $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A}$ is an ideal in $(\mathfrak{A}'', \Box)$. Then the following are equivalent:*

(a) *$\mathfrak{A}$ is weakly amenable;*
(b) *$\mathfrak{A}$ is $(2n+1)$-weakly amenable for some $n \in \mathbb{Z}^+$;*
(c) *$\mathfrak{A}$ is $(2n+1)$-weakly amenable for each $n \in \mathbb{Z}^+$.* ∎

We shall see later that there are Banach algebras $\mathfrak{A}$ such that: (i) $\mathfrak{A}$ is an ideal in $(\mathfrak{A}'', \Box)$ and $\mathfrak{A}$ does satisfy the equivalent conditions (a)–(c) (Corollary 5.4); (ii) $\mathfrak{A}$ is an ideal in $(\mathfrak{A}^{\#}, \Box)$, and yet $\mathfrak{A}$ is not n-weakly amenable for any $n \in \mathbb{N}$ (Example 3.8).

We continue this introduction with a remark on how n-weak amenability passes from an algebra to a closed ideal in the case of commutative Banach algebras. Let I be a closed ideal in a weakly amenable, commutative Banach algebra $\mathfrak{A}$. Then, by [13, Corollary 1.3], I is weakly amenable if and only if I^2 is dense in I.

PROPOSITION 1.15. *Let I be a closed ideal in a commutative Banach algebra $\mathfrak{A}$, and let $n \in \mathbb{N}$. Suppose that $\mathfrak{A}$ is $2n$-weakly amenable. Then the following conditions are equivalent:*

(a) *I is $2n$-weakly amenable;*
(b) *either I^2 is dense in I or $I \cdot I^{(2n-1)}$ is dense in $I^{(2n-1)}$.*

Proof. (b)⇒(a). Let $D \in \mathcal{Z}^1(I, I^{(2n)})$, and let j be the embedding of I into $\mathfrak{A}$. From [13, Theorem 1.1] we see that the map

$$x \mapsto j^{(2n)}(D(abx) - D(ab) \cdot x), \qquad \mathfrak{A} \to \mathfrak{A}^{(2n)},$$

is a derivation for every fixed $a, b \in \mathfrak{A}$. Since $\mathfrak{A}$ is assumed to be $2n$-weakly amenable, and since $j^{(2n)}$ is a monomorphism, it follows that

$$D(abc) = D(ab) \cdot c \qquad (a, b, c \in I),$$

and therefore $ab \cdot D(c) = 0$ $(a, b, c \in I)$. If either of the conditions in (b) hold, then $D \equiv 0$. Thus I is $2n$-weakly amenable.

(a)⇒(b). Let $\mathfrak{B}$ be any Banach algebra, and let X be any Banach $\mathfrak{B}$-bimodule. Any map $D : \mathfrak{B} \to X'$ admitting a factorization

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{\;Q\;} & \mathfrak{B}/\overline{\mathfrak{B}^2} \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle T} \\ X' & \xleftarrow{\;i'\;} & (\mathfrak{B} \cdot X + X \cdot \mathfrak{B})^{\perp} \end{array}$$

where Q is the quotient map and i is the embedding, is trivially a derivation. With $\mathfrak{B} = I$, $X = I^{(2n-1)}$, and both conditons in (b) failing we may choose T, and hence D, to be non-zero. Therefore I cannot be $2n$-weakly amenable. ∎

Finally in this section, we shall give two results on the calculation of $\mathcal{H}^1(\mathfrak{A}, X')$ that we shall use.

Let E and F be Banach spaces. Then $E \widehat{\otimes} F$ denotes the projective tensor product of E and F.

Let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule; we consider the standard complex

$$\ldots \to \mathfrak{A} \widehat{\otimes} \mathfrak{A} \widehat{\otimes} X \xrightarrow{d_2} \mathfrak{A} \widehat{\otimes} X \xrightarrow{d_1} X \to 0,$$

where the maps d_1 and d_2 are specified by the formulae:

$$(1.12) \qquad \begin{aligned} d_1(a \otimes x) &= a \cdot x - x \cdot a & (a \in \mathfrak{A},\ x \in X); \\ d_2(a \otimes b \otimes x) &= b \otimes x \cdot a - ab \otimes x + a \otimes b \cdot x & (a, b \in \mathfrak{A},\ x \in X). \end{aligned}$$

This complex is discussed in [20] and [19]; the first of our two results is essentially [20, Corollary 1.3] and [19, II.5.29].

PROPOSITION 1.16. *Let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule. Then $\mathcal{H}^1(\mathfrak{A}, X') = \{0\}$ if and only if both $\operatorname{im} d_1$ is closed in X and $\operatorname{im} d_2$ is dense in $\ker d_1$.* ∎

Let $\mathfrak{A}$ be a Banach algebra. A *bounded approximate identity* in $\mathfrak{A}$ is a bounded net (e_α) in $\mathfrak{A}$ such that $e_\alpha a \to a$ and $a e_\alpha \to a$ in $\mathfrak{A}$ for each $a \in \mathfrak{A}$.

PROPOSITION 1.17. *Let $\mathfrak{A}$ be a Banach algebra with a bounded approximate identity, and let X be a Banach $\mathfrak{A}$-bimodule. Let $D \in \mathcal{Z}^1(\mathfrak{A}, X')$, and suppose that there exists $\lambda_0 \in X'$ such that*

$$\langle a \cdot x \cdot b, D(c) \rangle = \langle a \cdot x \cdot b, c \cdot \lambda_0 - \lambda_0 \cdot c \rangle \qquad (a, b, c \in \mathfrak{A},\ x \in X).$$

Then $D \in N^1(\mathfrak{A}, X')$.

Proof. By replacing D with $D - \delta_{\lambda_0}$, we may suppose that

$$(1.13) \qquad \langle a \cdot x \cdot b, D(c) \rangle = 0 \qquad (a, b, c \in \mathfrak{A},\ x \in X).$$

Choose a bounded approximate identity (e_α) in $\mathfrak{A}$ such that the iterated weak-$*$ limit $\lambda = \lim_\alpha \lim_\beta (e_\alpha D(e_\beta) - D(e_\beta) e_\alpha)$ exists. Then, for $x \in X$ and $a \in \mathfrak{A}$, we have, using (1.13),

$$\begin{aligned} \langle x, D(a) \rangle &= \lim_\alpha \lim_\beta \langle x, D(e_\alpha a e_\beta) \rangle \\ &= \lim_\alpha \lim_\beta \langle x, D(e_\alpha) \cdot a e_\beta + e_\alpha a \cdot D(e_\beta) \rangle \\ &= \lim_\beta \langle x, D(e_\beta) \cdot a + a \cdot D(e_\beta) \rangle \\ &= \lim_\alpha \lim_\beta \langle x, a \cdot (e_\alpha \cdot D(e_\beta) - D(e_\beta) \cdot e_\alpha) \rangle \\ &\quad - \lim_\alpha \lim_\beta \langle x, (e_\alpha \cdot D(e_\beta) - D(e_\beta) \cdot e_\alpha) \cdot a \rangle \\ &= \langle x, a \cdot \lambda - \lambda \cdot a \rangle, \end{aligned}$$

where we have used several times the facts that $a \cdot D(b) \cdot c = 0$ $(a, b, c \in \mathfrak{A})$ and that bounded norm limits are interchangeable with weak-$*$ limits. Hence D is inner. ∎

2. C^*-algebras. We shall first discuss the class of C^*-algebras; in this case the situation is clear.

Recall first that, by a very deep theorem of Connes ([5]) and Haagerup ([18]), a C^*-algebra is amenable if and only if it is nuclear; not all C^*-algebras are amenable, and, in particular, the C^*-algebra $\mathcal{B}(H)$ is not nuclear, and hence not amenable, in the case where H is an infinite-dimensional Hilbert space. However, every C^*-algebra is weakly amenable ([18]).

Let A be a Banach algebra with a continuous involution $*$. Then $*$ defines a continuous linear involution on A''; we have

$$(\Phi \,\Box\, \Psi)^* = \Psi^* \diamond \Phi^* \quad (\Phi, \Psi \in A''),$$

and so the map $*$ is an involution on A'' if and only if A is Arens regular.

THEOREM 2.1. *Every C^*-algebra is permanently weakly amenable.*

Proof. Let $\mathfrak{A}$ be a C^*-algebra. It is standard that each C^*-algebra is Arens regular and that the iterated duals $\mathfrak{A}^{(2n)}$ are also C^*-algebras; indeed, they are von Neumann algebras, $\mathfrak{A}''$ being the enveloping von Neumann algebra of $\mathfrak{A}$ ([4], [3, 38.19]). Also, $\mathcal{H}^1(\mathfrak{B}, \mathfrak{B}) = \{0\}$ for each von Neumann algebra $\mathfrak{B}$ ([30, Theorem 4.1.8]). Hence it follows from Corollary 1.12 that $\mathfrak{A}$ is $2n$-weakly amenable for each $n \in \mathbb{N}$.

We now show that $\mathfrak{A}$ is $(2n+1)$-weakly amenable under the assumption that each C^*-algebra is $(2n-1)$-weakly amenable. Let $D \in \mathcal{Z}^1(\mathfrak{A}, \mathfrak{A}^{(2n+1)})$. Then D is, in particular, a continuous linear operator from $\mathfrak{A}$ into the predual of the von Neumann algebra $\mathfrak{A}^{(2n+2)}$; by [1, Corollary II.9], D is weakly compact, and so the range of D'' is contained in $\mathfrak{A}^{(2n+1)}$; clearly, we have $D'' \in \mathcal{Z}^1(\mathfrak{A}'', \mathfrak{A}^{(2n+1)})$. Since $\mathfrak{A}''$ is $(2n-1)$-weakly amenable, there exists $\Lambda \in \mathfrak{A}^{(2n+1)}$ such that

$$D''(\Phi) = \Phi \cdot \Lambda - \Lambda \cdot \Phi \quad (\Phi \in \mathfrak{A}''),$$

and again $D \in \mathcal{N}^1(\mathfrak{A}, \mathfrak{A}^{(2n+1)})$, as required. It follows by induction that $\mathfrak{A}$ is $(2n+1)$-weakly amenable for each $n \in \mathbb{Z}^+$.

Hence each C^*-algebra is permanently weakly amenable. ■

COROLLARY 2.2. *Let H be an infinite-dimensional Hilbert space. Then $\mathcal{B}(H)$ is permanently weakly amenable, but not amenable.* ■

3. Commutative Banach algebras. Let $(\mathfrak{A}, \|\cdot\|)$ be a commutative, semisimple Banach algebra with character space the locally compact space $\Omega = \Phi_{\mathfrak{A}}$; we regard $\mathfrak{A}$ as a subalgebra of $C_0(\Omega)$, so that $\|f\| \geq |f|_\Omega$, where $|\cdot|_\Omega$ is the uniform norm on Ω. In this case $\mathfrak{A}$ is a *Banach function algebra* on Ω; $\mathfrak{A}$ is a *uniform algebra* in the case where $\|\cdot\|$ is equivalent to $|\cdot|_\Omega$.

THEOREM 3.1. *Let $\mathfrak{A}$ be a uniform algebra. Then $\mathfrak{A}$ is $2n$-weakly amenable for each $n \in \mathbb{N}$.*

Proof. Set $\Omega = \Phi_{\mathfrak{A}}$. Then $\mathfrak{A}'' \subseteq C_0(\Omega)''$ as a closed subalgebra; $C_0(\Omega)''$ has the form $C(\widehat{\Omega})$ for a compact space $\widehat{\Omega}$, and so $\mathfrak{A}''$ is an Arens regular uniform algebra on $C(\widehat{\Omega})$. Continuing, each $\mathfrak{A}^{(2n)}$ is an Arens regular uniform algebra on a compact space for each $n \in \mathbb{N}$. Again by the Singer–Wermer theorem, $\mathcal{H}^1(\mathfrak{A}^{(2n)}, \mathfrak{A}^{(2n)}) = \{0\}$ for each $n \in \mathbb{N}$.

It follows from Corollary 1.12 that $\mathfrak{A}$ is $2n$-weakly amenable for each $n \in \mathbb{N}$. ■

COROLLARY 3.2. *Let $\mathfrak{A}$ be a uniform algebra. Then the following conditions are equivalent:*

(a) $\mathfrak{A}$ *is weakly amenable;*
(b) $\mathfrak{A}$ *is permanently weakly amenable;*
(c) $\mathfrak{A}$ *is $(2k+1)$-weakly amenable for some $k \in \mathbb{Z}^+$.* ■

Suppose that $\mathfrak{A}$ is a uniform algebra that is not weakly amenable. Then $\mathfrak{A}$ is $2n$-weakly amenable for each $n \in \mathbb{N}$, but $\mathfrak{A}$ is not $(2n+1)$-weakly amenable for any $n \in \mathbb{Z}^+$. It is conjectured that, for a uniform algebra $\mathfrak{A} \subseteq C(\Omega)$, $\mathfrak{A}$ is weakly amenable if and only if $\mathfrak{A} = C(\Omega)$; this question is open, but we note that it is a theorem of Sheĭnberg ([32]) that $\mathfrak{A} = C(\Omega)$ if and only if $\mathfrak{A}$ is amenable.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc, and let $A(\overline{\mathbb{D}})$ be the disc algebra. It follows from Theorem 3.1 that $A(\overline{\mathbb{D}})$ is $2n$-weakly amenable for each $n \in \mathbb{Z}$. We show this directly by an elementary argument.

For $k \in \mathbb{N}$, the map $T_k : A(\overline{\mathbb{D}}) \to A(\overline{\mathbb{D}})$ defined by

$$(T_k f)(z) = z^k f(z) \quad (f \in A(\overline{\mathbb{D}}),\ z \in \overline{\mathbb{D}})$$

is an isometry. Thus the $2n$th adjoint, $T_k^{(2n)}$, of T_k is an isometry on $A(\overline{\mathbb{D}})^{(2n)}$. Let $D \in \mathcal{Z}^1(A(\overline{\mathbb{D}}), A(\overline{\mathbb{D}})^{(2k)})$. Then

$$D(Z^k) = kZ^{k-1} \cdot D(Z) \quad (k \in \mathbb{N}),$$

and so, for each $k \in \mathbb{N}$,

$$\|D(Z)\| = \|T_{k-1}^{(2n)}(D(Z))\| = \|Z^{k-1} \cdot D(Z)\| = \frac{1}{k}\|D(Z^k)\| \leq \frac{1}{k}\|D\|.$$

Thus $D(Z) = 0$, and so $D = 0$.

In particular, $A(\overline{\mathbb{D}})$ is a 2-weakly amenable Banach function algebra which is not weakly amenable. In a similar way, the maximal ideal

$$M = \{f \in A(\overline{\mathbb{D}}) : f(0) = 0\}$$

of $A(\overline{\mathbb{D}})$ is 2-weakly amenable. This shows that, for a 2-weakly amenable commutative Banach algebra $\mathfrak{A}$, it is not necessarily the case that $\mathfrak{A}^2$ is dense in $\mathfrak{A}$ (cf. Proposition 1.3(i)).

We next consider some Banach function algebras which are not uniform algebras. We have in mind, in particular, Banach function algebras $\mathfrak{A}$ on a compact space $\Omega = \Phi_{\mathfrak{A}}$ which are dense in $(C(\Omega), |\cdot|_\Omega)$.

The algebra $C^{(1)}(\mathbb{I})$ consists of the continuously differentiable functions on the unit interval $\mathbb{I} = [0,1]$; $C^{(1)}(\mathbb{I})$ is a Banach function algebra on $\mathbb{I}$ with respect to the norm

$$\|f\|_1 = |f|_{\mathbb{I}} + |f'|_{\mathbb{I}} \quad (f \in C^{(1)}(\mathbb{I})).$$

Certainly, by Proposition 1.3(ii), $C^{(1)}(\mathbb{I})$ is not weakly amenable.

PROPOSITION 3.3. *The Banach function algebra $C^{(1)}(\mathbb{I})$ is Arens regular, but it is not 2-weakly amenable.*

Proof. We set $\mathfrak{A} = C^{(1)}(\mathbb{I})$.

We first define $\mathfrak{B} = C(\mathbb{I}) \oplus_1 C(\mathbb{I})$ as a Banach space with respect to the norm

$$\|(f,g)\| = |f|_{\mathbb{I}} + |g|_{\mathbb{I}} \quad (f,g \in C(\mathbb{I})).$$

A product in $\mathfrak{B}$ is defined by the formula

$$(f_1,g_1)(f_2,g_2) = (f_1f_2, f_1g_2 + g_1f_2) \quad (f_1,f_2,g_1,g_2 \in C(\mathbb{I})).$$

As in §1, $\mathfrak{B}$ is a Banach algebra, and $\mathfrak{B}$ is commutative. The map

$$f \mapsto (f,f'), \quad \mathfrak{A} \to \mathfrak{B},$$

is an isometric embedding of $\mathfrak{A}$ in $\mathfrak{B}$; we regard $\mathfrak{A}$ as a closed subalgebra of $\mathfrak{B}$.

The dual space of $\mathfrak{B}$ is $M \oplus_\infty M$, where $M = M(\mathbb{I})$ is the space of regular Borel measures on $\mathbb{I}$, and the second dual space of $\mathfrak{B}$ is

$$\mathfrak{B}'' = M' \oplus_1 M'.$$

Since $C(\mathbb{I})$ is Arens regular, it is easily seen that $\mathfrak{B}$ is Arens regular, and that multiplication in $\mathfrak{B}''$ for each Arens product is given by

$$(3.1) \qquad (\Phi_1,\Psi_1)(\Phi_2,\Psi_2) = (\Phi_1\Phi_2, \Phi_1\Psi_2 + \Psi_1\Phi_2),$$

where $\Phi\Psi$ is the Arens product of $\Phi,\Psi \in C(\mathbb{I})'' = M'$. It follows that $\mathfrak{A}$ is Arens regular, being a closed subalgebra of an Arens regular algebra.

We next define an element Ψ_0 of $M' = C(\mathbb{I})''$ by

$$\Psi_0(\mu) = \mu(\{0\}) \quad (\mu \in M).$$

Clearly, $(0,\Psi_0) \in \mathfrak{B}''$ and $\|\Psi_0\| = 1$.

We claim that $(0,\Psi_0) \in \mathfrak{A}''$; for this, we must show that $(0,\Psi_0)$ belongs to the weak-$*$ closure of $\mathfrak{A}$ in $\mathfrak{B}''$.

Take $f \in \mathfrak{A}$ to be such that $f(t) = t$ for $t \in [0,1/2]$, $f(1) = f'(1) = 0$, and $|f|_{\mathbb{I}} \le 1$, and then, for $n \in \mathbb{N}$, define

$$f_n(t) = \begin{cases} f(nt)/n & (0 \le t \le 1/n), \\ 0 & (1/n < t \le 1). \end{cases}$$

It is immediate that $f_n \in \mathfrak{A}$, that $\operatorname{supp} f_n \subseteq [0,1/n]$, that $|f_n|_{\mathbb{I}} \le 1/n$, that $f_n'(t) = 1$ for $t \in [0,1/2n]$, and that $|f_n'|_{\mathbb{I}} = |f'|_{\mathbb{I}}$. For each $\mu \in M$, we have

$$|\langle f_n, \mu\rangle| \le |f_n|_{\mathbb{I}}\|\mu\| \to 0 \quad \text{as } n \to \infty,$$

and so $f_n \to 0$ weak-$*$ closure in M'. We further see that

$$\begin{aligned} |\langle f_n', \mu\rangle - \mu([0,1/(2n)))| &\le |f_n'|_{\mathbb{I}}|\mu|([1/(2n),1/n]) \\ &= |f'|_{\mathbb{I}}|\mu|([1/(2n),1/n]) \quad (\mu \in M). \end{aligned}$$

Since $\mu([1/(2n),1/n]) \to 0$ and $\mu([0,1/(2n)]) \to \mu(\{0\})$ as $n \to \infty$ for each $\mu \in M$, we have $f_n' \to \Psi_0$ weak-$*$ in M'. It follows that $(f_n, f_n') \to (0,\Psi_0)$ weak-$*$ in $\mathfrak{B}''$, and so indeed $(0,\Psi_0) \in \mathfrak{A}''$.

We define a map $D : \mathfrak{A} \to \mathfrak{A}''$ as follows. First $D(1) = (0,0)$ and

$$D(Z) = (0,\Psi_0),$$

where Z is the coordinate functional $Z : t \mapsto t$ on $\mathbb{I}$. Then D is extended to be a derivation on the subalgebra $\mathbb{C}[Z]$ of polynomials: for each $p \in \mathbb{C}[Z]$,

$$D(p) = p' \cdot D(Z) = (p',p'')(0,\Psi_0) = (0,p'\Psi_0).$$

We see that

$$\|D(p)\| = \|p'\Psi_0\| \le |p'|_{\mathbb{I}}\|\Psi_0\| \le \|p\|_1 \quad (p \in \mathbb{C}[Z]).$$

Thus D is continuous on $(\mathbb{C}[Z], \|\cdot\|_1)$, and D has a continuous extension to a derivation $D : \mathfrak{A} \to \mathfrak{A}''$. Since $D(Z) = (0,\Psi_0)$, we see that $D \ne 0$.

It follows that $\mathcal{H}^1(\mathfrak{A},\mathfrak{A}'') \ne \{0\}$, and so $\mathfrak{A}$ is not 2-weakly amenable. ∎

It follows from Proposition 1.2 that $C^{(1)}(\mathbb{I})$ is not n-weakly amenable for any $n \in \mathbb{N}$.

The main relation involving n-weak amenability for commutative Banach algebras that is open so far is whether a 2-weakly amenable algebra is 4-weakly amenable; we cannot answer this question, but the following example gives some new information.

Let K be a compact metric space with metric d, and take α such that $0 < \alpha \le 1$. Then $\operatorname{Lip}_\alpha K$ is the space of complex-valued functions f on K such that

$$p_\alpha(f) = \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)^\alpha} : x,y \in K,\ x \ne y\right\}$$

is finite. For $f \in \operatorname{Lip}_\alpha K$, set

$$\|f\|_\alpha = |f|_K + p_\alpha(f).$$

Then $(\operatorname{Lip}_\alpha K, \|\cdot\|_\alpha)$ is a Banach function algebra on K. A function f belongs to $\operatorname{lip}_\alpha K$ if

$$\frac{|f(x)-f(y)|}{d(x,y)^\alpha} \to 0 \quad \text{as } d(x,y) \to 0;$$

$\operatorname{lip}_\alpha K$ is a closed subalgebra of $\operatorname{Lip}_\alpha K$. For studies of these Lipschitz algebras, see [33] and [2]. It is shown in [2, Theorem 3.8] that, in the case where $\alpha < 1$, $(\operatorname{lip}_\alpha K)''$ is isometrically isomorphic as a Banach space to $\operatorname{Lip}_\alpha K$, and that the two Arens products on $(\operatorname{lip}_\alpha K)''$ coincide with the given product in $\operatorname{Lip}_\alpha K$; in particular, $\operatorname{lip}_\alpha K$ is Arens regular.

PROPOSITION 3.4. *Let K be an infinite, compact metric space, and let $\alpha \in (0,1)$. Then:*

(i) $\operatorname{lip}_\alpha K$ *is not amenable;*
(ii) *in the case where $\alpha < 1/2$, $\operatorname{lip}_\alpha K$ is permanently weakly amenable;*
(iii) $\operatorname{lip}_\alpha K$ *is 2-weakly amenable.*

Proof. (i) This is [2, Theorem 3.9].

(ii) By [2, Theorem 3.10], $\mathrm{lip}_\alpha K$ is weakly amenable, and hence $\mathrm{lip}_\alpha K$ is permanently weakly amenable.

(iii) The algebra $\mathrm{lip}_\alpha K$ is Arens regular, and $(\mathrm{lip}_\alpha K)''$ is semisimple, and so this follows from Corollary 1.11. ■

We obtain a further result in the case where $K = \mathbb{T}$ with the usual metric.

PROPOSITION 3.5. *Let $\alpha \in (0,1)$. Then:*

(i) *in the case where $\alpha \le 1/2$, $\mathrm{lip}_\alpha \mathbb{T}$ is permanently weakly amenable;*

(ii) *in the case where $\alpha > 1/2$, $\mathrm{lip}_\alpha \mathbb{T}$ is not $(2k+1)$-weakly amenable for any $k \in \mathbb{Z}^+$.*

Proof. (i) This follows from [2, Theorem 3.13].

(ii) By [2, Theorem 3.11], $\mathrm{lip}_\alpha \mathbb{T}$ is not weakly amenable. ■

PROPOSITION 3.6. (i) *Let K be a compact metric space, and let $\alpha \in (0,1]$. Then $\mathrm{Lip}_\alpha K$ is Arens regular.*

(ii) *The algebra $(\mathrm{Lip}_\alpha \mathbb{I})''$ is not semisimple.*

(iii) *$\mathrm{Lip}_\alpha \mathbb{I}$ is not n-weakly amenable for any $n \in \mathbb{N}$.*

Proof. (i) Set $\mathfrak{A} = \mathrm{Lip}_\alpha K$, and define

$$\Delta = \{(x,y) \in K \times K : x = y\}, \qquad V = (K \times K) \setminus \Delta.$$

We consider the Banach algebra $X = C^b(V)$ of bounded, continuous functions on V with the uniform norm $|\cdot|_V$ (so that $X \cong C(\beta V)$).

For $F \in C(K)$ and $G \in X$, define

$$(F \cdot G)(x,y) = F(x)G(x,y), \quad (G \cdot F)(x,y) = G(x,y)F(y) \quad ((x,y) \in V).$$

Then $F \cdot G$, $G \cdot F \in X$, and X is a Banach $C(K)$-bimodule for the operations

$$(F,G) \mapsto F \cdot G, \qquad (F,G) \mapsto G \cdot F.$$

Form the corresponding Banach algebra

$$\mathfrak{B} = C(K) \oplus_1 X$$

as in §1; we note that $\mathfrak{B}$ is not commutative. Since $C(\beta V)$ is Arens regular, it follows that condition (1.11) is satisfied. Also $C(K)$ is Arens regular, and so $\mathfrak{B}$ is Arens regular, with the product in $\mathfrak{B}''$ given by equation (1.10).

Now let $F \in \mathfrak{A}$, and define $\widetilde{F} \in C^b(V)$ by the formula

$$\widetilde{F}(x,y) = \frac{F(y) - F(x)}{d(x,y)^\alpha} \qquad ((x,y) \in V).$$

Then the map

$$\theta : F \mapsto (F, \widetilde{F}), \qquad \mathfrak{A} \to \mathfrak{B},$$

is clearly an isometric linear map. Further, for $F, G \in \mathfrak{A}$, we have

$$\begin{aligned}(F \cdot \widetilde{G} + \widetilde{F} \cdot G)(x,y) &= F(x)\left(\frac{G(y)-G(x)}{d(x,y)^\alpha}\right) + \left(\frac{F(y)-F(x)}{d(x,y)^\alpha}\right)G(y) \\ &= \frac{(FG)(y) - (FG)(x)}{d(x,y)^\alpha} \\ &= (\widetilde{FG})(x,y) \qquad ((x,y) \in V),\end{aligned}$$

and so θ is a homomorphism. Thus we may regard $\mathfrak{A}$ as a closed subalgebra of $\mathfrak{B}$.

It follows that $\mathfrak{A}$ is Arens regular.

(ii) Let $\mathfrak{A}$ be as in (i). Clearly, $\mathrm{rad}\,\mathfrak{A}'' = \{(0,\Psi) \in \mathfrak{A}'' : \Psi \in X''\}$.

Consider the special case of this construction in which $K = \mathbb{I}$. For $k \in \mathbb{N}$, set $x_k = 2^{-2k-1}$ and $y_k = 2^{-2k}$. Define F_n for $n \in \mathbb{N}$ by requiring that

$$F_n(y_k) = 2^{-(2k+1)\alpha} \quad (k \ge n), \qquad F_n(x_k) = 0 \quad (k \in \mathbb{N}),$$

and that F_n be linear on all intervals between the points at which it has been defined (with $F_n(0) = 0$). For each $n \in \mathbb{N}$, we have $F_n \in \mathfrak{A}$ with $\|F_n\|_\alpha \le 2$, $|F_n|_{\mathbb{I}} = 2^{-(2n+1)\alpha}$, and $\widetilde{F}_n(x_k, y_k) = 1$ for $k \ge n$. Further, $\|F_n\|_\alpha \to 0$ as $n \to \infty$. The sequence $((F_n, \widetilde{F}_n) : n \in \mathbb{N})$ has a weak-$*$ accumulation point, say (Φ_0, Ψ_0), in $\mathfrak{A}''$. For each $\mu \in M(\mathbb{I})$, $|\langle F_n, \mu\rangle| \le \|F_n\|_\alpha \|\mu\| \to 0$, and so $\Phi_0 = 0$. Take $\mathfrak{p} \in \beta V \setminus V$ to be an accumulation point of the set $\{(x_k, y_k) : k \in \mathbb{N}\}$, and define

$$\varepsilon_{\mathfrak{p}} : G \mapsto G(\mathfrak{p}), \qquad C(\beta V) \to \mathbb{C}.$$

Then $\langle \widetilde{F}_n, \varepsilon_{\mathfrak{p}}\rangle = 1$ $(n \in \mathbb{N})$, and so $\langle \Psi_0, \varepsilon_{\mathfrak{p}}\rangle = 1$. Hence $(0, \Psi_0) \ne (0,0)$.

Since $(0,\Psi_0)^2 = (0,0)$ and $\mathfrak{A}''$ is commutative, $(0,\Psi_0) \in \mathrm{rad}\,\mathfrak{A}''$, and so $\mathfrak{A}''$ is not semisimple.

(iii) By [33], there are non-zero, continuous point derivations on $\mathfrak{A}$, and so, by Proposition 1.3(ii), $\mathfrak{A}$ is not weakly amenable.

Define

$$D : F \mapsto (0, \widetilde{F}(\mathfrak{p})\Psi_0), \qquad \mathfrak{A} \to \mathfrak{A}'',$$

where $\mathfrak{p}$ and Ψ_0 are as in (ii). Then D is a continuous linear operator. Clearly, $F \cdot \Psi_0 = F(0)\Psi_0$ for $F \in \mathfrak{A}$, and

$$\widetilde{FG}(\mathfrak{p}) = F(0)\widetilde{G}(\mathfrak{p}) + \widetilde{F}(\mathfrak{p})G(0)$$

for $F, G \in \mathfrak{A}$, and so D is a derivation.

Let F_1 be as in (ii). Then $\widetilde{F}_1(\mathfrak{p}) = 1$, and so $D(F_1) = (0, \Psi_0) \ne (0,0)$. This shows that D is not zero, and hence that $\mathfrak{A}$ is not 2-weakly amenable.

By Proposition 1.2, $\mathfrak{A}$ is not n-weakly amenable for any $n \in \mathbb{N}$. ■

Thus, in the case where $\alpha \le 1/2$, $\mathrm{lip}_\alpha \mathbb{T}$ is 4-weakly amenable, although $(\mathrm{lip}_\alpha \mathbb{T})'' = \mathrm{Lip}_\alpha \mathbb{T}$ is not 2-weakly amenable. We do not know whether or not $\mathrm{lip}_\alpha K$ is always 4-weakly amenable.

We record a related result. For a function $f \in L^1(\mathbb{T})$, the associated Fourier series is $(\widehat{f}(n) : n \in \mathbb{Z})$. For $\alpha > 0$, the associated Beurling algebra $A_\alpha(\mathbb{T})$ on $\mathbb{T}$ consists of the continuous functions f on $\mathbb{T}$ such that

$$\|f\|_\alpha = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|(1+|n|)^\alpha < \infty.$$

PROPOSITION 3.7. *Let $\alpha > 0$. Then:*

(i) *$A_\alpha(\mathbb{T})$ is Arens regular;*
(ii) *$A_\alpha(\mathbb{T})$ is weakly amenable if and only if $\alpha < 1/2$;*
(iii) *$A_\alpha(\mathbb{T})''$ is not semisimple;*
(iv) *$A_\alpha(\mathbb{T})$ is 2-weakly amenable if and only if $\alpha < 1$.*

Proof. (i) This is [6, Corollary 2].
(ii) This is [2, Theorem 2.4].
(iii) This is [25, Theorem 2.1.7].
(iv) This is [25, Theorems 3.1.1 and 3.1.3]. ■

We conclude this section with an example promised after Corollary 1.14 of a Banach algebra which is an ideal in its second dual, and yet is not n-weakly amenable for any $n \in \mathbb{N}$.

EXAMPLE 3.8. First observe that, as we remarked after Proposition 1.2, a commutative Banach algebra $\mathfrak{A}$ such that $\mathcal{H}^1(\mathfrak{A},\mathfrak{A}) \neq \{0\}$ is not n-weakly amenable for any $n \in \mathbb{N}$.

Let $\mathcal{V}$ be the Volterra algebra, i.e., $\mathcal{V} = L^1(0,1)$, with the convolution product

$$(f \star g)(x) = \int_0^x f(x-t)g(t)\,dt \quad (f,g \in \mathcal{V},\ x \in (0,1)).$$

Then $\mathcal{H}^1(\mathcal{V},\mathcal{V}) \neq \{0\}$ ([24]), and so $\mathcal{V}$ is not n-weakly amenable for any $n \in \mathbb{N}$. For each $f \in \mathcal{V}$, the operator

$$g \mapsto f \star g, \quad \mathcal{V} \to \mathcal{V},$$

is compact (see [11, Theorem 1]), and hence weakly compact. This shows that $\mathcal{V}$ is an ideal in its second dual, and so $\mathcal{V}$ is a Banach algebra with the desired properties.

We remark that it follows from [12] that the Volterra algebra $\mathcal{V}$ is not Arens regular.

4. Group algebras. Let G be a locally compact group. We have noted that $L^1(G)$ is amenable if and only if G is an amenable group, and that $L^1(G)$ is always weakly amenable; we shall extend this latter result. Note that $L^1(G)$ is not Arens regular unless G is finite (see [36]).

THEOREM 4.1. *Let G be a locally compact group. Then $L^1(G)$ is $(2k+1)$-weakly amenable for each $k \in \mathbb{Z}^+$.*

Proof. Set $\mathfrak{A} = L^1(G)$. Then $\mathfrak{A}$ contains a bounded approximate identity (e_γ) with $\|e_\gamma\| \le 1$ for each γ. Thus there exists $E \in \mathfrak{A}''$ such that $\|E\| = 1$ and E is a right identity for $(\mathfrak{A}'', \Box)$ ([3, Proposition 28.7]).

Let $M(G)$ be the Banach algebra of all measures on G, with convolution multiplication; it is standard that $\mathfrak{A}$ is a closed ideal in $M(G)$, and so $(\mathfrak{A}'', \Box)$ is a closed ideal in $(M(G)'', \Box)$. The map

$$\theta : \mu \mapsto E \Box \mu, \quad M(G) \to (\mathfrak{A}'', \Box),$$

is an isometric embedding. We write E_s for $E \Box \delta_s$ when $s \in G$, so that $E_{st} = E_s E_t$ $(s,t \in G)$.

Our proof is now a development of the argument in [7] that $\mathfrak{A}$ is weakly amenable. The result in [7] establishes the case where $k = 0$, and so we may suppose that $k \in \mathbb{N}$.

Set $X = \mathfrak{A}^{(2k+2)}$. Then X' is the underlying space of a commutative von Neumann algebra, and hence it is an L^∞-space; we have the standard notion of the real part of an element in X'. The real-valued functions in X' form the space $X'_{\mathbb{R}}$; $X'_{\mathbb{R}}$ is a complete lattice in the sense that every non-empty subset of $X'_{\mathbb{R}}$ which is bounded above has a supremum. It is immediately checked that, for each $s \in G$ and $\Lambda \in X'$,

$$\mathrm{Re}(E_s \cdot \Lambda) = E_s \cdot (\mathrm{Re}\,\Lambda). \tag{4.1}$$

Let $D \in \mathcal{Z}^1(\mathfrak{A}, X')$. Then $D'' : (\mathfrak{A}'', \Box) \to X'''$ is a continuous derivation by Proposition 1.7. For $s,t \in G$, we have

$$D''(E_{st}) = D''(E_s) \cdot E_t + E_s \cdot D''(E_t),$$

and so

$$E_{(st)^{-1}} \cdot D''(E_{st}) = E_{t^{-1}} \cdot (E_{s^{-1}} \cdot D''(E_s)) \cdot E_t + E_{t^{-1}} \cdot D''(E_t). \tag{4.2}$$

We next show that, for each $r \in G$ and each bounded subset A of $X'_{\mathbb{R}}$, we have

$$E \cdot \sup\{E_r \cdot \Lambda : \Lambda \in A\} = E_r \cdot \sup\{E \cdot \Lambda : \Lambda \in A\}. \tag{4.3}$$

Indeed, set $\alpha = \sup\{E \cdot \Lambda : \Lambda \in A\}$ and $\beta = \sup\{E_r \cdot \Lambda : \Lambda \in A\}$. Then $E_r \cdot \Lambda \le E_r \cdot \alpha$, and so $E \cdot \beta \le E_r \cdot \alpha$. For the reverse inequality, we have

$$\alpha = \sup\{E_{r^{-1}} \cdot E_r \cdot E \cdot \Lambda : \Lambda \in A\} \le E_{r^{-1}} \cdot E \cdot \beta,$$

and so $E_r \cdot \alpha \le E \cdot \beta$. Thus (4.3) holds. Similarly, we have

$$\sup\{E_r \cdot \Lambda : \Lambda \in A\} \cdot E = \sup\{E \cdot \Lambda : \Lambda \in A\} \cdot E_r. \tag{4.4}$$

Define

$$\Phi = \sup\{E_{s^{-1}} \cdot \mathrm{Re}\, D''(E_s) : s \in G\},$$

the supremum being taken in the complete lattice $X'_{\mathbb{R}}$. Let $t \in G$. Then it follows from (4.1)–(4.4) that

$$E \cdot \Phi \cdot E = E_{t^{-1}} \cdot \Phi \cdot E_t + E_{t^{-1}} \cdot \operatorname{Re} D''(E_t) \cdot E.$$

Hence

$$E \cdot \operatorname{Re} D''(E_t) \cdot E = E_t \cdot \Phi \cdot E - E \cdot \Phi \cdot E_t.$$

A similar result holds for the imaginary part of $D''(E_t)$, and so we see that there exists $\Psi \in X'''$ such that

$$E \cdot D''(E_t) \cdot E = E_t \cdot \Psi \cdot E - E \cdot \Psi \cdot E_t \quad (t \in G).$$

It follows that, for each discrete measure $\nu \in \ell^1(G)$, we have

$$E \cdot D''(E \,\Box\, \nu) \cdot E = (E \,\Box\, \nu) \cdot \Psi \cdot E - E \cdot \Psi \cdot \nu \cdot E.$$

Now let $f, g \in \mathfrak{A}$, and recall that E is a right identity of $(\mathfrak{A}'', \Box)$; we have

$$f \cdot D''(E \,\Box\, \nu) \cdot g = (f * \nu) \cdot \Psi \cdot g - f \cdot \Psi \cdot (\nu * g). \tag{4.5}$$

Next, take $h \in \mathfrak{A}$. Then there is a net (ν_γ) of discrete measures such that $\nu_\gamma \to h$ in the (two-sided) strong operator topology on $\mathfrak{A}$, that is, $\lim_\gamma (f * \nu_\gamma) = f * h$ and $\lim_\gamma (\nu_\gamma * g) = h * g$ for each $f, g \in \mathfrak{A}$. Let $f, g \in \mathfrak{A}$. Then

$$\begin{aligned}\lim_\gamma f \cdot D''(E \,\Box\, \nu_\gamma) \cdot g &= \lim_\gamma (D''(f * \nu_\gamma) \cdot g - D''(f) \cdot (\nu_\gamma * g)) \\ &= D''(f * h) \cdot g - D''(f) \\ &= f \cdot D''(h) \cdot g,\end{aligned}$$

and so, from (4.5),

$$\begin{aligned}f \cdot D''(h) \cdot g &= (f * h) \cdot \Psi \cdot g - f \cdot \Psi \cdot (h * g) \\ &= f \cdot (h \cdot \Psi - \Psi \cdot h) \cdot g.\end{aligned}$$

Let $P : X''' \to X' = \mathfrak{A}^{(2k+1)}$ be the natural projection, so that P is an $\mathfrak{A}$-bimodule morphism. We have $D = P \circ D''$. Set $\Psi_0 = P(\Psi)$. Then

$$f \cdot D(h) \cdot g = f \cdot (h \cdot \Psi_0 - \Psi_0 \cdot h) \cdot g \quad (f, g, h \in \mathfrak{A}),$$

and so

$$\langle f \cdot x \cdot g, D(h) \rangle = \langle f \cdot x \cdot g, h \cdot \Psi_0 - \Psi_0 \cdot h \rangle \quad (f, g, h \in \mathfrak{A},\ x \in X).$$

It now follows from Proposition 1.17 that $D \in N^1(\mathfrak{A}, X')$, and so $\mathfrak{A}$ is $(2k+1)$-weakly amenable. ■

We should next discuss the $2k$-weak amenability, and especially the 2-weak amenability, of $L^1(G)$ in the case where G is non-amenable. Unfortunately, we cannot resolve this question for any non-amenable group G. In particular, we cannot resolve the question whether or not $\ell^1(\mathbb{F}_2)$ is 2-weakly amenable, where $\mathbb{F}_2$ is the free group on two generators. The space $\ell^1(\mathbb{F}_2)''$ is naturally identified with $M(\beta\mathbb{F}_2)$, the Banach space of all measures on the Stone–Čech compactification $\beta\mathbb{F}_2$ of $\mathbb{F}_2$, with the induced module actions from $\ell^1(\mathbb{F}_2)$; we are asking whether or not

$$\mathcal{H}^1(\ell^1(\mathbb{F}_2), M(\beta\mathbb{F}_2)) = \{0\}.$$

Let $\ell^1(\beta\mathbb{F}_2)$ be the submodule of $M(\beta\mathbb{F}_2)$ consisting of the discrete measures. Then it is true that $\mathcal{H}^1(\ell^1(\mathbb{F}_2), \ell^1(\beta\mathbb{F}_2)) = \{0\}$; this can be proved by an application of Proposition 3.7 of [20], taking the auxiliary uniformly convex norm p on $\ell^1(\beta\mathbb{F}_2)$ to be the relative norm from $\ell^2(\beta\mathbb{F}_2)$.

5. Algebras of operators. We shall consider certain Banach algebras which are subalgebras of the Banach algebra $\mathcal{B}(E)$ for a Banach space E.

So far, we have no counter-example to the possibility that a weakly amenable Banach algebra is permanently weakly amenable; for example, the implication holds for commutative Banach algebras; all C^*-algebras are permanently weakly amenable; it is possible that all group algebras are permanently weakly amenable. However, we shall shortly exhibit a Banach subalgebra of $\mathcal{B}(E)$ which is weakly amenable, but not 2-weakly amenable.

Let E and F be Banach spaces. We denote by $\mathcal{F}(E,F)$ the linear subspace of $\mathcal{B}(E,F)$ consisting of the continuous, finite-rank operators. Then $\mathcal{F}(E,F)$ is identified with $F \otimes E'$; the rank-one operator corresponding to $y \otimes \lambda$ (where $y \in F$ and $\lambda \in E'$) is the map

$$y \otimes \lambda : x \mapsto \langle x, \lambda \rangle y, \quad E \to F.$$

In particular, $E \otimes E'$ is identified with $\mathcal{F}(E)$; the corresponding product in $E \otimes E'$ is defined by

$$(x_1 \otimes \lambda_1)(x_2 \otimes \lambda_2) = \langle x_2, \lambda_1 \rangle\ x_1 \otimes \lambda_2.$$

The completion $E \widehat{\otimes} E'$ of $E \otimes E'$ in the projective norm $\|\cdot\|_\pi$ is the *tensor algebra* of E; the tensor algebra is a Banach algebra ([19, II. 2.20]).

The natural identification of $F \otimes E'$ with $\mathcal{F}(E,F)$ extends to a continuous linear map $\mathcal{R} : F \widehat{\otimes} E' \to \mathcal{B}(E,F)$; the range of this map, with the quotient norm, is the Banach space $(\mathcal{N}(E,F), \|\cdot\|_{\mathcal{N}})$ of *nuclear operators* from E to F. Let $S \in \mathcal{N}(E,F)$, and let G be a Banach space. If $T \in \mathcal{B}(G,E)$, then $S \circ T \in \mathcal{N}(G,F)$ and $\|S \circ T\|_{\mathcal{N}} \le \|S\|_{\mathcal{N}} \|T\|$; if $T \in \mathcal{B}(F,G)$, then $S \circ T \in \mathcal{N}(E,G)$ and $\|T \circ S\|_{\mathcal{N}} \le \|S\|_{\mathcal{N}} \|T\|$.

In particular, $\mathcal{R} : E \widehat{\otimes} E' \to \mathcal{N}(E)$ identifies the nuclear operators on E as a Banach algebra; the map $\mathcal{R}$ is an injection (and so $\mathcal{N}(E)$ is isometrically isomorphic to the tensor algebra $E \widehat{\otimes} E'$) if and only if E has the approximation property (AP). The Banach algebra $(\mathcal{N}(E), \|\cdot\|_{\mathcal{N}})$ is a Banach $\mathcal{B}(E)$-bimodule. We note that E has AP when E' has AP. For further details concerning these remarks, see [26, 1.7.10/11].

Let E be a Banach space. For $T \in \mathcal{B}(E')$, define $\widehat{T} \in (E \widehat{\otimes} E')'$ by the requirement that

$$\widehat{T}(x \otimes \lambda) = \langle x, T(\lambda) \rangle \quad (x \in E,\ \lambda \in E').$$

Then

$$T \mapsto \widehat{T}, \quad \mathcal{B}(E') \to (E \widehat{\otimes} E')',$$

is an isometric linear homomorphism. The linear functional corresponding to the identity operator is called the *canonical trace* on $E \widehat{\otimes} E'$ and is denoted by tr. For details, see [26, 1.7.11].

A Banach algebra A is *simplicially trivial* if $\mathcal{H}^n(A, A') = \{0\}$ for each $n \in \mathbb{N}$; in this case A is weakly amenable.

THEOREM 5.1. *Let E be a Banach space. Then the tensor algebra $E \widehat{\otimes} E'$ is weakly amenable.*

Proof. It is a theorem of Selivanov ([31], [19, IV.5.11]) that $E \widehat{\otimes} E'$ is biprojective. Every biprojective Banach algebra is weakly amenable; indeed, such an algebra is simplicially trivial. ■

THEOREM 5.2. *Let E be an infinite-dimensional Banach space with E' having the approximation property. Then the weakly amenable Banach algebra $E \widehat{\otimes} E' = \mathcal{N}(E)$ is not 2-weakly amenable.*

Proof. Set $\mathfrak{A} = E \widehat{\otimes} E' = \mathcal{N}(E)$, and set $X = \mathfrak{A}' = \mathcal{B}(E')$ (with the above identification). It is easily seen that the module actions of $\mathfrak{A}$ on $\mathfrak{A}'$ are given by

$$S \cdot T = T \circ S', \quad T \cdot S = S' \circ T \quad (S \in \mathfrak{A},\ T \in X).$$

We shall prove that $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}'') \neq \{0\}$ by showing that $\operatorname{im} d_1$ is not closed in X, where d_1 is the map defined in equation (1.12); the result will then follow from Proposition 1.16.

In the present notation, the map $d_1 : \mathfrak{A} \widehat{\otimes} X \to X$ is defined by the condition that

$$d_1 : S \otimes T \mapsto T \circ S' - S' \circ T, \quad \mathfrak{A} \widehat{\otimes} X \to X.$$

To obtain a contradiction, assume that d_1 has closed range in $\mathcal{B}(E')$.

We first note that $\operatorname{im} d_1 \subseteq \mathcal{N}(E')$. For let $U \in \mathfrak{A} \widehat{\otimes} X$. Then we can write $U = \sum_{n=1}^{\infty} S_n \otimes T_n$, with $S_n \in \mathfrak{A}$, $T_n \in X$, and $\sum_{n=1}^{\infty} \|S_n\|_{\mathcal{N}} \|T_n\| < \infty$. But then

$$\sum_{n=1}^{\infty} \|S_n' \circ T_n - T_n \circ S_n'\|_{\mathcal{N}} \le 2 \sum_{n=1}^{\infty} \|S_n\|_{\mathcal{N}} \|T_n\| < \infty,$$

and so $d_1(U) \in \mathcal{N}(E')$, as required.

Consider the map $\Phi : \mathfrak{A} \widehat{\otimes} \mathfrak{A} \to \mathcal{B}(E)$ given by

$$\Phi(S \otimes U) = S \circ U - U \circ S \quad (S, U \in \mathfrak{A}).$$

Clearly, $\{A' : A \in \operatorname{im} \Phi\} \subseteq \operatorname{im} d_1$. We want to show that $\operatorname{im} \Phi$ is closed in $\mathcal{B}(E)$. Towards this, take (A_n) in $\operatorname{im} \Phi$ such that $A_n \to A$ in the operator norm on E. Since $\operatorname{im} d_1$ is closed by our assumption, and since the map

$$T \mapsto T', \quad \mathcal{B}(E) \to \mathcal{B}(E'),$$

is an isometry, it follows that A' is nuclear. But then A is nuclear since E' has AP [8, Theorem 7, p. 243]. Since $\operatorname{tr} A = \operatorname{tr} A' = 0$, it follows that $A \in \operatorname{im} \Phi$. To see this, we write $A = \mathcal{R}(\sum x_n \otimes \lambda_n)$, where the map $\mathcal{R} : E \widehat{\otimes} E' \to \mathcal{N}(E)$ was defined at the beginning of this section. Choose $x_0 \in E$ and $\lambda_0 \in E'$ such that $\langle x_0, \lambda_0 \rangle = 1$. Then

$$A = \Phi\Big(\sum \mathcal{R}(x_n \otimes \lambda_0) \otimes \mathcal{R}(x_0 \otimes \lambda_n)\Big) \in \operatorname{im} \Phi.$$

Thus $\operatorname{im} \Phi$ is closed. Since E' has AP, and therefore E has AP, we may suppose that $\operatorname{im} \Phi \not\supseteq \mathcal{F}(E)$.

It now follows that there is a continuous linear functional Λ on $\mathcal{B}(E)$ such that $\Lambda | \operatorname{im} \Phi = 0$ and $\Lambda | \mathcal{F}(E) \neq 0$. But then Λ is a non-zero bounded trace on $\mathcal{F}(E)$. This is a contradiction of [27, Theorem 1.14].

We deduce that $\operatorname{im} d_1$ is not closed in $\mathfrak{B}(E')$, and so $\mathcal{N}(E)$ is not 2-weakly amenable. ■

REMARK. Suppose that E' fails to have the approximation property. Then the map

$$d_1 : \mathcal{N}(E) \widehat{\otimes} \mathcal{B}(E') \to \mathcal{N}(E')$$

is surjective. To see this first note that, since E' fails to have AP, there exists $u_0 \in E' \widehat{\otimes} E''$ such that $\mathcal{R}(u_0) = 0$, but $\operatorname{tr} u_0 \neq 0$. Let $u \in E' \widehat{\otimes} E''$ be arbitrary, and define

$$v = u - \frac{\operatorname{tr} u}{\operatorname{tr} u_0} u_0.$$

Then $\mathcal{R}(v) = \mathcal{R}(u)$ and $\operatorname{tr} v = 0$. Write

$$v = \sum \lambda_n \otimes \Lambda_n \quad (\lambda_n \in E',\ \Lambda_n \in E''),$$

and then choose $x_0 \in E$ and $\lambda_0 \in E'$ such that $\langle x_0, \lambda_0 \rangle = 1$. Then

$$v = \sum_n ((\lambda_n \otimes \widehat{x}_0)(\lambda_0 \otimes \Lambda_n) - (\lambda_0 \otimes \Lambda_n)(\lambda_n \otimes \widehat{x}_0)),$$

so that $\mathcal{R}(v) \in \operatorname{im} d_1$. There are infinite-dimensional Banach spaces for which all approximable operators are nuclear [27, Theorem 10.6]. Thus it seems that the hypothesis that E' has AP cannot be essentially relaxed.

We can obtain somewhat stronger results if we assume a little more about the Banach space E.

PROPOSITION 5.3. *Let E be a Banach space.*

(i) *Suppose that $E = F'$ for a Banach space F, and suppose that E has AP and the Radon–Nikodym property. Then $\mathcal{N}(E)$ is Arens regular.*

(ii) *Suppose that E is a reflexive space with AP. Then $\mathcal{N}(E)$ is a closed ideal in $(\mathcal{N}(E)'', \Box)$.*

Proof. Throughout, set $\mathfrak{A} = \mathcal{N}(E)$.

(i) Let $\mathcal{A}(F)$ denote the Banach algebra of approximable operators on F. Under the stated conditions on E, $\mathcal{A}(F)'$ is identified with $\mathcal{N}(E)$ (see [8, p. 248]): the duality is "trace duality", given by

$$\langle S, T\rangle = \operatorname{tr}(TS') \quad (S \in \mathcal{A}(E),\ T \in \mathfrak{A}),$$

where tr denotes the trace. Also, $\mathfrak{A}'$ is identified with $\mathcal{B}(E')$, so that $\mathcal{A}(F)''$ is identified with $\mathcal{B}(E')$.

We apply Proposition 1.6, taking X to be the canonical image of $\mathcal{A}(F)$ in $\mathcal{B}(E')$. The embedding of $\mathcal{A}(F)$ in $\mathcal{B}(E')$ is just the map $T \mapsto T'' \in \mathcal{A}(F'')$, and so certainly $\mathfrak{A} \cdot \mathfrak{A}' + \mathfrak{A}' \cdot \mathfrak{A} \subseteq X$. Hence, by Proposition 1.6, $\mathfrak{A}$ is Arens regular.

(ii) As we remarked in §1, $\mathcal{N}(E)$ is an ideal in $(\mathcal{N}(E)'', \Box)$ if and only if both the maps $L_T : S \mapsto TS$ and $R_T : S \mapsto ST$ from $\mathfrak{A}$ to $\mathfrak{A}$ are weakly compact for each $T \in \mathfrak{A}$. Since $\mathcal{F}(E)$ is dense in $(\mathfrak{A}, \|\cdot\|_{\mathcal{N}})$ and the set of weakly compact operators on a Banach space Y is a closed ideal in $\mathcal{B}(Y)$, it is sufficient to show that L_T and R_T are both weakly compact in the case where T is a rank-one operator, say $T = x_0 \otimes \lambda_0$, with $x_0 \in E$ and $\lambda_0 \in E'$. We note that

$$(x_0 \otimes \lambda_0) \circ S = x_0 \otimes S'(\lambda_0), \qquad S \circ (x_0 \otimes \lambda_0) = S(x_0) \otimes \lambda_0$$

for $S \in \mathcal{B}(E)$.

Let (S_γ) be a bounded net in $\mathcal{N}(E)$.

First, $(S'_\gamma(\lambda_0))$ is a bounded net in E', and so, by passing to a subnet, we may suppose that $S'_\gamma(\lambda_0) \to \lambda_1$ in $(E', \sigma(E', E))$. For each $U \in \mathcal{B}(E')$,

$$\langle x_0 \otimes S'_\gamma(\lambda_0), U\rangle = \langle x_0, US'_\gamma(\lambda_0)\rangle = \langle U'x_0, S'_\gamma(\lambda_0)\rangle \to \langle U'x_0, \lambda_1\rangle,$$

noting that $U' \in \mathcal{B}(E)$ because E is reflexive. Thus $L_{x_0 \otimes \lambda_0}$ is weakly compact.

Second, $(S_\gamma(x_0))$ is a bounded net in E, and so we may suppose that $S_\gamma(x_0) \to \Lambda_1$ in $(E, \sigma(E, E'))$. For each $U \in \mathcal{B}(E')$,

$$\langle S_\gamma(x_0) \otimes \lambda_0, U\rangle = \langle S_\gamma(x_0), U(\lambda_0)\rangle \to \langle \Lambda_1, U(\lambda_0)\rangle.$$

Thus $R_{x_0 \otimes \lambda_0}$ is weakly compact.

It follows that $\mathcal{N}(E)$ is an ideal in $\mathcal{N}(E)''$. ∎

The next statement combines some previous results; the result contrasts $\mathcal{N}(E)$ with the disc algebra $A(\overline{\mathbb{D}})$, in that the Banach algebra $\mathcal{N}(E)$ is k-weakly amenable precisely in the cases where $A(\overline{\mathbb{D}})$ is not k-weakly amenable.

COROLLARY 5.4. *Let E be a reflexive space with AP. Then:*

(i) *$\mathcal{N}(E)$ is Arens regular and $\mathcal{N}(E)$ is a closed ideal in $(\mathcal{N}(E)'', \Box)$;*

(ii) *$\mathcal{N}(E)$ is $(2n+1)$-weakly amenable for each $n \in \mathbb{Z}^+$;*

(iii) *$\mathcal{N}(E)$ is not $2n$-weakly amenable for any $n \in \mathbb{N}$.*

Proof. Again, set $\mathfrak{A} = \mathcal{N}(E)$.

(i) Each reflexive space has the Radon–Nikodym property, and so $\mathfrak{A}$ is Arens regular by Proposition 5.3(i). The second part of the statement is Proposition 5.3(ii).

(ii) By Theorem 5.2, $\mathcal{N}(E)$ is weakly amenable. The result now follows from (i) and Corollary 1.14.

(iii) Since E is reflexive and has AP, E' also has AP. Hence $\mathfrak{A}$ is not 2-weakly amenable by Theorem 5.2. The result follows from Proposition 1.2. ∎

Thus, for special Banach spaces E, $\mathcal{N}(E)$ is 3-weakly amenable; we do not know if this is the case just under the hypothesis that E' has AP.

We make one further remark about $\mathcal{N}(E)$: for each Banach space E,

$$\mathcal{H}^1(\mathcal{N}(E), \mathcal{N}(E)) = \mathcal{B}(E)/(\mathcal{N}(E) + \mathbb{C}I_E).$$

This is a special case of [19, III.4.19].

We shall now give some results on the Banach algebra of approximable operators $\mathcal{A}(E)$ on a Banach space E. Recall that $\mathcal{A}(E)$ is defined as the closure of $\mathcal{F}(E)$ in $(\mathcal{B}(E), \|\cdot\|)$, and that the dual $\mathcal{A}(E)'$ is identified with the space of integral operators $\mathcal{I}(E')$ on E' by means of trace duality:

$$\langle F, T\rangle = \operatorname{tr}(F'T) \quad (F \in \mathcal{F}(E),\ T \in \mathcal{I}(E')).$$

The integral norm $\|T\|_{\mathcal{I}}$ of $T \in I(E')$ is the norm of T as a bounded functional on $\mathcal{A}(E)$. (For details, see [26, 1.7.12].) Apart from the cases where $\mathcal{A}(E)$ is actually amenable [17], little is known about the weak amenability of $\mathcal{A}(E)$. We start by giving some instances where $\mathcal{A}(E)$ is weakly amenable, but not amenable.

We let $\mathcal{W}(E)$ denote the space of weakly compact operators on E; $\mathcal{W}(E)$ is a closed two-sided ideal of $\mathcal{B}(E)$.

LEMMA 5.5. *Let E be a Banach space such that E' has AP. Then there is a natural isometric embedding of $\mathcal{W}(E)$ into $\mathcal{A}(E)''$; this embedding is a Banach algebra homomorphism when $\mathcal{A}(E)''$ is equipped with either of its Arens products.*

Proof. Let $T \in \mathcal{A}(E)' = \mathcal{I}(E')$, and let $W \in \mathcal{W}(E)$. Then $W' \in \mathcal{W}(E')$. By [8, Lemma 8 and Corollary 9, pp. 250–251], there is a reflexive space Y

and a factorization

$$\begin{array}{ccc} E' & \xrightarrow{\;W'\;} & E' \\ & {\scriptstyle B}\searrow \quad \nearrow{\scriptstyle A} & \\ & Y & \end{array}$$

A close inspection of the proof shows that $\|W'\| = \inf\{\|A\|\cdot\|B\|\}$, where the infimum is taken over all such factorizations. Since the map B has its range in a reflexive space, BT is nuclear by [8, Theorem 8, p. 175]. Furthermore, the nuclear and integral norms coincide: $\|BT\|_{\mathcal{N}} = \|BT\|_{\mathcal{I}}$. (This is essentially the argument in [8, p. 252].) Since E' is assumed to have AP, the application of tr is valid. Using this, we see that

$$|\mathrm{tr}(W'T)| = |\mathrm{tr}(ABT)| \le \|A\|\,\|BT\|_{\mathcal{N}} = \|A\|\,\|BT\|_{\mathcal{I}} \le \|A\|\,\|B\|\,\|T\|_{\mathcal{I}},$$

so that $|\mathrm{tr}(W'T)| \le \|W\|\,\|T\|_{\mathcal{I}}$. It follows that W defines a linear functional of norm not exceeding $\|W\|$ on $\mathcal{A}(E)'$. By applying W to a rank-one operator, we see that the norm of the functional defined by W is at least $\|W\|$. We have thus shown that the embedding of $W(E)$ into $\mathcal{A}(E)''$ is an isometry.

To show that the above embedding is a homomorphism when $\mathcal{A}(E)''$ has either of its Arens products, take $V, W \in \mathcal{W}(E)$ and $T \in \mathcal{I}(E')$, and let (F_α), (G_β) be two nets in $\mathcal{A}(E)$ converging in the weak-$*$ topology of $\mathcal{A}(E)''$ to V and W, respectively. Then

$$\begin{aligned}
\lim_\alpha \lim_\beta \langle T, F_\alpha G_\beta\rangle &= \lim_\alpha \lim_\beta \mathrm{tr}(G'_\beta(F'_\alpha T)) = \lim_\alpha \lim_\beta \langle F'_\alpha T, G_\beta\rangle \\
&= \lim_\alpha \langle F'_\alpha T, W\rangle = \lim_\alpha \mathrm{tr}(W' F'_\alpha T) \\
&= \lim_\alpha \mathrm{tr}(F'_\alpha T W') = \lim_\alpha \langle TW', F_\alpha\rangle \\
&= \langle TW', V\rangle = \mathrm{tr}(V'TW') = \mathrm{tr}(W'V'T) \\
&= \langle T, VW\rangle = \lim_\beta \mathrm{tr}(G'_\beta V'T) \\
&= \lim_\beta \mathrm{tr}(V'(TG'_\beta)) = \lim_\beta \langle TG'_\beta, V'\rangle \\
&= \lim_\beta \lim_\alpha \langle TG'_\beta, F_\alpha\rangle = \lim_\beta \lim_\alpha \langle T, F_\alpha G_\beta\rangle.
\end{aligned}$$

We have here several times used the fact that, if $T\colon E' \to E'$ is integral and $U: E' \to E'$ is weakly compact, then UT and TU are both nuclear, and therefore $\mathrm{tr}(RUT) = \mathrm{tr}(UTR)$ and $\mathrm{tr}(RTU) = \mathrm{tr}(TUR)$ for any bounded operator $R: E' \to E'$.

By the formulae in §1, $VW = V \,\square\, W = V \diamond W$, and so the embedding is indeed a homomorphism. ∎

THEOREM 5.6. *Let E be a Banach space such that E' has the approximation property, and let $D: \mathcal{A}(E) \to \mathcal{A}(E)'$ be a derivation. Then there exists*

a derivation $\widetilde{D}: \mathcal{W}(E) \to \mathcal{W}(E)'$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{A}(E) & \xhookrightarrow{\;i\;} & \mathcal{W}(E) \\ {\scriptstyle D}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{D}} \\ \mathcal{A}(E)' & \xleftarrow{\;i'\;} & \mathcal{W}(E)' \end{array}$$

In particular, $\mathcal{A}(E)$ is weakly amenable in the case where $\mathcal{W}(E)$ is weakly amenable.

Proof. To keep track of the various dual spaces involved, we stipulate the following versions of our notation. The canonical projection from $\mathcal{A}(E)'''$ onto $\mathcal{A}(E)'$ is $P: \mathcal{A}(E)''' \to \mathcal{A}(E)'$, whereas $\widehat{\ }: \mathcal{A}(E)' \to \mathcal{A}(E)'''$ is the canonical embedding. The isometric embedding given by the previous lemma is $j: \mathcal{W}(E) \to \mathcal{A}(E)''$. Note that $P = i' \circ j'$ and that $P(\widehat{\lambda}) = \lambda$ ($\lambda \in \mathcal{A}(E)'$).

By Proposition 1.7, the map $D'': \mathcal{A}(E)'' \to \mathcal{A}(E)'''$ is a continuous derivation when $\mathcal{A}(E)'''$ has the $(\mathcal{A}(E)'', \square)$-module structure given in (1.5) and (1.9). We start by examining the module multiplication when it is restricted to $j(\mathcal{W}(E))$. Let $F, G \in \mathcal{W}(E)$ and let $\Lambda \in \mathcal{A}(E)'''$. Choose (a_α) in $\mathcal{A}(E)$ and (λ_β) in $\mathcal{A}(E)'$ such that $a_\alpha \to j(F)$ in $\sigma(\mathcal{A}(E)'', \mathcal{A}(E)')$ and $\lambda_\beta \to \Lambda$ in $\sigma(\mathcal{A}(E)''', \mathcal{A}(E)'')$. Then

$$\begin{aligned}
\langle j(G), j(F)\cdot\Lambda\rangle &= \lim_\alpha \lim_\beta \langle a_\alpha \lambda_\beta, j(G)\rangle = \lim_\alpha \lim_\beta \langle G\cdot a_\alpha, \lambda_\beta\rangle \\
&= \lim_\alpha \langle G\cdot a_\alpha, \Lambda\rangle = \lim_\alpha \langle G\cdot a_\alpha, P(\Lambda)\rangle \\
&= \langle P(\Lambda), j(G)\diamond j(F)\rangle = \langle P(\Lambda), j(GF)\rangle,
\end{aligned}$$

where the application of P follows because $G\cdot a_\alpha \in \mathcal{W}(E)\cdot\mathcal{A}(E) \subseteq \mathcal{A}(E)$, and the last step follows because j is an algebra homomorphism into $(\mathcal{A}(E)'', \diamond)$. Similarly, we have

$$\begin{aligned}
\langle j(G), \Lambda\cdot j(F)\rangle &= \lim_\beta \lim_\alpha \langle \lambda_\beta a_\alpha, j(G)\rangle \\
&= \lim_\beta \langle \lambda_\beta, j(F)\,\square\, j(G)\rangle = \langle j(FG), \Lambda\rangle.
\end{aligned}$$

It follows that $P \circ D'' \circ j: \mathcal{W}(E) \to \mathcal{A}(E)'$ is a derivation when $\mathcal{A}(E)'$ has its natural $\mathcal{W}(E)$-bimodule structure arising from the fact that $\mathcal{A}(E)$ is an ideal in $\mathcal{W}(E)$.

Now, as above, let $F, G \in \mathcal{W}(E)$ and $\lambda \in \mathcal{A}(E)'$. Then

$$\begin{aligned}
\langle F, j'(\widehat{G\cdot\lambda})\rangle &= \langle j(F), \widehat{G\cdot\lambda}\rangle = \lim_\alpha \langle a_\alpha, G\cdot\lambda\rangle \\
&= \langle \lambda, j(F)\,\square\, j(G)\rangle = \langle FG, j'(\widehat{\lambda})\rangle.
\end{aligned}$$

Similarly, $\langle F, j'(\widehat{\lambda\cdot G})\rangle = \langle GF, j'(\widehat{\lambda})\rangle$. We have shown that the map

$$\lambda \mapsto j'(\widehat{\lambda}), \qquad \mathcal{A}(E)' \to \mathcal{W}(E)',$$

is a $\mathcal{W}(E)$-bimodule map. The map given by

$$\widetilde{D}(F) = j'((P \circ D'' \circ j)(F)^{\frown}) \quad (F \in \mathcal{W}(E))$$

is therefore a derivation satisfying $i' \circ \widetilde{D} \circ i = D$. ■

We now give a condition ensuring that $\mathcal{W}(E)$ is weakly amenable.

PROPOSITION 5.7. *Suppose that the Banach space E is reflexive and has the form $E = \ell_p(Y)$ for some (reflexive) Banach space Y and that $1 < p < \infty$. Then $\mathcal{W}(E)$ $(= \mathcal{B}(E))$ is simplicially trivial.*

Proof. Since $E \cong \ell_p(E)$, we may copy the proof of [35, Proposition 5] for the case where E is a Hilbert space to show that the Hochschild homology $\mathcal{H}_n(\mathcal{W}(E), \mathcal{W}(E)) = \{0\}$ for $n \in \mathbb{Z}^+$. The conclusion now follows from [20, Corollary 1.3]. ■

COROLLARY 5.8. *Suppose that E, in addition to satisfying the hypotheses of Proposition 5.7, has the approximation property. Then $\mathcal{A}(E)$ is weakly amenable.* ■

EXAMPLE. Let C_p, $p = 0$ or $1 \le p \le \infty$, be one of the spaces defined by W. B. Johnson [23]. These spaces have the property that every approximable operator between Banach spaces factors through C_p, i.e., if $S : E \to F$ is approximable, then there are approximable operators $T_1 : E \to C_p$, and $T_2 : C_p \to F$ such that $S = T_2 \circ T_1$. Furthermore, for $1 < p < \infty$, C_p satisfies the hypotheses of Corollary 5.8. As in Example IV.11 of [16] it then follows that all Banach algebras $\mathcal{A}(E \oplus C_p)$, where E' has the bounded approximation property and $p = 0$ or $1 \le p < \infty$, are Morita equivalent. In particular, they are all weakly amenable. Note that $\mathcal{A}(C_p)$ is not amenable for $p = 0, 1$ ([17, Question 7.7]).

We now turn to the 2-weak amenability of $\mathcal{A}(E)$. First we seek a description of $\mathcal{H}^1(\mathcal{A}(E), \mathcal{A}(E)'')$. We start by a description of $\mathcal{N}(E)'$ in the case where E may lack the approximation property. Let K_E be the kernel of the canonical map $\mathcal{R} : E \widehat{\otimes} E' \to \mathcal{N}(E)$. Then

$$\mathcal{N}(E)' = K_E^{\perp} = \{T \in \mathcal{B}(E') : \langle u, T \rangle = 0 \ (\mathcal{R}u = 0)\},$$

a closed subspace of $\mathcal{B}(E')$. We note that $\mathcal{A}(E') \subseteq K_E^{\perp}$. To see this, let $\lambda \otimes \Lambda$ be a rank-one operator in $\mathcal{B}(E')$, and let $u \in K_E$, say $u = \sum x_n \otimes \lambda_n$ with $\sum \langle x, \lambda_n \rangle x_n = 0$ for all $x \in E$. Then

$$\langle u, \lambda \otimes \Lambda \rangle = \sum \langle x_n, \lambda \rangle \langle \lambda_n, \Lambda \rangle = 0.$$

It follows that, for $S \in \mathcal{N}(E)$ and $F \in \mathcal{A}(E')$, the trace $\operatorname{tr}(S' \circ F)$ is well-defined.

THEOREM 5.9. *Let $i : \mathcal{N}(E') \to \mathcal{I}(E')$ be the canonical embedding. Then*

$$\mathcal{H}^1(\mathcal{A}(E), \mathcal{A}(E)'') = \mathcal{B}(E'')/(i'(\mathcal{A}(E)'') + \mathbb{C}1_{E''}).$$

Proof. Consider the dual map $i' : \mathcal{I}(E')' \to \mathcal{N}(E')'$ of i, formed by taking restrictions. We make the isometric identifications $\mathcal{A}(E)'' = \mathcal{I}(E')'$ and $\mathcal{N}(F')' = K_{E'}^{\perp} = \mathcal{B}(E'')$, so that i' is regarded as a map from $\mathcal{A}(E)''$ into $\mathcal{B}(E'')$.

Let $D \in \mathcal{Z}^1(\mathcal{A}(E), \mathcal{A}(E)'')$, and take $T \in \mathcal{A}(E)' = \mathcal{I}(E')$ and $F, G \in \mathcal{A}(E)$. Then

$$\langle T, D(F \circ G) \rangle = \langle T \cdot F, D(G) \rangle + \langle G \cdot T, D(F) \rangle.$$

Since $\mathcal{A}(E) \cdot \mathcal{I}(E') + \mathcal{I}(E') \cdot \mathcal{A}(E) \subset \mathcal{N}(E')$, it follows that

$$\begin{aligned}\langle T, D(F \circ G) \rangle &= \operatorname{tr}((T \cdot F)' \circ i'(D(G))) + \operatorname{tr}((G \cdot T)' \circ i'(D(F))) \\ &= \operatorname{tr}(T' \circ i'(D(F \circ G))).\end{aligned}$$

However, operators of the form $F \circ G$, where $F, G \in \mathcal{A}(E)$, form a dense subset of $\mathcal{A}(E)$ because every finite-rank operator U can be expressed as $U = P \circ U$ for a finite-rank projection P. It follows that the range of $i' \circ D$ is contained in $\mathcal{A}(E'')$ and that

$$\langle T, D(F) \rangle = \operatorname{tr}(T' \circ i'(D(F))) \quad (T \in \mathcal{I}(E'),\ F \in \mathcal{A}(E)).$$

Using this, we see that

$$\begin{aligned}\|D(F)\| &= \sup\{|\langle T, D(F) \rangle| : \|Tt\|_{\mathcal{I}(E)} = 1\} \\ &= \sup\{|\operatorname{tr}(T' \circ i'(D(F))| : \|T\|_{\mathcal{I}(E)} = 1\} \\ &= \sup\{|\operatorname{tr}(T'' \circ (i'(D(F)))'| : \|T\|_{\mathcal{I}(E)} = 1\} \\ &= \|i'(D(F))\|;\end{aligned}$$

for the last equality, we have used the fact that the map $T \mapsto T''$ is an isometry from $\mathcal{I}(E')$ to $\mathcal{I}(E''')$. It follows that we may regard D as a continuous derivation from $\mathcal{A}(E)$ into $K_{E'}^{\perp} \subset \mathcal{B}(E'')$, and this we shall do.

As in the proof of the Kaliman–Selivanov theorem [19, Proposition III.4.16 and Theorem III.4.17], we see that E' is a projective right module over $\mathcal{A}(E)$. Hence E'' is an injective left module, so that

$$\mathcal{H}^n(\mathcal{A}(E), \mathcal{B}(E'')) = \operatorname{Ext}^n_{\mathcal{A}(E)}(E'', E'') = \{0\} \quad (n \in \mathbb{N}).$$

In particular, it follows that there is an element $U \in \mathcal{B}(E'')$ such that

$$\langle T, D(F) \rangle = \operatorname{tr}(T' \circ (F'' \circ U - U \circ F'')) \quad (T \in \mathcal{I}(E'),\ F \in \mathcal{A}(E)).$$

Conversely, each $U \in \mathcal{B}(E'')$ defines a continuous derivation from $\mathcal{A}(E)$ to $\mathcal{A}(E)''$ by the above formula. Such a derivation will be inner precisely when $U \in i'(\mathcal{A}(E)'') + \mathbb{C}I_{E''}$. ■

COROLLARY 5.10. *If* $\dim K_{E'} > 1$, *then* $\mathcal{A}(E)$ *is not* 2-*weakly amenable.*
If $\dim K_{E'} \le 1$ *and* E' *has the Radon–Nikodym property, then* $\mathcal{A}(E)$ *is* 2-*weakly amenable.*

Proof. If $\dim K_{E'} > 1$, then $K_{E'}^{\perp} + \mathbb{C}I_{E''} \ne \mathcal{B}(E'')$. Since we have $i'(\mathcal{A}(E)'') \subseteq K_{E'}^{\perp}$, the algebra $\mathcal{A}(E)$ cannot be 2-weakly amenable. If E' has the Radon–Nikodym property, then $\mathcal{N}(E') = \mathcal{I}(E')$ isometrically, and so we have $i'(\mathcal{A}(E)'') = K_{E'}^{\perp}$. ■

6. Summary. Let us write "n-WA" as an abbreviation for "n-weakly amenable". We have been concerned with the relations between m-WA and n-WA for Banach algebras in particular classes, where $m, n \in \mathbb{N}$. We briefly summarize our results and the questions that we have left open.

It is always the case that, for each $n \in \mathbb{N}$,

$$(n+2)\text{-WA} \Rightarrow n\text{-WA}.$$

In the case where $\mathfrak{A}$ is commutative,

$$1\text{-WA} \Rightarrow n\text{-WA} \quad \text{for each } n \in \mathbb{N}.$$

In the other direction, the disc algebra $A(\overline{\mathbb{D}})$ is a uniform algebra which shows that

$$2\text{-WA} \not\Rightarrow 1\text{-WA},$$

and the algebra $\mathcal{N}(E)$ for certain Banach spaces E shows that

$$1\text{-WA} \not\Rightarrow 2\text{-WA}$$

(cf. Theorem 5.2). In fact, $A(\widetilde{\mathbb{D}})$ is n-WA if and only if n is even, and $\mathcal{N}(E)$ (in the case where E is reflexive and has AP) is n-WA if and only if n is odd.

The algebra $C^{(1)}(\mathbb{I})$ is an Arens regular Banach function algebra which is not n-WA for any $n \in \mathbb{N}$; the Volterra algebra $\mathcal{V}$ has the same property, and $\mathcal{V}$ is an ideal in its second dual.

We do not know whether or not 2-WA $\Rightarrow$ 4-WA for an arbitrary Banach algebra; in particular, we do not know whether or not this holds for each commutative Banach algebra.

We do not know whether or not 1-WA $\Rightarrow$ 3-WA for an arbitrary (non-commutative) Banach algebra; the tensor algebra $E\widehat{\otimes}E'$ is 1-WA for each Banach space E, and it is possible that it is not 3-WA for some Banach space E.

C^*-algebras are n-WA for each $n \in \mathbb{N}$ (but they are not necessarily amenable); the group algebras $L^1(G)$ are n-WA for each odd n, but we do not know whether or not $l^1(\mathbb{F}_2)$ is 2-WA.

Addendum. A similar construction to that showing that X'' is a Banach $(A'', \Box)$-bimodule has also been given by F. Gourdeau, *Amenability and the second dual of a Banach algebra*, Studia Math. 125 (1997), 75–81.

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Tauberian theorems for vector-valued Fourier and Laplace transforms

by

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Abstract. Let X be a Banach space and $f \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$ be absolutely regular (i.e. integrable when divided by some polynomial). If the distributional Fourier transform of f is locally integrable then f converges to 0 at infinity in some sense to be made precise. From this result we deduce some Tauberian theorems for Fourier and Laplace transforms, which can be improved if the underlying Banach space has the analytic Radon–Nikodym property.

0. Introduction. In the last decade Tauberian theorems for vector-valued Laplace transforms attained much attention because of the intimate relation with the asymptotic behavior of Cauchy problems in Banach spaces ([1]–[6], [20], [21]). A typical Tauberian theorem, essentially Ingham's theorem, says the following: Let $f : \mathbb{R}_+ \to X$ be uniformly continuous (where X is a Banach space) and assume that the Laplace transform has a continuous extension to $\overline{\mathbb{C}}_+$. Then $\lim_{t\to\infty} f(t) = 0$ (cf. [4, Thm. 3.5]).

The proof in [4] uses a tricky contour argument from Korevaar [18], which has been exploited in most of the cited papers. In the first section of this paper we present a new approach to Ingham's theorem via Fourier transforms. Our proofs are not more difficult, and moreover, this approach allows us to relax the Tauberian hypothesis considerably and to go beyond the most recent results even for asymptotically almost periodic functions. To give an example, under suitable ergodic conditions on f it suffices in Ingham's theorem to suppose a continuous extension of the Laplace transform to the imaginary axis minus a closed, countable set. This Tauberian hypothesis arises naturally in Volterra equations ([4]). In particular, our result answers a problem asked in the introduction of [2].

In the second section we derive a Tauberian theorem in a similar way to Section 1, but for a larger class of Laplace transformable functions (than

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