Contents of Volume 128, Number 1

STUDIA MATHEMATICA

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (1998): Vols. 127–131 (15 issues); $32 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by

02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

PRINTED IN POLAND

ISSN 0039-3223

On the dependence of the orthogonal projector on deformations of the scalar product

by

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Abstract. We consider scalar products on a given Hilbert space parametrized by bounded positive and invertible operators defined on this space, and orthogonal projectors onto a fixed closed subspace of the initial Hilbert space corresponding to these scalar products. We show that the projector is an analytic function of the scalar product, we give the explicit formula for its Taylor expansion, and we prove some algebraic formulas for projectors.

1. Introduction. One of the most important directions of modern mathematical physics is the explanation of the role of *complex objects* and the notion of *holomorphy* in the description of fundamental laws of physics. It is particularly well seen in quantum theories, where complex objects appear in the most natural way.

The notions important from the complex analysis point of view which have been applied in quantum theories include, in particular, *reproducing kernels*. They are a generalization of the *Bergman function* (see [1] or [2]). They became the basis of models of quantum field theory described in [4] or [5] (see also [10]). More precisely, we have in mind reproducing kernels defined for the Hilbert spaces of all holomorphic and square integrable (with respect to suitable measures and hermitian structures) sections of holomorphic vector bundles (see [8], [9] or [10]). To be able to use *perturbation methods* in such quantum models, investigation of the dependence of the reproducing kernels on deformations of measures and hermitian structures is necessary.

In the simplest case of the trivial bundle over a domain in $\mathbb{C}^n$, both the measure and hermitian structure are together represented by *a weight of integration* (with respect to the Lebesgue measure). The dependence of the reproducing kernels on the weight of integration was investigated in [7].

1991 *Mathematics Subject Classification*: 46C99, 47A55, 47A56, 47A60, 47A62, 47B65.
Key words and phrases: scalar product, orthogonal projector, dependence of projectors on scalar products.

They were shown to depend on the weights *analytically*. We recall that those reproducing kernels are simultaneously the integral kernels of *the orthogonal projectors* of the space of all functions square integrable with respect to a given weight onto the subspace of holomorphic functions. Therefore an important step of the considerations in [7] was the investigation of the *dependence of orthogonal projectors on weights of integration.*

To extend the results of [7] to general vector bundles (in the applications to physics so far, only line bundles are needed) the dependence of suitable orthogonal projectors on measures defined on the base manifold of the bundle and on hermitian structures defined on the bundle should be investigated. It turns out that a description of that dependence can be obtained from a general theorem on *dependence of orthogonal projectors of an arbitrary Hilbert space onto a fixed subspace on scalar products.* The main goal of our paper is to prove that general theorem.

Section 2 recalls facts concerning linear and multilinear operators defined on Banach spaces and positive definite operators on Hilbert spaces. It includes some definitions and notation needed in the other sections. Section 3 contains the proof of the theorem on *analytic dependence of orthogonal projectors onto a finite-dimensional subspace on scalar products* (Theorem 3.2). In this theorem an explicit formula for an appropriate Taylor expansion is given. Most of the results of Section 3 are generalizations of the results obtained in Section 4 of [7]. Some of the proofs are similar to those of [7]; however, they are included for the convenience of the reader.

The results of Section 3 are used in Section 4, where some *algebraic formulas* for orthogonal projectors (onto an arbitrary closed subspace of the initial Hilbert space) are derived. They express orthogonal projectors in terms of one fixed projector and positive definite operators determined by scalar products (Theorems 4.1 and 4.2). Those results have no equivalent in [7].

In Section 5 we apply the algebraic formulas in the proof of a generalization of Theorem 3.2, establishing *analytic dependence of orthogonal projectors onto an arbitrary closed subspace on scalar products* (Theorem 5.1).

In this paper no examples are given. They can be found in [4], [5], [7] or [10].

2. Preliminaries. If $\mathbf{X}$, $\mathbf{Y}$ are normed spaces then the standard normed space of all linear bounded operators on $\mathbf{X}$ into $\mathbf{Y}$ will be denoted by $L(\mathbf{X},\mathbf{Y})$. If $X = Y$ we write $L(\mathbf{X})$ instead of $L(\mathbf{X},\mathbf{Y})$. We use the symbol $F\circ G$ for the superposition of arbitrary maps F and G, and the symbol AB for the superposition of *linear operators* A and B.

The normed space of all k-linear bounded operators on a Cartesian product $X_1\times\ldots\times X_k$ of normed spaces into Y will be denoted by $L^k(X_1,\ldots,X_k;Y)$. If U is an open subset of X and $T: U\to Y$ is differentiable up to order k then we denote its kth derivative at $x\in U$ by $D^{(k)}T(x)$. We consider it as a k-linear bounded operator on X^k into Y, i.e., $D^{(k)}T(x)\in L^k(\mathbf{X},\ldots,\mathbf{X};\mathbf{Y})$. If $\mathbf{X}=X_1\times\ldots\times X_m$ then $D_{x_j}T(x)$ is the derivative of T at $x\in U$ with respect to the variable $x_j\in X_j$, $1\le j\le m$.

In this paper we will consider analytic mappings between normed spaces. We adopt the following definition of analyticity.

DEFINITION 2.1. Let U be an open subset of a normed space $\mathbf{X}$ and let $\mathbf{Y}$ be a normed space. We say that a map $F: U\to\mathbf{Y}$ is *analytic* on U iff for any $x\in U$ there exists a ball $B\subset\mathbf{X}$ with center at $0\in\mathbf{X}$ such that $x+B\subset U$ and for any $h\in B$,

$$F(x+h)=F(x)+\sum_{m=1}^{\infty}a_m(\underbrace{h,\ldots,h}_{m\ \text{times}}),$$

where $a_m:\mathbf{X}^m\to\mathbf{Y}$ is a continuous m-linear function for $m=1,2,\ldots$ and the series on the right hand side converges uniformly on B.

Let $\mathbf{X}$ (or $(\mathbf{X},\langle\cdot\mid\cdot\rangle)$) be an arbitrary Hilbert space (real or complex) with the norm $\|\cdot\|$ given by the scalar product $\langle\cdot\mid\cdot\rangle$. Let $H(\mathbf{X})\subset L(\mathbf{X})$ denote the $\mathbb{R}$-linear Banach space of all hermitian bounded operators on $\mathbf{X}$, $S(\mathbf{X})$ the cone of all elements of $H(\mathbf{X})$ which are positive definite, and $S_0(\mathbf{X})$ the set of all $A\in S(\mathbf{X})$ such that

$$i(A):=\inf_{\|x\|=1}\langle x\mid Ax\rangle>0.$$

For any $A\in S_0(\mathbf{X})$ and any $x\in\mathbf{X}$ we have

$$\langle x\mid Ax\rangle\ge i(A)\|x\|^2,\qquad x\in\mathbf{X}. \tag{1}$$

It is known that any operator $A\in S_0(\mathbf{X})$ is invertible. Moreover, (1) and the Schwarz inequality imply that for any $x\in\mathbf{X}$,

$$\|Ax\|\cdot\|x\|\ge\langle x\mid Ax\rangle\ge i(A)\|x\|^2.$$

Then for each $y\in\mathbf{X}$, $\|A^{-1}y\|\le i(A)^{-1}\|y\|$ and therefore $\|A^{-1}\|\le i(A)^{-1}$. In fact, one can easily prove the equality

$$\|A^{-1}\|=\frac{1}{i(A)},\qquad A\in S_0(\mathbf{X}). \tag{2}$$

We will identify elements of $S(\mathbf{X})$ with scalar products on X by assigning to any $A\in S(\mathbf{X})$ the scalar product

$$\langle x\mid y\rangle_A:=\langle x\mid Ay\rangle,\qquad x,y\in\mathbf{X}.$$

Note that for any $x\in\mathbf{X}$,

$$\|x\|_A^2=\langle x\mid Ax\rangle\le\|A\|\cdot\|x\|^2.$$

If $A \in S_0(\mathbf{X})$ then combining (1) and the above inequality we obtain

$$\frac{1}{C_{I,A}}\|x\| \le \|x\|_A \le C_{A,I}\|x\|, \tag{3}$$

where

$$C_{A,I} := \sqrt{\|A\|} \tag{4}$$

and

$$C_{I,A} := \sqrt{\frac{1}{i(A)}}. \tag{5}$$

Let us consider the dependence of $i(A)$ on deformations of $A \in S_0(\mathbf{X})$. If $h \in H(\mathbf{X})$ and $\|h\| < i(A)$ then for any $x \in \mathbf{X}$ such that $\|x\| = 1$,

$$\langle x \mid (A+h)x\rangle = \langle x \mid Ax\rangle + \langle x \mid hx\rangle \ge i(A) - \|h\|.$$

Hence

$$i(A+h) \ge i(A) - \|h\| > 0, \tag{6}$$

which means that $A + h \in S_0(\mathbf{X})$.

For any $A \in S_0(\mathbf{X})$, define

$$\mathbf{B}_A := B_{H(\mathbf{X})}(0, i(A)/2) = \{h \in H(\mathbf{X}) : \|h\| < i(A)/2\}. \tag{7}$$

If $h \in \mathbf{B}_A$, then by (6),

$$i(A+h) > i(A)/2 > 0. \tag{8}$$

Consequently, the ball $A + \mathbf{B}_A = B_{H(\mathbf{X})}(A, i(A)/2) \subset S_0(\mathbf{X})$, which implies that $S_0(\mathbf{X})$ is an open subset of $H(\mathbf{X})$.

It is well known that $A \mapsto \langle\cdot \mid \cdot\rangle_A$ is a one-to-one map of $S(\mathbf{X})$ onto the set of all scalar products on $\mathbf{X}$ which are continuous on $\mathbf{X} \times \mathbf{X}$ with respect to the initial topology defined by the norm $\|\cdot\|$ on $\mathbf{X}$. A scalar product $\langle\cdot \mid \cdot\rangle_A$ is equivalent to $\langle\cdot \mid \cdot\rangle$ iff $A \in S_0(\mathbf{X})$.

Let us consider three maps: $F : \mathbf{X} \times \mathbf{X} \times H(\mathbf{X}) \to \mathbb{C}$, $F_1 : H(\mathbf{X}) \to L(H(\mathbf{X}), L(\mathbf{X}))$, and $F_2 : S_0(\mathbf{X}) \to S_0(\mathbf{X})$, where

$$\begin{aligned} F(x,y,A) &:= \langle x \mid Ay\rangle, && x, y \in \mathbf{X},\ A \in H(\mathbf{X}),\\ F_1(A)h &:= Ah, && A, h \in H(\mathbf{X}),\\ F_2(A) &:= A^{-1}, && A \in S_0(\mathbf{X}). \end{aligned} \tag{9}$$

The map F is C^∞ as a bounded 3-linear form over $\mathbb{R}$; F_1 is linear and bounded and therefore it is C^∞. The map F_2 is a C^∞ diffeomorphism of $S_0(\mathbf{X})$. We have

$$DF_2(A)h = -A^{-1}hA^{-1}, \quad A \in S_0(\mathbf{X}),\ h \in H(\mathbf{X}).$$

We can now define the map $M : S_0(\mathbf{X}) \to L(H(\mathbf{X}), L(\mathbf{X}))$ by the formula

$$M := F_1 \circ F_2, \tag{10}$$

i.e., for any $A \in S_0(\mathbf{X})$ and any $h \in H(\mathbf{X})$,

$$M(A)h = A^{-1}h \in L(\mathbf{X}). \tag{11}$$

This map is C^∞ as a superposition of C^∞ maps. If $h \in H(\mathbf{X})$ is fixed then we have

$$D_A(M(A)h)h_1 = -A^{-1}h_1A^{-1}h, \quad A \in S_0(\mathbf{X}),\ h_1 \in H(\mathbf{X}). \tag{12}$$

Moreover,

$$\|M(A)h\| = \|A^{-1}h\| \le \|A^{-1}\| \cdot \|h\| = \frac{\|h\|}{i(A)}. \tag{13}$$

3. Orthogonal projectors onto a fixed finite-dimensional subspace. Let (f_j) be an arbitrary (finite or infinite) sequence of linearly independent elements of a Hilbert space $\mathbf{X}$. Denote by $(e_j(A))$ the sequence obtained from (f_j) by the orthonormalization procedure with respect to the scalar product $\langle\cdot \mid \cdot\rangle_A$, where $A \in S_0(\mathbf{X})$. We have

$$\begin{aligned} e_1(A) &= \frac{f_1}{\|f_1\|_A},\\ e_j(A) &= \frac{f_j - \sum_{i=1}^{j-1}\langle e_i(A) \mid f_j\rangle_A e_i(A)}{\|f_j - \sum_{i=1}^{j-1}\langle e_i(A) \mid f_j\rangle_A e_i(A)\|_A}. \end{aligned} \tag{14}$$

LEMMA 3.1. *For any $j \in \mathbf{N}$ the map $e_j : S_0(\mathbf{X}) \to \mathbf{X}$ given by the formula* (14) *is C^∞. Moreover, for every $A \in S_0(\mathbf{X})$ the range of the operator $De_j(A)$ is contained in the subspace of $\mathbf{X}$ spanned by the vectors $f_1, \ldots, f_j$.*

Proof. One can easily prove the first part of this lemma by induction. The second part is an immediate consequence of the fact that for every $j \in \mathbf{N}$, $e_j : S_0(\mathbf{X}) \to \operatorname{span}\{f_1, \ldots, f_j\}$. We leave the details to the reader. ∎

Let V be an arbitrary but fixed closed subspace of $\mathbf{X}$. We now examine the analyticity of the map

$$S_0(\mathbf{X}) \ni A \mapsto P(A) \in L(\mathbf{X}),$$

where $P(A)$ is the projection of $\mathbf{X}$ onto V, orthogonal with respect to the scalar product $\langle\cdot \mid \cdot\rangle_A$. To do this we first consider the finite-dimensional case.

LEMMA 3.2. *Let V be a finite-dimensional subspace of $\mathbf{X}$* ($\dim V = m < \infty$)*. Then for any $x \in \mathbf{X}$ the map*

$$S_0(\mathbf{X}) \ni A \mapsto T_x(A) := P(A)x \in V \subset \mathbf{X}$$

is C^∞. Moreover, for any $A \in S_0(\mathbf{X})$ and any $h \in H(\mathbf{X})$,

$$DT_x(A)h = [P(A) \circ (M(A)h) \circ (I - P(A))]x,$$

where M is given by (10) *and I is the identity operator on $\mathbf{X}$.*

Proof. Fix $x \in \mathbf{X}$ and some basis $f_1, \ldots, f_m$ in V. Let $e_1(A), \ldots, e_m(A)$ be given by (14). Then

$$T_x(A) = \sum_{j=1}^{m} \langle e_j(A) \mid x \rangle_A e_j(A) = \sum_{j=1}^{m} F(e_j(A), x, A) e_j(A),$$

where F is defined by (9). Hence the smoothness of T_x follows from Lemma 3.1. Now fix $A_0 \in S_0(\mathbf{X})$. For any $A \in S_0(\mathbf{X})$ we have

$$T_x(A) = P(A)x = P(A)[P(A_0)x + (I - P(A_0))x] = P(A_0)x + P(A)x_0,$$

where $x_0 := (I - P(A_0))x$. Since the first term does not depend on A we obtain

$$\begin{aligned} DT_x(A_0)h &= D_A[P(\cdot)x_0](A_0)h \\ &= \sum_{j=1}^{m} F(De_j(A_0)h, x_0, A_0) e_j(A_0) \\ &\quad + \sum_{j=1}^{m} F(e_j(A_0), x_0, h) e_j(A_0) \\ &\quad + \sum_{j=1}^{m} F(e_j(A_0), x_0, A_0) De_j(A_0)h \\ &= \sum_{j=1}^{m} F(e_j(A_0), x_0, h) e_j(A_0), \qquad h \in H(\mathbf{X}). \end{aligned}$$

We have used the fact that x_0 is orthogonal to V with respect to $\langle \cdot \mid \cdot \rangle_{A_0}$, and $De_j(A_0)h \in V$ for $j = 1, \ldots, m$. Hence

$$\begin{aligned} DT_x(A_0)h &= \sum_{j=1}^{m} F(e_j(A_0), x_0, A_0 A_0^{-1} h) e_j(A_0) \\ &= \sum_{j=1}^{m} \langle e_j(A_0) \mid (A_0^{-1}h)x_0 \rangle_{A_0} e_j(A_0) \\ &= \sum_{j=1}^{m} \langle e_j(A_0) \mid (M(A_0)h)x_0 \rangle_{A_0} e_j(A_0) = P(A_0)[M(A_0)h]x_0 \\ &= [P(A_0) \circ (M(A_0)h) \circ (I - P(A_0))]x. \ \blacksquare \end{aligned}$$

LEMMA 3.3. *Under the assumption of Lemma 3.2 the map P is continuous. More precisely, for any $A \in S_0(\mathbf{X})$ there exists a constant $K_A > 0$ such that for every $h \in \mathbf{B}_A$* (see (7)),

$$\|P(A+h) - P(A)\| \le K_A \|h\|. \tag{15}$$

K_A does not depend on V.

Proof. By Lemma 3.2 for every $x \in \mathbf{X}$ and $h \in \mathbf{B}_A$,

$$\begin{aligned} &\|P(A+h)x - P(A)x\| \\ &\qquad = \left\| \int_0^1 \frac{d}{dt}(T_x(A+th))\,dt \right\| \\ &\qquad \le \int_0^1 \|[P(A+th)(M(A+th)h)(I - P(A+th))]x\|\,dt. \end{aligned}$$

Note that by (3),

$$\begin{aligned} &\|[P(A+th)(M(A+th)h)(I - P(A+th))]x\| \\ &\qquad \le C_{I,A+th} \|[P(A+th)(M(A+th)h)(I - P(A+th))]x\|_{A+th} \\ &\qquad \le C_{I,A+th} \|M(A+th)h\|_{A+th} \|x\|_{A+th}, \end{aligned}$$

where we have used the fact that for any $t \in [0;1]$,

$$\|P(A+th)\|_{A+th} = \|I - P(A+th)\|_{A+th} = 1. \tag{16}$$

Since for any $x \in \mathbf{X}$, any $B \in S_0(\mathbf{X})$ and any $h \in H(\mathbf{X})$,

$$\begin{aligned} \|[M(B)h]x\|_B^2 &= \langle B^{-1}hx \mid BB^{-1}hx \rangle \le \|B^{-1}\| \cdot \|h\|^2 \|x\|^2 \\ &= \frac{1}{i(B)} \|h\|^2 \|x\|^2 \le \frac{1}{i(B)} C_{I,B}^2 \|h\|^2 \|x\|_B^2 = \frac{1}{i(B)^2} \|h\|^2 \|x\|_B^2 \end{aligned}$$

(see (2) and (3)) we obtain

$$\|M(B)h\|_B \le \frac{\|h\|}{i(B)}. \tag{17}$$

Hence for any $h \in \mathbf{B}_A$,

$$\begin{aligned} &\|[P(A+th)(M(A+th)h)(I - P(A+th))]x\| \\ &\qquad \le C_{I,A+th} \frac{\|h\|}{i(A+th)} \|x\|_{A+th} \le \frac{\|h\|}{(i(A+th))^{3/2}} C_{A+th,I} \|x\| \\ &\qquad \le \frac{\sqrt{\|A+th\|}\,\|h\|}{(i(A) - t\|h\|)^{3/2}} \|x\| \le \frac{\sqrt{\|A\| + t\|h\|}\,\|h\|}{(i(A) - i(A)/2)^{3/2}} \|x\| \\ &\qquad \le \frac{\sqrt{\|A\| + i(A)/2}\,\|h\|}{(i(A)/2)^{3/2}} \|x\|. \end{aligned}$$

and therefore

$$\|(P(A+h) - P(A))x\| \le \int_0^1 K_A \|h\| \cdot \|x\|\,dt = K_A \|h\| \cdot \|x\|,$$

where

$$K_A := \frac{\sqrt{\|A\| + i(A)/2}}{(i(A)/2)^{3/2}}. \blacksquare$$

THEOREM 3.1. *Let V be a finite-dimensional subspace of the Hilbert space $\mathbf{X}$. Then the mapping P which assigns to any $A \in S_0(\mathbf{X})$ the projector $P(A)$ of $\mathbf{X}$ onto V (orthogonal with respect to the scalar product $\langle \cdot \mid \cdot \rangle_A$) is differentiable. Moreover, for any $h \in H(\mathbf{X})$ we have*

$$DP(A)h = P(A) \circ (M(A)h) \circ (I - P(A)), \tag{18}$$

where $M(A)$ is given by (10).

Proof. It is enough to show that for any $A \in S_0(\mathbf{X})$ there exists a number $N_A > 0$ which has the following property: for every finite-dimensional subspace V of $\mathbf{X}$ and any $h \in \mathbf{B}_A$,

$$\|P(A+h) - P(A) - P(A)(M(A)h)(I - P(A))\| \le N_A \|h\|^2. \tag{19}$$

Fix $A \in S_0(\mathbf{X})$. For every $h \in \mathbf{B}_A$ and every $x \in \mathbf{X}$,

$$\begin{aligned}
&\|[P(A+h) - P(A) - P(A)(M(A)h)(I - P(A))]x\| \\
&= \Big\| \int_0^1 [DT_x(A+th)h - P(A)(M(A)h)(I - P(A))x]\,dt \Big\| \\
&\le \int_0^1 \|P(A+th)[(M(A+th)h)(I - P(A+th)) - (M(A)h)(I - P(A))]x\|\,dt \\
&\quad + \int_0^1 \|(P(A+th) - P(A))(M(A)h)(I - P(A))x\|\,dt.
\end{aligned} \tag{20}$$

Note that by (3) and (16),

$$\begin{aligned}
&\|P(A+th)[(M(A+th)h)(I - P(A+th)) - (M(A)h)(I - P(A))]x\| \\
&\le C_{I,A+th} \|[M(A+th)h - M(A)h]x \\
&\quad + [(M(A)h)P(A) - (M(A+th)h)P(A+th)]x\|_{A+th} \\
&\le C_{I,A+th} (\|[M(A+th)h - M(A)h]x\|_{A+th} \\
&\quad + \|[(M(A)h)P(A) - (M(A+th)h)P(A+th)]x\|_{A+th}) \\
&=: a(A,t,h,x).
\end{aligned} \tag{21}$$

Since

$$\begin{aligned}
\|[M(A+th)h - M(A)h]x\| &= \|[(A+th)^{-1}h - A^{-1}h]x\| \\
&\le \|(A+th)^{-1}h - A^{-1}h\| \cdot \|x\|
\end{aligned}$$

and

$$\begin{aligned}
\|(A+th)^{-1}h - A^{-1}h\| &= \Big\| \int_0^1 D[M(A+sth)h]h\,ds \Big\| \\
&\le \int_0^1 \|(A+sth)^{-1}h(A+sth)^{-1}h\|\,ds \\
&\le \int_0^1 \|(A+sth)^{-1}\|^2 \|h\|^2\,ds \\
&\le \|h\|^2 \int_0^1 \frac{1}{i(A+sth)^2}\,ds \le \|h\|^2 \int_0^1 \frac{4}{i(A)^2}\,ds \\
&\le \frac{4}{i(A)^2} \|h\|^2
\end{aligned}$$

(see (12) and (8)) we obtain

$$\begin{aligned}
&\|[M(A+th)h - M(A)h]x\|_{A+th} \\
&\le C_{A+th,I} \frac{4}{i(A)^2} \|h\|^2 \|x\| \le \frac{4\sqrt{\|A\| + i(A)/2}}{i(A)^2} \|h\|^2 \|x\| = N_{A,1} \|h\|^2 \|x\|,
\end{aligned} \tag{22}$$

where

$$N_{A,1} := \frac{4\sqrt{\|A\| + i(A)/2}}{i(A)^2}.$$

Then we have

$$\|[M(A+th)h - M(A)h]x\|_{A+th} \le C_{I,A+th} N_{A,1} \|h\|^2 \|x\|_{A+th}$$

and consequently

$$\|[M(A+th)h - M(A)h]\|_{A+th} \le C_{I,A+th} N_{A,1} \|h\|^2.$$

By the above inequalities, by (13) and by Lemma 3.3,

$$\begin{aligned}
&\|[(M(A)h)P(A) - (M(A+th)h)P(A+th)]x\|_{A+th} \\
&\le C_{A+th,I} \|[(M(A)h)(P(A) - P(A+th))]x\| \\
&\quad + \|[(M(A)h - M(A+th)h)P(A+th)]x\|_{A+th} \\
&\le C_{A+th,I} \|M(A)h\| \cdot \|[P(A) - P(A+th)]x\| \\
&\quad + \|[M(A)h - M(A+th)h]\|_{A+th} \|x\|_{A+th} \\
&\le \frac{C_{A+th,I}\|h\|}{i(A)} K_A \|h\| \cdot \|x\| + C_{I,A+th} N_{A,1} \|h\|^2 C_{A+th,I} \|x\|.
\end{aligned}$$

Since

$$\frac{C_{A+th,I}}{i(A)}K_A = \frac{\sqrt{\|A+th\|}}{i(A)}K_A \le \frac{\sqrt{\|A\|+i(A)/2}}{i(A)}K_A$$

and

$$C_{I,A+th}N_{A,1}C_{A+th,I} = \sqrt{\frac{\|A+th\|}{i(A+th)}}N_{A,1} \le \sqrt{\frac{2\|A\|+i(A)}{i(A)}}N_{A,1}$$

we obtain

$$\|[(M(A)h)P(A)-(M(A+th)h)P(A+th)]x\|_{A+th} \le N_{A,2}\|h\|^2\|x\|, \tag{23}$$

where

$$N_{A,2} := \frac{\sqrt{\|A\|+i(A)/2}}{i(A)}K_A + \sqrt{\frac{2\|A\|+i(A)}{i(A)}}N_{A,1}.$$

Setting (22) and (23) in (21) we obtain

$$\begin{aligned} a(A,t,h,x) &\le C_{I,A+th}(N_{A,1}+N_{A,2})\|h\|^2\|x\| \\ &= \sqrt{\frac{1}{i(A+th)}}(N_{A,1}+N_{A,2})\|h\|^2\|x\| \le N_{A,3}\|h\|^2\|x\|, \end{aligned} \tag{24}$$

where

$$N_{A,3} := \sqrt{\frac{2}{i(A)}}(N_{A,1}+N_{A,2}).$$

Moreover, by (13), (15) and (16),

$$\|(P(A+th)-P(A))(M(A)h)(I-P(A))x\| \le \frac{K_A}{i(A)}\|h\|^2\|x\|. \tag{25}$$

Setting (24) and (25) in (20) we obtain (19), where

$$N_A := N_{A,3} + \frac{K_A}{i(A)}. \ \blacksquare$$

LEMMA 3.4. *Let $X, X_1, \ldots, X_k$ and Y be normed spaces over the same field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) and let $Q : X \times X_1 \times \ldots \times X_k \to Y$ be a bounded $(k+1)$-linear map. Then the map $Q^0 : X_1 \times \ldots \times X_k \to L(X,Y)$ given by the formula*

$$Q^0(x_1,\ldots,x_k)x := Q(x,x_1,\ldots,x_k), \quad x \in X,\ (x_1,\ldots,x_k) \in X_1 \times \ldots \times X_k,$$

is k-linear and bounded. $\blacksquare$

THEOREM 3.2. *Under the assumptions of Theorem 3.1 the mapping P is analytic on $S_0(\mathbf{X})$. More precisely, for any $A \in S_0(\mathbf{X})$ and $h \in \mathbf{B}_A$* (see (7)) *we have*

$$P(A+h) = P(A) + \sum_{k=1}^{\infty}(-1)^{k-1}[P(A)(M(A)h)]^k(I-P(A)) \tag{26}$$

and the series on the right hand side converges uniformly in h on each ball $B_{H(\mathbf{X})}(0,r)$, where $r < i(A)/2$.

Proof. First we prove that P is C^∞. Define $Q : L(\mathbf{X})^3 \to L(\mathbf{X})$ by

$$Q(T,T_1,T_2) := T_1 \circ T \circ T_2, \quad T, T_1, T_2 \in L(\mathbf{X}).$$

Let $Q^0 : L(\mathbf{X})^2 \to L(L(\mathbf{X}))$ be defined as in Lemma 3.4. Then Q^0 is C^∞, being a bounded bilinear map. Moreover, by Theorem 3.1 we have

$$DP(A) = Q^0(P(A), I-P(A)) \circ M(A).$$

Suppose that P is of class C^k. Then DP is also C^k as a superposition of C^k-maps and bounded bilinear maps. Now we can use induction. Since by Theorem 3.1, P is of class C^0 we see that it is C^∞.

We now show that for any $h \in H(\mathbf{X})$,

$$D^{(k)}P(A)h^{(k)} = (-1)^{k-1}k![P(A)(M(A)h)]^k(I-P(A)), \tag{27}$$

where

$$h^{(k)} := (\underbrace{h,\ldots,h}_{k\text{ times}}) \in H(\mathbf{X}), \quad A \in S_0(\mathbf{X}).$$

By (18) it is true for $k=1$. Assume that (27) holds for $k=m$. We have

$$\begin{aligned} &D_A[P(A)(M(A)h)]h_1 \\ &\quad = P(A)(M(A)h_1)(I-P(A))(M(A)h) - P(A)A^{-1}h_1A^{-1}h \\ &\quad = P(A)A^{-1}h_1A^{-1}h - P(A)A^{-1}h_1P(A)A^{-1}h - P(A)A^{-1}h_1A^{-1}h \\ &\quad = -P(A)A^{-1}h_1P(A)A^{-1}h, \end{aligned}$$

where $h, h_1 \in H(\mathbf{X})$ (see (12)). For $h = h_1$ we obtain

$$D_A[P(A)(M(A)h)]h = -[P(A)(M(A)h)]^2.$$

Then

$$\begin{aligned} &D^{(m+1)}P(A)h^{(m+1)} \\ &\quad = D_A\{(-1)^{m-1}m![P(A)(M(A)h)]^m(I-P(A))\}h \\ &\quad = (-1)^{m-1}m!\{D_A[P(A)(M(A)h)]^m h\}(I-P(A)) \\ &\qquad + (-1)^m m![P(A)(M(A)h)]^m P(A)(M(A)h)(I-P(A)) \\ &\quad = (-1)^m m!m[P(A)(M(A)h)]^{m+1}(I-P(A)) \\ &\qquad + (-1)^m m![P(A)(M(A)h)]^{m+1}(I-P(A)) \\ &\quad = (-1)^m(m+1)![P(A)(M(A)h)]^{m+1}(I-P(A)). \end{aligned}$$

We conclude that (27) holds for every natural k.

Fix $A \in S_0(\mathbf{X})$. If $h \in \mathbf{B}_A$ then for any $m \in \mathbb{N}$,

$$P(A+h) = P(A) + \sum_{k=1}^{m} \frac{1}{k!} D^{(k)} P(A) h^{(k)} + \int_0^1 \frac{(1-t)^m}{m!} D^{(m+1)} P(A+th) h^{(m+1)}\, dt \tag{28}$$

(the Taylor formula, see [6]). Since for each $x \in \mathbf{X}$,

$$\begin{aligned}\left\| \left[\frac{1}{p!} D^{(p)} P(A+th) h^{(p)}\right] x \right\| &\le C_{I,A+th} \|\{[P(A+th)(M(A+th)h)]^p (I - P(A+th))\} x\|_{A+th} \\ &\le C_{I,A+th} \|(M(A+th))h\|^p_{A+th} \|x\|_{A+th} \\ &\le C_{I,A+th} C_{A+th,I} \left(\frac{\|h\|}{i(A+th)}\right)^p \|x\| \\ &= \sqrt{\frac{\|A+th\|}{i(A+th)}} \left(\frac{\|h\|}{i(A+th)}\right)^p \|x\| \le \sqrt{\frac{2\|A\| + i(A)}{i(A)}} \left(\frac{2\|h\|}{i(A)}\right)^p \|x\|\end{aligned}$$

(see (4), (5), (8) and (17)) we have

$$\left\| \frac{1}{p!} D^{(p)} P(A+th) h^{(p)} \right\| \le L_A \left(\frac{2\|h\|}{i(A)}\right)^p, \quad p \in \mathbb{N},$$

where

$$L_A := \sqrt{\frac{2\|A\| + i(A)}{i(A)}}.$$

This gives

$$\left\| \int_0^1 \frac{(1-t)^m}{m!} D^{(m+1)} P(A+th) h^{(m+1)}\, dt \right\| \le L_A \left(\frac{2\|h\|}{i(A)}\right)^{m+1} \int_0^1 (1-t)^m\, dt \le L_A \left(\frac{2\|h\|}{i(A)}\right)^{m+1}.$$

Since $h < i(A)/2$ we see that the remainder term in (28) converges to zero. ■

4. Algebraic formulas for the map P. In this section we assume that V is an arbitrary closed subspace of a Hilbert space $\mathbf{X}$ and for any $A \in S_0(\mathbf{X})$, $P(A)$ denotes the projector of $\mathbf{X}$ onto V, orthogonal with respect to the scalar product $\langle \cdot \mid \cdot \rangle_A$.

THEOREM 4.1. *For any $A \in S_0(\mathbf{X})$ and any $h \in \mathbf{B}_A$* (see (7)),

$$P(A+h) = P(A)[I + A^{-1} h (I - P(A+h))]. \tag{29}$$

Proof. Assume that $\dim V < \infty$. By (26) we have

$$\begin{aligned}P(A+h) &= P(A)\Big\{I + (M(A)h)\Big[I - P(A) - \sum_{k=2}^{\infty} (-1)^k [P(A)(M(A)h)]^{k-1} (I - P(A))\Big]\Big\} \\ &= P(A)\Big\{I + (M(A)h)\Big[I - \Big(P(A) - \sum_{k=1}^{\infty} (-1)^{k+1} [P(A)(M(A)h)]^k (I - P(A))\Big)\Big]\Big\} \\ &= P(A)[I + (M(A)h)(I - P(A+h))].\end{aligned}$$

Suppose now that V is an arbitrary closed subspace of $\mathbf{X}$. For a given $A \in S_0(\mathbf{X})$, $h \in \mathbf{B}_A$ and $x \in \mathbf{X}$ denote by V_0 the subspace of $\mathbf{X}$ spanned by the vectors $P(A+h)x, P(A)x$ and $P(A)A^{-1}h(I - P(A+h))x$. For any $B \in S_0(\mathbf{X})$ let $P_0(B)$ denotes the projector of $\mathbf{X}$ onto V_0, orthogonal with respect to the scalar product $\langle \cdot \mid \cdot \rangle_B$. Since V_0 is finite-dimensional we have

$$P_0(A+h)x = P_0(A)[I + A^{-1}h(I - P_0(A+h))]x.$$

On the other hand, $V_0 \subset V$, which implies $P_0(A+h)x = P(A+h)x$ and consequently

$$\begin{aligned}P_0(A)[I + A^{-1}h(I - P_0(A+h))]x &= P_0(A)x + P_0(A)A^{-1}h(I - P_0(A+h))x \\ &= P(A)x + P_0(A)A^{-1}h(I - P(A+h))x \\ &= P(A)x + P(A)A^{-1}h(I - P(A+h))x.\end{aligned}$$

Then

$$P(A+h)x = P(A)[I + A^{-1}h(I - P(A+h))]x. \ ■$$

Using the above theorem we will prove the next two algebraic formulas for $P(A+h)$.

THEOREM 4.2. *For any $A \in S_0(\mathbf{X})$ and any $h \in \mathbf{B}_A$* (see (7)) *such that $\|h\| < i(A)^{3/2}$,*

$$P(A+h) = [I + P(A)A^{-1}h]^{-1} P(A)(I + A^{-1}h) \tag{30}$$

(see also Remark 5.1). *If moreover $\|h\| < \frac{1}{3}\min\{i(A), i(A)^{3/2}\}$ then*

$$P(A+h) = P(A)[I - (A+h)^{-1}h(I - P(A))]^{-1}. \tag{31}$$

Proof. By (29) we have

$$P(A+h) = P(A) + P(A)A^{-1}h - P(A)A^{-1}hP(A+h).$$

Then

$$(I+P(A)A^{-1}h)P(A+h)=P(A)(I+A^{-1}h). \tag{32}$$

Since $\|h\|<i(A)^{3/2}$ we obtain

$$\begin{aligned}\|P(A)A^{-1}h\| &\le \|P(A)\|\cdot\|A^{-1}\|\cdot\|h\|\\ &\le C_{I,A}\|P(A)\|_A\frac{1}{i(A)}\|h\|\le\frac{1}{i(A)^{3/2}}\|h\|<1\end{aligned}$$

(see (5)). Hence $I+P(A)A^{-1}h$ is an invertible operator. Multiplying both sides of (32) by $[I+P(A)A^{-1}h]^{-1}$ we obtain (30).

Suppose now that $\|h\|<\frac13\min\{i(A),i(A)^{3/2}\}$. We have

$$\begin{aligned}\|(A+h)^{-1}h(I-P(A))\| &\le \frac{\|h\|}{i(A+h)}C_{I,A}\|I-P(A)\|_A\\ &=\frac{\|h\|}{i(A+h)i(A)^{1/2}}\le\frac{\|h\|}{(i(A)-\|h\|)i(A)^{1/2}}\\ &\le\frac{(1/3)i(A)^{3/2}}{(2/3)i(A)^{3/2}}=\frac12\end{aligned}$$

and therefore the operator $I-(A+h)^{-1}h(I-P(A))$ is invertible. Thus the right hand side of (31) is well defined. Since

$$\frac12 i(A+h)\ge\frac12(i(A)-\|h\|)\ge\frac12\cdot\frac23 i(A)=\frac13 i(A)$$

and $\|-h\|<\frac13 i(A)$ we see that $-h\in\mathbf{B}_{A+h}$. Hence, by Theorem 4.1,

$$\begin{aligned}P(A)&=P((A+h)-h)\\ &=P(A+h)[I+(A+h)^{-1}(-h)(I-P((A+h)-h)]\\ &=P(A+h)[I-(A+h)^{-1}h(I-P(A))],\end{aligned}$$

which implies (31). ■

5. Orthogonal projectors onto an arbitrary closed subspace. Theorem 4.2 allows us to prove Theorem 3.2 without the assumption that V is finite-dimensional. More precisely, we have the following

THEOREM 5.1. *Let V be an arbitrary closed subspace of the Hilbert space $\mathbf{X}$. Then the mapping P which assigns to any $A\in S_0(\mathbf{X})$ the projector $P(A)$ of $\mathbf{X}$ onto V (orthogonal with respect to the scalar product $\langle\cdot\mid\cdot\rangle_A$) is analytic on $S_0(\mathbf{X})$. For any $A\in S_0(\mathbf{X})$ and $h\in\mathbf{B}_A$ we have the Taylor expansion*

$$P(A+h)=P(A)+\sum_{k=1}^{\infty}(-1)^{k-1}[P(A)(M(A)h)]^k(I-P(A)), \tag{33}$$

where the series on the right hand side converges uniformly in h on each ball $B_{H(\mathbf{X})}(0,r)$ such that $r<i(A)/2$.

Proof. Assume that $\|h\|<i(A)^{3/2}$, i.e. $h\in B_{H(\mathbf{X})}(0,r(A))$, where $r(A):=\min\{i(A)/2,i(A)^{3/2}\}$. Then we can use formula (30) from Theorem 4.2 and the equality

$$[I+P(A)A^{-1}h]^{-1}=\sum_{k=0}^{\infty}(-1)^k[P(A)A^{-1}h]^k, \tag{34}$$

where the series on the right hand side converges uniformly in h on any ball $B_{H(\mathbf{X})}(0,r)$ with $r<r(A)$. We obtain

$$\begin{aligned}P(A+h)&=\sum_{k=0}^{\infty}(-1)^k[P(A)A^{-1}h]^kP(A)(I+A^{-1}h)\\ &=P(A)(I+A^{-1}h)+\sum_{k=1}^{\infty}(-1)^k[P(A)A^{-1}h]^kP(A)\\ &\quad+\sum_{k=1}^{\infty}(-1)^k[P(A)A^{-1}h]^{k+1}\\ &=P(A)+P(A)A^{-1}h+\sum_{k=2}^{\infty}(-1)^{k-1}[P(A)A^{-1}h]^k\\ &\quad-\sum_{k=1}^{\infty}(-1)^{k-1}[P(A)A^{-1}h]^kP(A)\\ &=P(A)+\sum_{k=1}^{\infty}(-1)^{k-1}[P(A)A^{-1}h]^k\\ &\quad-\sum_{k=1}^{\infty}(-1)^{k-1}[P(A)A^{-1}h]^kP(A)\\ &=P(A)+\sum_{k=1}^{\infty}(-1)^{k-1}[P(A)A^{-1}h]^k(I-P(A)),\end{aligned}$$

and the last series converges uniformly on $B_{H(\mathbf{X})}(0,r)$, $r<r(A)$. This means that P is analytic and (33) holds for $h\in B_{H(\mathbf{X})}(0,r(A))$. In particular, equalities (27) hold for any $k\in\mathbb{N}$ and therefore, using the same arguments as in the proof of Theorem 3.2 (for the remainder in the Taylor expansion), one can show (33) for each $h\in\mathbf{B}_A$. ■

REMARK 5.1. The assumption that $\|h\|<i(A)^{3/2}$ in Theorem 4.2 is not essential. It is not difficult to show (using (17) and arguments similar to the proof of Theorem 3.2) that the equality (34) holds for any $h\in\mathbf{B}_A$. Then (30) also holds for any $h\in\mathbf{B}_A$. We leave the details to the reader. ■

REMARK 5.2. For a given $A \in S_0(\mathbf{X})$ let

$$F(h) := P(A)A^{-1}h(I + P(A)A^{-1}h)^{-1}, \quad h \in \mathbf{B}_A.$$

Since $\|P(A)A^{-1}h\| < \frac{1}{2}$ (see (2) and (7)) we see that the map F is well defined and analytic from $\mathbf{B}_A$ to the Banach algebra $L(\mathbf{X})$ and $F(0) = 0$. Moreover, $P(A)$ is idempotent and $[F(0), P(A)] = 0$. We can now refer to the result obtained in [3]. Suppose that Theorem 1 in [3] holds with $h \in \mathbf{B}_A$ in place of $z \in U \subset \mathbb{C}$. Then we would have:

There exists an open neighborhood V of 0 in $\mathbf{B}_A$ and two analytic mappings $P_0(h)$ and $R(h)$ of V in $L(\mathbf{X})$ such that:

(i) $P_0(0) = P(A)$ *and for all* $h \in V$, $P_0(h)$ *is idempotent*;

(ii) *for all* $h \in V$, $[R(h), P(A)] = 0$;

(iii) *for all* $h \in V$, $F(h) = P_0(h) + R(h)$;

(iv) *the pair of mappings* (P_0, R) *is uniquely determined by properties* (i) *to* (iii).

The map P_0 is given by the formula (2) of [3]. Using this formula we obtain

$$(35)\quad P_0(h) = P(A) + [[F(h), P(A)], P(A)] = P(A) + F(h)(I - P(A)) = P(A) + \sum_{k=1}^{\infty}(-1)^{k-1}[P(A)(M(A)h)]^k(I - P(A))$$

(we have $[F(h), P(A)][[F(h), P(A)], P(A)] = (F(h)(I - P(A)))^2 = 0$). Note that the last expression in (35) is identical with the right hand side of (33). Then one can hope that it is possible to obtain Theorem 5.1 using (a little modified) results of [3]. However, we do not know how to prove the equality $P_0(h) = P(A + h)$, $h \in \mathbf{B}_A$, without using considerations contained in Sections 2–4 of our paper.

Acknowledgments. I express my thanks to A. Odzijewicz and S. Janeczko for the inspiration and very useful discussions during writing this paper. I also thank S. Woronowicz and J. Janas for conversations on the topic of the study.

Special thanks are due to my wife for her help.

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Received March 6, 1996
Revised version March 6, 1997 and September 22, 1997 (3626)