References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, 1972.

[2] S. Angenent, *Local existence and regularity for a class of degenerate parabolic equations*, Math. Ann. 280 (1988), 465–482.

[3] A. Attalienti and S. Romanelli, *On some classes of analytic semigroups on $C([a,b])$ related to* **R**- *or Γ-admissible mappings*, in: Evolution Equations, G. Ferreyra, G. Goldstein and F. Neubrander (eds.), Lecture Notes in Pure and Appl. Math. 168, Dekker, 1995, 29–34.

[4] H. Brezis, W. Rosenkrantz and B. Singer, *On a degenerate elliptic-parabolic equation occurring in the theory of probability*, Comm. Pure Appl. Math. 24 (1971), 395–416.

[5] M. Campiti and G. Metafune, *Ventcel's boundary conditions and analytic semigroups*, Arch. Math. (Basel), to appear.

[6] M. Campiti, G. Metafune and D. Pallara, *Degenerate self-adjoint evolution equations on the unit interval*, Semigroup Forum, to appear.

[7] P. Clément and C. A. Timmermans, *On C_0-semigroups generated by differential operators satisfying Ventcel's boundary conditions*, Indag. Math. 89 (1986), 379–387.

[8] A. Favini, J. Goldstein and S. Romanelli, *Analytic semigroups on $L^p_w(0,1)$ generated by some classes of second order differential operators*, preprint, 1996.

[9] A. Favini and S. Romanelli, *Analytic semigroups on $C([0,1])$ generated by some classes of second order differential operators*, Semigroup Forum, to appear.

[10] W. Feller, *Two singular diffusion problems*, Ann. of Math. 54 (1951), 173–182.

[11] —, *The parabolic differential equations and the associated semi-groups of transformations*, ibid. 55 (1952), 468–519.

[12] G. Fichera, *On a degenerate evolution problem*, in: Partial Differential Equations with Real Analysis, H. Begeher and A. Jeffrey (eds.), Pitman Res. Notes Math. Ser. 263, Longman, 1992, 15–42.

[13] R. Nagel (ed.), *One-Parameter Semigroups of Positive Operators*, Springer, 1986.

[14] K. Taira, *On the existence of Feller semigroups with boundary conditions, II*, J. Funct. Anal. 129 (1995), 108–131.

[15] C. A. Timmermans, *On C_0-semigroups in a space of bounded continuous functions in the case of entrance or natural boundary points*, in: Approximation and Optimization, J. A. Gómez Fernández *et al.* (eds.), Lecture Notes in Math. 1354, Springer, 1988, 209–216.

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Intrinsic characterizations of distribution spaces on domains

by

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Abstract. We give characterizations of Besov and Triebel–Lizorkin spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ in smooth domains $\Omega \subset \mathbb{R}^n$ via convolutions with compactly supported smooth kernels satisfying some moment conditions. The results for $s \in \mathbb{R}$, $0 < p,q \le \infty$ are stated in terms of the mixed norm of a certain maximal function of a distribution. For $s \in \mathbb{R}$, $1 \le p \le \infty$, $0 < q \le \infty$ characterizations without use of maximal functions are also obtained.

1. Introduction. The Besov and Triebel–Lizorkin spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p,q \le \infty$, are well-known scales of spaces of tempered distributions on $\mathbb{R}^n$, covering classical Hölder–Zygmund spaces, fractional Sobolev spaces, local Hardy spaces and their duals.

After being introduced in the 60s–70s in the pioneering papers by

O. V. Besov [Bes1,2] (B^s_{pq} spaces, $s > 0$, $1 \le p,q \le \infty$),
M. H. Taibleson [Tai] (B^s_{pq} spaces, $s \in \mathbb{R}$, $1 \le p,q \le \infty$),
P. I. Lizorkin [Liz1,2] (F^s_{pq} spaces, $s > 0$, $1 < p,q < \infty$),
H. Triebel [Tri1] (F^s_{pq} spaces, $s \in \mathbb{R}$, $1 < p,q < \infty$),
J. Peetre [P1,2] (extensions of B^s_{pq} and F^s_{pq} to all $0 < p,q \le \infty$),

these spaces were studied in detail. General references for the theory of B^s_{pq} and F^s_{pq} spaces are two monographs by H. Triebel [Tri2,3], and the fundamental paper by M. Frazier and B. Jawerth [FrJ].

In this paper B^s_{pq} and F^s_{pq} spaces on domains are studied. Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary. The natural way (also used here) to introduce distribution spaces $B^s_{pq}(\Omega), F^s_{pq}(\Omega) \subset \mathcal{D}'(\Omega)$ is to define them as restrictions of corresponding spaces from $\mathbb{R}^n$ to Ω. Then the problem of finding intrinsic characterizations of these spaces arises.

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Intrinsic characterizations of $B^s_{pq}(\Omega)$ with $s > 0$, $1 \le p, q \le \infty$ were proved by O. V. Besov [Bes3]. G. A. Kalyabin [Kal1,2] proved intrinsic characterizations of $F^s_{pq}(\Omega)$ with $s > 0$, $1 < p, q < \infty$. Extensions to $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ with $s > n\max(0, 1/p - 1)$, $0 < p, q \le \infty$ are due to A. Seeger [See] and H. Triebel [Tri4] (see [Tri3, 1.10, 5.2]).

We would like to stress two points. First, in all the above-mentioned cases the spaces under consideration consist of regular distributions (locally summable functions). Second, all the above-mentioned characterizations are given in quite constructive terms, such as differences and oscillations of functions.

However, if we ask for constructive intrinsic characterizations of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ with, say, $s < 0$, it turns out that the question for a considerable part remains open. Indeed, the problem for Besov spaces $B^s_{pq}(\Omega)$ with $s \in \mathbb{R}$, $1 \le p, q \le \infty$, and for Sobolev spaces $W^s_p(\Omega) = F^s_{p2}(\Omega)$ with $s \in \mathbb{R}$, $1 < p < \infty$ was solved by T. Muramatu [Mur] by constructing certain integral representations. As far as general F^s_{pq} spaces or $0 < p < 1$ are concerned, the only relevant work known to us is the paper by H. Triebel and H. Winkelvoß [TrW], where intrinsic characterizations of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ by means of atomic decompositions for all $s \in \mathbb{R}$ (and even for more general domains) are given. But these characterizations, in spite of their usefulness for some applications, are apparently not constructive in the usual sense of the word (see a more detailed discussion in Section 5(b)).

In this paper we give rather satisfactory constructive intrinsic characterizations of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ in the full range of indices s, p and q. To write an explicit equivalent quasi-norm of a distribution $f \in \mathcal{D}'(\Omega)$ in a corresponding space, we use means of f via some kernels belonging to $\mathcal{D}(\Omega)$.

It has already been known for a long time that means of $f \in \mathcal{S}'$ via compactly supported smooth kernels can be used to introduce equivalent quasi-norms in $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ (see [Tri3, 2.4.6]). But the rather delicate question whether these quasi-norms give characterizations of the corresponding spaces was answered only recently by H.-Q. Bui, M. Paluszyński, and M. H. Taibleson [BPT1,2]. Fortunately, the answer is yes. Our considerations will be based to a great extent on ideas from [Tri3], [BPT1,2].

The rest of the paper is organized as follows. In Section 2 necessary definitions are given and main results are formulated and discussed. Proofs are contained in Sections 3 and 4. Section 5 is devoted to remarks as well as to relations with other results in the literature.

The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}^n$, $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$ (where Ω is a domain in $\mathbb{R}^n$), L_p $(0 < p \le \infty)$ have their usual meaning.

For $x \in \mathbb{R}^n$ we write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

For the sake of brevity we omit $\mathbb{R}^n$ in notations of all spaces, quasi-norms and integrals on $\mathbb{R}^n$.

For $q = \infty$ an expression $(\sum |a_k|^q)^{1/q}$ always denotes $\sup |a_k|$.

For $\lambda \in \mathcal{D}$ and $j \in \mathbb{N}$ we write $\lambda_j(x) = 2^{jn}\lambda(2^j x)$.

As usual, c denotes an unimportant positive constant, which can change from one estimate to another.

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2. Definitions and results. From the variety of known equivalent definitions of B^s_{pq} and F^s_{pq} spaces, the following one is the most close to the spirit of our paper. H. Triebel [Tri3, 2.4.6] called it "characterization via local means".

DEFINITION 2.1. Let $0 < p, q \le \infty$ ($p < \infty$ in the F^s_{pq} case), $s \in \mathbb{R}$ and $M \in \mathbb{N}_0$, $2M > s$. Let $\lambda_0 \in \mathcal{D}$ and $\int \lambda_0(x)\,dx \ne 0$. Let $\lambda = \Delta^M \lambda_0$, where Δ stands for the Laplacian. Then

$$B^s_{pq} = \Big\{ f \in \mathcal{S}' : \|f|B^s_{pq}\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|\lambda_j * f(\cdot)|L_p\|^q \Big)^{1/q} < \infty \Big\},$$

$$F^s_{pq} = \Big\{ f \in \mathcal{S}' : \|f|F^s_{pq}\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |\lambda_j * f(\cdot)|^q \Big)^{1/q} \Big| L_p \Big\| < \infty \Big\}.$$

Remark 2.1. (a) A reader familiar with the original Fourier analytical definition of B^s_{pq} and F^s_{pq} (see e.g. [Tri3, 2.3.1]) will notice that all its main features (the sequence of convolutions, the factor 2^{jsq}, $l_q(L_p)$ and $L_p(l_q)$ quasi-norms) are preserved in Definition 2.1. The only (but crucial) difference lies in the convolution kernels λ_j having compact support in Definition 2.1. This was by no means the case in the original definition, where the Fourier transforms $\widehat{\lambda}_j$ rather than the λ_j were compactly supported. The equivalence of the two definitions follows from more general results of [BPT1,2]; see also [Tri3, 2.4.6] for earlier versions.

(b) Defining F^s_{pq} spaces, we excluded the case $p = \infty$. The spaces $F^s_{\infty q}$ will not be treated in this paper. We only note that Definition 2.1 does not work in this case and should be replaced by a more sophisticated construction. The details can be found in [Tri2, 2.3.4] and [FrJ].

Given a space of distributions on $\mathbb{R}^n$, one can easily construct the corresponding space on an arbitrary domain $\Omega \subset \mathbb{R}^n$ by a restriction procedure. The exact definition of B^s_{pq} and F^s_{pq} spaces on domains is the following.

DEFINITION 2.2. Let Ω be a domain in $\mathbb{R}^n$. Let $0 < p, q \le \infty$ ($p < \infty$ in the F^s_{pq} case) and $s \in \mathbb{R}$. Then $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ are the restrictions of

B^s_{pq} and F^s_{pq} to Ω, quasi-normed by

$$\|f|B^s_{pq}(\Omega)\| = \inf\{\|g|B^s_{pq}\| : g \in B^s_{pq},\ g|_\Omega = f \text{ in the sense of } \mathcal{D}'(\Omega)\},$$
$$\|f|F^s_{pq}(\Omega)\| = \inf\{\|g|F^s_{pq}\| : g \in F^s_{pq},\ g|_\Omega = f \text{ in the sense of } \mathcal{D}'(\Omega)\}.$$

Remark 2.2. Being defined in this way, spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ luckily inherit some good properties of spaces on $\mathbb{R}^n$, e.g. the completeness in the topology induced by the quasi-norm, and the embedding assertions.

Some other properties either do not have direct analogues in the case of spaces on $\mathbb{R}^n$, or should be given a separate proof. Besides that, generally speaking, a certain smoothness of the boundary $\partial\Omega$ is needed. If $\partial\Omega$ is smooth enough (say, C^∞), then $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ have the following properties (which we demonstrate on the example of the F^s_{pq} scale).

(a) *The extension property* ([Tri3, 4.5.5]). There exists a linear bounded extension operator $\mathrm{ext} : F^s_{pq}(\Omega) \to F^s_{pq}$.

(b) *Estimates for elliptic operators* (J. Franke and T. Runst [FrR]). Let $s > (n-1)\max(0, 1/p - 1) + 1/p$. Then the Dirichlet problem

$$\Delta u = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

has a unique solution $u \in F^s_{pq}(\Omega)$ for every right-hand side $f \in F^{s-2}_{pq}(\Omega)$.

(c) *Intrinsic characterizations.* We have already discussed this question in the Introduction. The aim of the paper is to contribute in just this field of knowledge about $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$.

Our main results will be formulated for the case of Ω being an open set above the graph of a smooth function $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$. The exact definition of that class of domains runs as follows.

Definition 2.3. Let $L \in \mathbb{N}$. A *special C^L-domain* $\Omega \subset \mathbb{R}^n$ is a domain of the type

$$\Omega = \{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n > \omega(x')\},$$

where the function $\omega(x')$ has continuous derivatives up to order L, and $|D^\alpha\omega(x')| \le C < \infty$ for all multi-indices α with $0 < |\alpha| \le L$ and all $x' \in \mathbb{R}^{n-1}$.

We want to give characterizations of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ inspired by Definition 2.1. It means that we should provide ourselves with a sequence of averaging kernels having sufficiently many vanishing moments (as $\lambda = \Delta^M\lambda_0$ does). Moreover, convolutions of these kernels with distributions from $\mathcal{D}'(\Omega)$ should make sense. These two crucial properties are summarized in the following definition.

Definition 2.4. We say that a function $\eta_0 \in \mathcal{D}$ is a *proper kernel of order* $M \in \mathbb{N}_0$ if

$$\int \eta_0(x)\,dx = 1,$$

and the function η defined by $\eta(x) = \eta_0(x) - 2^{-n}\eta_0(x/2)$ has vanishing moments up to order M, i.e.

$$\int x^\alpha\eta(x)\,dx = 0 \quad \text{for all multi-indices } \alpha \text{ with } |\alpha| \le M.$$

Furthermore, we say that the kernel η_0 is *suitable* for a special C^L-domain Ω if η_0 is supported in a cone

$$K = \{x : |x'| < B^{-1}|x_n|,\ x_n < 0\},$$

where $\sup|\nabla\omega(x')| \le B < \infty$.

The condition $\operatorname{supp}\eta_0 \subset K$ implies that, for every $x \in \Omega$ and each $j \in \mathbb{N}_0$,

$$\operatorname{supp}\eta_j(x - \cdot) \subset x - K \subset \Omega$$

and thus the convolution $\eta_j * f(x)$ is correctly defined for every $f \in \mathcal{D}'(\Omega)$.

Given a number $M \in \mathbb{N}_0$ and a special C^L-domain Ω, one can construct a proper kernel of order M, suitable for Ω, in the following way. First, there exists a function $\eta_0 \in \mathcal{D}$ supported in the lower halfspace $\{x : x_n < 0\}$ and such that

$$\int \eta_0(x)\,dx = 1, \quad \int x^\alpha\eta_0(x)\,dx = 0, \quad 0 < |\alpha| \le M$$

(see [Tri3, 3.3, pp. 173–175] for details). Then $A^{n-1}\eta_0(Ax', x_n)$ is a proper kernel of order M for every $A > 0$. For sufficiently large A this kernel will also be suitable for Ω.

Now we are in a position to formulate our results. These are two theorems below. Recall that $\eta_j = 2^{jn}\eta(2^j x)$ for $j \in \mathbb{N}$.

Theorem 2.1. *Let $s \in \mathbb{R}$, $0 < p, q \le \infty$ and $a > n/p$ ($p < \infty$ and $a > n/\min(p,q)$ in the F^s_{pq} case). Let $M = [|s| + 2a]$. Let Ω be a special C^{M+1}-domain, and η_0 be a proper kernel of order M, suitable for Ω. For $f \in \mathcal{D}'(\Omega)$ and $j \in \mathbb{N}_0$ introduce the maximal functions*

$$\eta^\Omega_{j,a}f(x) = \sup_{y\in\Omega}\frac{|\eta_j * f(y)|}{(1 + 2^j|x-y|)^a}, \quad x \in \Omega. \tag{2.1}$$

Then

$$B^s_{pq}(\Omega) = \Big\{f \in \mathcal{D}'(\Omega) :$$
$$\|f|B^s_{pq}(\Omega)\|' = \Big(\sum_{j=0}^\infty 2^{jsq}\|\eta^\Omega_{j,a}f(\cdot)|L_p(\Omega)\|^q\Big)^{1/q} < \infty\Big\},$$

$$F^s_{pq}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) : \|f|F^s_{pq}(\Omega)\|' = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \eta^{\Omega}_{j,a} f(\cdot)^q \Big)^{1/q} \Big| L_p(\Omega) \Big\| < \infty \Big\},$$

and the quantities $\|f|B^s_{pq}(\Omega)\|'$ *and* $\|f|F^s_{pq}(\Omega)\|'$ *are equivalent quasi-norms in the corresponding spaces.*

THEOREM 2.2. *Let* $s \in \mathbb{R}$, $1 \le p \le \infty$ ($1 \le p < \infty$ *in the* F^s_{pq} *case) and* $0 < q \le \infty$. *Let* Ω *be a special* C^{L+1}*-domain, and* η_0 *be a proper kernel of order* M, *suitable for* Ω, *where*

$$L = [5|s| + 4n/\min(1,q)], \qquad M = [8|s| + 8n/\min(1,q)].$$

Then

$$B^s_{pq}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) : \|f|B^s_{pq}(\Omega)\|'' = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|\eta_j * f(\cdot)|L_p(\Omega)\|^q \Big)^{1/q} < \infty \Big\},$$

$$F^s_{pq}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) : \|f|F^s_{pq}(\Omega)\|'' = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |\eta_j * f(\cdot)|^q \Big)^{1/q} \Big| L_p(\Omega) \Big\| < \infty \Big\},$$

and the quantities $\|f|B^s_{pq}(\Omega)\|''$ *and* $\|f|F^s_{pq}(\Omega)\|''$ *are equivalent quasi-norms in the corresponding spaces.*

Theorems 2.1 and 2.2 give the desired intrinsic characterizations of $F^s_{pq}(\Omega)$ and $B^s_{pq}(\Omega)$ for domains Ω of the indicated type. Theorem 2.1 is applicable in the full range of parameters s, p, q. Its result can be called a "characterization via maximal local means". On the other hand, Theorem 2.2 supplies a "characterization via local means", which is very close to that of Definition 2.1 for spaces on $\mathbb{R}^n$. However, in this case we have not been able to cover the "gap" $0 < p < 1$.

Remark 2.3. (a) The maximal functions (2.1) are nothing but modifications of Peetre's maximal functions

$$\eta^*_{j,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\eta_j * f(y)|}{(1 + 2^j|x - y|)^a}, \tag{2.2}$$

introduced by J. Peetre [P2]. In the same paper the first characterizations of F^s_{pq} spaces by means of these maximal functions were obtained. Far-reaching generalizations of these results have recently been proved in [BPT1].

(b) We note that with the help of a smooth partition of unity our two theorems can be used to write down intrinsic characterizations for spaces

$B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ on an arbitrary bounded domain Ω with smooth boundary of class C^L, L being sufficiently large. We omit the exact formulation of this result, but we believe that the main idea should be quite clear.

(c) We realize that the limitations $M = [|s| + 2a]$ and $M = [8|s| + 8n/\min(1,q)]$, imposed on the number of vanishing moments of kernels in Theorems 2.1 and 2.2, look unnatural in comparison with Definition 2.1, where (roughly speaking) only s vanishing moments are required. The contrast becomes even more evident if $s < 0$ is taken, and $M = 0$ is admissible in Definition 2.1.

Theorems 2.1 and 2.2 are proved in Sections 3 and 4, respectively. In the proofs we consider only the case of F^s_{pq} spaces. For B^s_{pq} spaces the corresponding arguments are quite analogous and generally simpler.

3. Proof of Theorem 2.1. To prove Theorem 2.1, it is necessary and sufficient to show the following two assertions.

(A) Let $g \in F^s_{pq}$ and $f = g|_\Omega$. Then

$$\|f|F^s_{pq}(\Omega)\|' \le c\|g|F^s_{pq}\|.$$

(B) Let $f \in \mathcal{D}'(\Omega)$ and $\|f|F^s_{pq}(\Omega)\|' < \infty$. Then there exists a distribution $\operatorname{ext} f \in F^s_{pq}$ such that $\operatorname{ext} f|_\Omega = f$ and

$$\|\operatorname{ext} f|F^s_{pq}\| \le c\|f|F^s_{pq}(\Omega)\|'.$$

Step 1. Let us prove (A). Immaterial modifications of [BPT1, Theorem 3.1] (see also [Tri3, 2.4.1, Corollary 2]) show that if η has vanishing moments up to order $M \ge [s]$ (which is true in our context), then, for every $g \in F^s_{pq}$,

$$\Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \eta^*_{j,a} g(\cdot)^q \Big)^{1/q} \Big| L_p \Big\| \le c\|g|F^s_{pq}\|,$$

where $\eta^*_{j,a} g(x)$ are Peetre's maximal functions (see (2.2)). Let $f = g|_\Omega$. Since the kernel η_0 is suitable for Ω, we see that, for all $x \in \Omega$ and $j \in \mathbb{N}_0$,

$$\eta^{\Omega}_{j,a} f(x) \le \eta^*_{j,a} g(x).$$

This implies that $\|f|F^s_{pq}(\Omega)\|' \le c\|g|F^s_{pq}\|$.

Step 2. To prove (B), we should choose an appropriate extension operator ext.

Let $1 < u_1 < u_2 < \ldots < u_{2M+3}$. Let $v_1, \ldots, v_{2M+3}$ be the uniquely determined real numbers such that

$$\sum_{m=1}^{2M+3} v_m (1 - u_m)^k = 1, \qquad k = -(M+2), \ldots, 0, \ldots, M \tag{3.1}$$

(Vandermonde's determinant). In the sequel, the index m will always run through the set $\{1,\ldots,2M+3\}$, and $\sum_m$ will always denote $\sum_{m=1}^{2M+3}$.

First, let f be a continuous function in $\overline{\Omega}$ ($f\in C(\overline{\Omega})$). For such a function we put

$$\operatorname{ext} f(x)=\begin{cases} f(x) & \text{if } x\in\overline{\Omega},\\ \sum_m v_m f(x', x_n+u_m(\omega(x')-x_n)) & \text{if } x\in\mathbb{R}^n\setminus\overline{\Omega},\end{cases}\tag{3.2}$$

where $\omega:\mathbb{R}^{n-1}\to\mathbb{R}$ is the function describing $\partial\Omega$. Note that (3.1) with $k=0$ immediately implies that $\operatorname{ext} f\in C(\mathbb{R}^n)$.

Remark 3.1. The construction of the extension operator in the form (3.1)–(3.2) goes back to M. Hestenes [Hes], at least in the case of Ω being the halfspace. In fact, conditions (3.1) with nonnegative k guarantee that $\operatorname{ext} f\in C^M(\mathbb{R}^n)$ for each $f\in C^M(\overline{\Omega})$. However, we shall use these conditions in another context. Conditions (3.1) with negative k are especially important for extensions of spaces with negative smoothness; they were introduced by H. Triebel (see, e.g., [Tri3, 4.5.5]).

Now let us consider the general case: $f\in\mathcal{D}'(\Omega)$ and $\|f|F^s_{pq}(\Omega)\|'<\infty$. For $j\in\mathbb{N}_0$ we define $f_j(x)=\eta_j*f(x)$, $x\in\Omega$. It is clear that $f_j\in C(\overline{\Omega})$, and thus the extensions $\operatorname{ext} f_j$ can be defined by (3.4). For $i\in\mathbb{N}$ we put

$$\operatorname{ext}^i f=\sum_{j=0}^{i}\operatorname{ext} f_j.$$

Below we shall prove the uniform boundedness of the sequence $\{\operatorname{ext}^i f\}$ in F^s_{pq}:

$$\|\operatorname{ext}^i f|F^s_{pq}\|\le c\|f|F^s_{pq}(\Omega)\|',\quad i\in\mathbb{N},\tag{3.3}$$

and this will be enough to finish the proof. Indeed, by (3.3) and the continuous embedding $F^s_{pq}\subset\mathcal{S}'$, the sequence $\{\operatorname{ext}^i f\}\subset\mathcal{S}'$ is uniformly bounded on a neighborhood of 0 in a separable topological vector space $\mathcal{S}$. Now by a variant of the Banach–Alaoglu theorem (see, e.g., W. Rudin [Rud, Theorem 3.17]) there exists a subsequence $\{\operatorname{ext}^{i_k} f\}$ converging in $\mathcal{S}'$. Denoting its limit by $\operatorname{ext} f$, we see by (3.3) that

$$\|\operatorname{ext} f|F^s_{pq}\|\le c\|f|F^s_{pq}(\Omega)\|'.$$

Moreover, since $\sum_{j=0}^{i}\eta_j(x)=2^{in}\eta_0(2^i x)\to\delta$ (Dirac's δ-function) as $i\to\infty$, we have $\sum_{j=0}^{\infty}f_j=f$ in $\mathcal{D}'(\Omega)$, and thus $\operatorname{ext} f|_\Omega=f$.

The proof of (3.3) will be divided into 4 steps.

Step 3. Let us note the following identity. Let $\lambda\in\mathcal{D}$ and $f\in C(\Omega)$. Then straightforward calculations, based on (3.2), show that

$$\int\lambda(x)\operatorname{ext} f(x)\,dx=\int_\Omega\widetilde{\lambda}(x)f(x)\,dx,\tag{3.4}$$

where

$$\widetilde{\lambda}(x)=\lambda(x)-\sum_m\frac{v_m}{1-u_m}\lambda\Big(x',\omega(x')+\frac{x_n-\omega(x')}{1-u_m}\Big),\quad x\in\overline{\Omega}.\tag{3.5}$$

We establish some properties of $\widetilde{\lambda}$. Taking into account the smoothness of ω, stated in Definition 2.3, we see that

$$\widetilde{\lambda}\in C^{M+1}(\overline{\Omega}),\tag{3.6}$$

i.e., all derivatives $D^\alpha\widetilde{\lambda}$, $|\alpha|\le M+1$, are continuous and bounded in $\overline{\Omega}$. Moreover, using the identity

$$\sum_m\frac{v_m}{(1-u_m)^{i+1}}\Big(1-\frac{1}{1-u_m}\Big)^j=0,\quad i,j\ge0,\ i+j\le M+1,$$

following from (3.1) with negative k by linearity, it is not difficult to see that

$$D^\alpha\widetilde{\lambda}|_{\partial\Omega}=0,\quad |\alpha|\le M+1.\tag{3.7}$$

This implies, in particular, that the function $\widetilde{\lambda}_0$, obtained by the extension of $\widetilde{\lambda}$ from $\overline{\Omega}$ to the whole $\mathbb{R}^n$ by zero, belongs to $C^{M+1}(\mathbb{R}^n)$.

Another useful circumstance (clarifying the role of (3.1) with positive k) is the following. Suppose additionally that λ has vanishing moments up to order M. Then the same is true for $\widetilde{\lambda}_0$. To see this, we note the following easy consequence of (3.1) with nonnegative k: if $f(x)=x^\alpha$, $x\in\overline{\Omega}$, where $|\alpha|\le M$, then $\operatorname{ext} f=x^\alpha$, $x\in\mathbb{R}^n$. From this fact and (3.4) it follows that, for $|\alpha|\le M$,

$$\int_\Omega x^\alpha\widetilde{\lambda}(x)\,dx=\int x^\alpha\lambda(x)\,dx,$$

and thus

$$\int x^\alpha\lambda(x)dx=0,\ |\alpha|\le M\ \Rightarrow\ \int x^\alpha\widetilde{\lambda}_0(x)dx=0,\ |\alpha|\le M.\tag{3.8}$$

Step 4. Now let again $f\in\mathcal{D}'(\Omega)$ and $\|f|F^s_{pq}(\Omega)\|'<\infty$. To estimate $\|\operatorname{ext}^i f|F^s_{pq}\|$, we shall use the following lemma, which is essentially known (combine Theorem 3.3.3 from [Tri3] with results of [BPT2]).

LEMMA 3.1. *Let $0<p<\infty$, $0<q\le\infty$, and $s\in\mathbb{R}$. Let μ_0 be a proper kernel of order $M>s$. There exists a natural number $J=J(\mu_0)$ such that,* *for all* $g \in \mathcal{S}'$,

$$\|g|F^s_{pq}\| \le c\Big\|\Big(\sum_{j=-J}^{\infty} 2^{jsq}|\mu_j * g(\cdot)|^q\Big)^{1/q}\Big|L_p\Big\|,$$

where for negative j *we put* $\mu_j(x) = 2^{jn}\mu_0(2^j x)$.

Our way to the application of Lemma 3.1 begins with the inequality

$$(3.9)\qquad |\eta_l * \operatorname{ext}^i f(x)| \le \sum_{j=0}^{\infty} |\eta_l * \operatorname{ext} f_j(x)| =: \mathbf{S}_l f(x),$$
$$x \in \mathbb{R}^n,\ i \in \mathbb{N},\ l \ge -J,\ J = J(\eta_0).$$

According to (3.4)–(3.8), we have the identity

$$(3.10)\qquad \mathbf{S}_l f(x) = \sum_{j=0}^{\infty} \Big|\int K_l(x,y)\,\eta_j * f(y)\,dy\Big|,$$

where the kernel $K_l(x,y)$ is given by

$$K_l(x,y) = \begin{cases} \eta_l(x-y) - \sum_m \frac{v_m}{1-u_m}\eta_l\Big(x'-y', x_n - \omega(y') - \frac{y_n - \omega(y')}{1-u_m}\Big), & y \in \Omega,\\ 0 \quad \text{otherwise}, \end{cases}$$

and has the properties

$$(3.11)\qquad K_l(x,\cdot) \in C^{M+1}(\mathbb{R}^n),\quad x \in \mathbb{R}^n;$$

$$(3.12)\qquad \int y^\alpha K_l(x,y)\,dy = 0,\quad |\alpha| \le M,\ x \in \mathbb{R}^n,\ l \ge 1.$$

We add to these the following observation:

$$(3.13)\qquad |D^\alpha_y K_l(x,y)| \le c2^{l(n+|\alpha|)},\quad |\alpha| \le M+1,\ x,y \in \mathbb{R}^n.$$

Moreover, the *localization* properties of $K_l(x,y)$ are important for us. They can be summarized as follows.

There exist constants $c, C > 0$, *independent of* $l \ge -J$ *and* $x \in \mathbb{R}^n$, *such that*

(a) *if* $x \in \Omega$, *then*

$$\operatorname{supp} K_l(x,\cdot) \subset \overline{\Omega} \cap \{y : |x-y| \le c2^{-l}\};$$

(b) *if* $x \in \mathbb{R}^n \setminus \Omega$ *and* $-C2^{-l} \le x_n - \omega(x') \le 0$, *then*

$$\operatorname{supp} K_l(x,\cdot) \subset \overline{\Omega} \cap \{y : |\widetilde{x} - y| \le c2^{-l}\},$$

where $\widetilde{x} = (x', 2\omega(x') - x_n) \in \overline{\Omega}$;

(c) *if* $x \in \mathbb{R}^n \setminus \Omega$ *and* $x_n - \omega(x') < -C2^{-l}$, *then*

$$K_l(x,y) = \sum_m K_{lm}(x,y),\qquad \operatorname{supp} K_{lm}(x,\cdot) \subset \overline{\Omega} \cap \{y : |\widetilde{x}^m - y| \le c2^{-l}\},$$

where $\widetilde{x}^m = (x', x_n + u_m(\omega(x') - x_n)) \in \Omega$, *and the functions* $K_{lm}(x,y)$ *satisfy the same conditions* (3.11)–(3.13) *as* $K_l(x,y)$. *In other words*, (3.11)–(3.13) *remain true with* $K_l(x,y)$ *replaced by* $K_{lm}(x,y)$.

Let us verify these assertions. Part (a) is obvious, as $K_l(x,y) = \eta_l(x-y)$ for $x \in \Omega$. Furthermore, as soon as $C > 0$ is chosen and fixed, (b) is plainly true with sufficiently large c, depending on C. It remains to choose $K_{lm}(x,y)$ and C so that (c) is true. We take

$$K_{lm}(x,y) = \frac{v_m}{1-u_m}\eta_l\Big(x'-y', x_n - \omega(y') - \frac{y_n - \omega(y')}{1-u_m}\Big)$$
$$= \frac{v_m}{1-u_m}\eta_l\Big(x'-y', \frac{(\widetilde{x}^m)_n - y_n - u_m(\omega(x') - \omega(y'))}{1-u_m}\Big).$$

The localization of $K_{lm}(x,y)$ around $\widetilde{x}^m$ is clear. Take C so large that, for x with $x_n - \omega(x') < -C2^{-l}$,

$$\operatorname{supp} \eta_l(x - \cdot) \subset \mathbb{R}^n \setminus \Omega,\qquad \operatorname{supp} K_{lm}(x,\cdot) \subset \Omega.$$

This implies that $K_l(x,y) = \sum_m K_{lm}(x,y)$ for these x. Finally, it is clear that (3.11)–(3.13) are true with $K_{lm}(x,y)$ in place of $K_l(x,y)$.

Remark 3.2. The idea to consider three different cases of the localization of $K_l(x,y)$ is due to H. Triebel [Tri3, 4.5.5], where it appeared in the context of extension theorems for the halfspace.

Step 5. Now we are ready to derive the estimates of $\mathbf{S}_l f(x)$, which are the essence of the proof.

From (3.10), using the identity $f = \sum_{i=0}^{\infty} \eta_i * f$ in $\mathcal{D}'(\Omega)$, we come to

$$(3.14)\qquad \mathbf{S}_l f(x) \le \sum_{i,j\ge 0} \Big|\int K_l(x,y)\,\eta_i * \eta_j * f(y)\,dy\Big|$$
$$\le 2 \sum_{i,j\ge 0:\, |i-l| \ge |j-l|} \Big|\int K_l(x,y)\,\eta_i * \eta_j * f(y)\,dy\Big|,$$

where the latter inequality follows from the commutativity $\eta_i * \eta_j = \eta_j * \eta_i$.

As we have seen in the previous step, for every $x \in \mathbb{R}^n$ the kernel $K_l(x,y)$ is localized around some point $\widetilde{x} \in \overline{\Omega}$ (cases (a), (b)), or can be represented as a finite sum of kernels so localized (case (c)). Having in mind necessary modifications, which should be done in the latter situation, we assume that the former case occurs.

Our plan is to estimate the summands on the right-hand side of (3.14) by values of maximal functions at the point $\widetilde{x}$. This can be done as follows:

$$(3.15)\qquad \Big|\int K_l(x,y)\,\eta_i * \eta_j * f(y)\,dy\Big| = \Big|\int K_l(x,y)\int \eta_i(y-\widetilde{x}+z)\,\eta_j * f(\widetilde{x}-z)\,dz\,dy\Big|$$
$$\le \int\Big|\int K_l(x,y)\,\eta_i(y-\widetilde{x}+z)\,dy\Big|\cdot|\eta_j * f(\widetilde{x}-z)|\,dz \le I_{ijl}\cdot \eta^{\Omega}_{j,a}f(\widetilde{x}).$$

Here, in passing to $\eta^{\Omega}_{j,a}f(\widetilde{x})$, we have used the inclusion $\operatorname{supp} K_l(x,\cdot)\subset\overline{\Omega}$ and the fact that $\eta_j(y-\widetilde{x}+z)=0$ for $y\in\Omega$, $\widetilde{x}-z\notin\Omega$, and we have set

$$(3.16)\qquad I_{ijl}=\int(1+2^j|z|)^a\Big|\int K_l(x,\widetilde{x}-z+y)\,\eta_i(y)\,dy\Big|\,dz.$$

To establish appropriate estimates of I_{ijl}, we need (3.11)–(3.13), the localization properties of $K_l(x,\cdot)$, and the moment conditions on η_i. The main idea is to expand either $K_l(x,\cdot)$ or η_i by Taylor's formula. The terms with lower derivatives then vanish because of the moment conditions on the other function, and the necessary estimates arise. It is intuitively clear that one should expand the function which is more "flat", i.e. has a smaller subscript.

Remark 3.3. This idea can be traced back to N. J. H. Heideman [Hei]. Estimates, somewhat less general than we need, but proved in a similar fashion, can be found in [BPT1].

Let us turn to concrete estimates of I_{ijl}. Consider the case $i\ge l$, $i\ge 1$. We begin with Taylor's expansion

$$K_l(x,\widetilde{x}-z+y)=\sum_{|\alpha|\le M}\frac{y^\alpha}{\alpha!}D^\alpha_y K_l(x,\widetilde{x}-z) + \sum_{|\alpha|=M+1}c_\alpha y^\alpha\int_0^1(1-\tau)^M D^\alpha_y K_l(x,\widetilde{x}-z+\tau y)\,d\tau.$$

We put it into (3.16) and use the moment conditions on η_i to get

$$I_{ijl}=\int(1+2^j|z|)^a \times\Big|\sum_{|\alpha|=M+1}c_\alpha\int y^\alpha\int_0^1(1-\tau)^M D^\alpha_y K_l(x,\widetilde{x}-z+\tau y)\,d\tau\,\eta_i(y)\,dy\Big|\,dz.$$

Now rough estimation suffices. By the localization properties of $K_l(x,\cdot)$ and η_i, the integrand is zero if $|z|\ge c2^{-l}$ or $|y|\ge c2^{-i}$ (recall that $i\ge l$). Taking into account this fact and (3.13), we have

$$(3.17)\qquad I_{ijl}\le c\max(1,2^{(j-l)a})2^{-i(M+1)}2^{l(n+M+1)}2^{in}2^{-in}2^{-ln} = c\max(1,2^{(j-l)a})2^{(l-i)(M+1)},\quad i\ge l,\ i\ge 1.$$

Note that this estimate is also true for $-J\le l\le 0$, $i=0$ with a constant depending on J.

Quite analogously to (3.17), by Taylor's formula for η_i and the moment conditions (3.12) on $K_l(x,\cdot)$, we also obtain

$$(3.18)\qquad I_{ijl}\le c\max(1,2^{(j-i)a})2^{(i-l)(M+1)},\quad i\le l.$$

Putting (3.14)–(3.18) together, we arrive at

$$\mathbf{S}_l f(x)\le c\sum_{j=0}^{l-1}\eta^{\Omega}_{j,a}f(\widetilde{x})\Big(\sum_{i=0}^{j}2^{(j-i)a}2^{(i-l)(M+1)}+\sum_{i=2l-j}^{\infty}2^{(l-i)(M+1)}\Big)$$
$$+c\sum_{j=l}^{\infty}\eta^{\Omega}_{j,a}f(\widetilde{x})\Big(\sum_{i=0}^{2l-j}2^{(j-i)a}2^{(i-l)(M+1)}+\sum_{i=j}^{\infty}2^{(j-l)a}2^{(l-i)(M+1)}\Big)$$
$$\le c\sum_{j=0}^{l-1}\eta^{\Omega}_{j,a}f(\widetilde{x})2^{(j-l)(M+1)}+c\sum_{j=l}^{\infty}\eta^{\Omega}_{j,a}f(\widetilde{x})2^{(l-j)(M+1-2a)}.$$

In view of $M>|s|+2a-1$, this implies

$$(3.19)\qquad \mathbf{S}_l f(x)\le\sum_{j=0}^{\infty}2^{-|j-l|\sigma}\eta^{\Omega}_{j,a}f(\widetilde{x}),$$

where $\sigma>|s|$.

This is the crucial estimate. Let us write it in its full form, specializing the position of $\widetilde{x}\in\overline{\Omega}$ in dependence on $x\in\mathbb{R}^n$ (cases (a)–(c), Step 4). The necessary modification to be done in case (c) is clear: $\eta^{\Omega}_{j,a}f(\widetilde{x})$ in (3.19) must be changed to $\sum_m\eta^{\Omega}_{j,a}f(\widetilde{x}^m)$. Thus, we have

$$(3.20)\qquad \sum_{j=0}^{\infty}|\eta_l * \operatorname{ext} f_j(x)|\le\sum_{j=0}^{\infty}2^{-|j-l|\sigma}\times\begin{cases}\eta^{\Omega}_{j,a}f(x) & \text{if } x\in\Omega,\\ \eta^{\Omega}_{j,a}f(x',2\omega(x')-x_n) & \text{if } -C2^{-l}\le x_n-\omega(x')\le 0,\\ \sum_m\eta^{\Omega}_{j,a}f(x',x_n+u_m(\omega(x')-x_n)) & \text{if } x_n-\omega(x')<-C2^{-l}.\end{cases}$$

Step 6. Having (3.20), it is not difficult to complete the proof of (3.3). Indeed, by Lemma 3.1 (with $\mu_0=\eta_0$), (3.9), (3.20), and well-known inequalities for l_q, we have, with $r=\min(1,q)$,

$$\|\mathrm{ext}^i f|F^s_{pq}\| \le c\Big\|\Big(\sum_{l=-J}^{\infty} 2^{lsq}\Big(\sum_{j=0}^{\infty}|\mu_l * \mathrm{ext}\, f_j(\cdot)|\Big)^q\Big)^{1/q}\Big|L_p\Big\|$$
$$\le c\Big(\sum_{m=-\infty}^{\infty} 2^{-|m|(\sigma-|s|)r}\Big)^{1/r}\Big\|\Big(\sum_{j=0}^{\infty}2^{jsq}\eta^{\Omega}_{j,a}f(\cdot)^q\Big)^{1/q}\Big|L_p(\Omega)\Big\|$$
$$\le c\|f|F^s_{pq}(\Omega)\|'.$$

The proof of Theorem 2.1 is complete.

4. Proof of Theorem 2.2. The proof itself is given at the end of the section. Its main point is to establish the inequality

$$\|f|F^s_{pq}(\Omega)\|' \le c\|f|F^s_{pq}(\Omega)\|'', \tag{4.1}$$

which enables us to reduce Theorem 2.2 to Theorem 2.1. The proof of (4.1) uses the ideas of [BPT2] and is based on some pointwise estimates for $\eta^{\Omega}_{j,a}f$, collected in Lemma 4.1 below.

Let $\mathcal{M}$ denote the Hardy–Littlewood maximal operator and let χ_Ω stand for the characteristic function of Ω. If a function g is defined on Ω, then its extension to the whole $\mathbb{R}^n$ by zero is formally denoted as a product $g\cdot\chi_\Omega$.

LEMMA 4.1. *Let Ω be a special C^L-domain with some $L\in\mathbb{N}$. Let η_0 be a proper kernel of order $M\in\mathbb{N}_0$, suitable for Ω.*

(a) *Let $a>2n$ and $M\ge 2a-n-1$. Then, for all $f\in\mathcal{D}'(\Omega)$, $l\in\mathbb{N}_0$ and $x\in\Omega$,*

$$\eta^{\Omega}_{l,a}f(x) \le c\sum_{j=l}^{\infty}2^{(l-j)a/2}\mathcal{M}(|\eta_j * f|\cdot\chi_\Omega)(x). \tag{4.2}$$

(b) *Let $0<r<1$, $a>2n/r$, $1\le p<\infty$, $s\in\mathbb{R}$, $M>2\max(2a,n/p-s)-n-1$. Let $f\in\mathcal{D}'(\Omega)$ be such that, for some constant B and all $j\in\mathbb{N}_0$,*

$$\|\eta_j * f|L_p(\Omega)\| \le B2^{-js}. \tag{4.3}$$

Then, for all $l\in\mathbb{N}_0$ and $x\in\Omega$,

$$\eta^{\Omega}_{l,a}f(x)^r \le c\sum_{j=l}^{\infty}2^{(l-j)ar/2}\mathcal{M}(|\eta_j * f|^r\cdot\chi_\Omega)(x), \tag{4.4}$$

the constant c being independent of B.

Remark 4.1. This is an analogue of [BPT2, Lemma 2]. The latter dealt with the case of the whole $\mathbb{R}^n$, and the authors were able to prove it without the a priori assumption (4.3). In Section 5(a) we discuss some specific difficulties, arising in our context. To overcome them one requires (4.3).

Proof. *Step 1.* In this step we prove the following preparatory estimate. Let $a\ge 0$ and $M\ge 2a-n-1$. Then, for all $f\in\mathcal{D}'(\Omega)$, $l\in\mathbb{N}_0$ and $x\in\Omega$,

$$|\eta_l * f(x)| \le c\sum_{j=l}^{\infty}2^{(l-j)a}2^{jn}\int_{\Omega}\frac{|\eta_j * f(y)|}{(1+2^j|x-y|)^a}\,dy. \tag{4.5}$$

We begin with a representation

$$f=\sum_{i,j\ge 0}\eta_i*\eta_j*f=\Big(\sum_{i,j>l}+\sum_{i\le l \text{ or } j\le l}\Big)\eta_i*\eta_j*f. \tag{4.6}$$

We note that

$$\sum_{i\le l \text{ or } j\le l}\eta_i*\eta_j=\Big(\sum_{i=0}^{l}\sum_{j=0}^{\infty}+\sum_{j=0}^{l}\sum_{i=0}^{\infty}-\sum_{i=0}^{l}\sum_{j=0}^{l}\Big)\eta_i*\eta_j \tag{4.7}$$
$$=2\sum_{j=0}^{l}\eta_j-\Big(\sum_{i=0}^{l}\eta_i\Big)*\Big(\sum_{j=0}^{l}\eta_j\Big)$$
$$=2(\eta_0)_l-(\eta_0)_l*(\eta_0)_l$$
$$=(2\eta_0-\eta_0*\eta_0)_l=\Psi_l,$$

where $\Psi_0=\Psi=2\eta_0-\eta_0*\eta_0$. By (4.6), (4.7), and the fact that $\eta_l*\eta_i$, η_0, and $\eta_0*\eta_0$ have support in K, for every $x\in\Omega$,

$$|\eta_l*f(x)|\le\sum_{i,j>l}|\eta_l*\eta_i*\eta_j*f(x)|+|\eta_l*\Psi_l*f(x)| \tag{4.8}$$
$$\le 2\sum_{j=l+1}^{\infty}\sum_{i=j}^{\infty}\int_{\Omega}|\eta_l*\eta_i(x-y)|\cdot|\eta_j*f(y)|\,dy$$
$$+\int_{\Omega}|\Psi_l(x-y)|\cdot|\eta_l*f(y)|\,dy.$$

It is clear that

$$\Psi_l(x-y)=2^{ln}\Psi(2^l(x-y))\le\frac{c2^{ln}}{(1+2^l|x-y|)^a}. \tag{4.9}$$

We want to establish analogous estimates for $|\eta_l*\eta_i(x-y)|$, $i\ge j\ge l+1$. By Taylor's formula and the moment conditions on η_i we have

$$\eta_l*\eta_i(x)=\sum_{|\alpha|=M+1}c_\alpha\int(-y)^\alpha\int_0^1(1-\tau)^M D^\alpha\eta_l(x-\tau y)\,d\tau\,\eta_i(y)\,dy.$$

Using $i>l$, we see that the integrand is zero if $|x|\ge c2^{-l}$ or $|y|\ge c2^{-i}$. This implies

$$|\eta_l*\eta_i(x)|\begin{cases}\le c2^{-i(M+1)}2^{l(n+M+1)}2^{in}2^{-in}=c2^{ln}2^{(l-i)(M+1)},\\ =0 \quad \text{if } |x|\ge c2^{-l}.\end{cases}$$

Hence it follows that, for $i \ge j \ge l+1$,

$$(4.10)\qquad |\eta_l * \eta_i(x-y)| \le \frac{c2^{ln}2^{(l-i)(M+1)}}{(1+2^l|x-y|)^a} \le 2^{(j-l)a}\frac{c2^{ln}2^{(l-i)(M+1)}}{(1+2^j|x-y|)^a}.$$

Now we put (4.9) and (4.10) into (4.8), sum over i and arrive at (4.5), provided $M \ge 2a-n-1$.

Step 2. Estimate (4.2) can be derived from (4.5) by the following transformations, very similar to those used in [BPT2] for $\mathbb{R}^n$.

By (4.5), for every $x+y \in \Omega$,

$$(4.11)\qquad |\eta_l * f(x+y)| \le c\sum_{j=l}^{\infty} 2^{(l-j)a}2^{jn}\int_{\Omega}\frac{|\eta_j * f(z)|}{(1+2^j|x+y-z|)^a}\,dz.$$

In view of the inequality

$$(1+2^j|x+y-z|)^a \ge (1+2^l|x+y-z|)^a \ge \frac{(1+2^l|x-z|)^a}{(1+2^l|y|)^a}$$
$$\ge \frac{(1+2^l|x-z|)^{a/2}}{(1+2^l|y|)^a}$$
$$\ge \frac{(1+2^j|x-z|)^{a/2}}{(1+2^l|y|)^a}2^{(l-j)a/2}, \qquad j \ge l,$$

it follows from (4.11) that

$$(4.12)\qquad \frac{|\eta_l * f(x+y)|}{(1+2^l|y|)^a} \le c\sum_{j=l}^{\infty} 2^{(l-j)a/2}2^{jn}\int_{\Omega}\frac{|\eta_j * f(z)|}{(1+2^j|x-z|)^{a/2}}\,dz.$$

Decomposing Ω into a union of concentric annuli:

$$\Omega = \{z \in \Omega : |x-z| \le 2^{-j}\} \cup \bigcup_{k=1}^{\infty}\{z \in \Omega : 2^{k-j-1} \le |x-z| \le 2^{k-j}\},$$

we obtain (4.2) from (4.12), provided $a > 2n$.

Step 3. In this (last) step we prove that, under the conditions of (b), for all $l \in \mathbb{N}_0$ and $x \in \Omega$,

$$(4.13)\qquad |\eta_l * f(x)|^r \le c\sum_{j=l}^{\infty} 2^{(l-j)ar}2^{jn}\int_{\Omega}\frac{|\eta_j * f(y)|^r}{(1+2^j|x-y|)^{ar}}\,dy,$$

the constant c being independent of B. Estimate (4.4) follows from (4.13) in the same fashion as (4.2) followed from (4.5).

For $L \in \mathbb{N}_0$, $a \ge 0$ and $f \in \mathcal{D}'(\Omega)$ satisfying (4.3), we introduce the maximal function

$$M_a(x,L) = \sup_{j\ge L,\, z\in\Omega}\frac{|\eta_j * f(z)|}{(1+2^L|x-z|)^a}2^{(L-j)a}, \qquad x \in \Omega.$$

Let $b > \max(2a, n/p - s)$ be such that $M \ge 2b-n-1$. Note that $b > n/p'$. Applying Hölder's inequality to (4.5) with b instead of a, and using (4.3), we see that, for every $x \in \Omega$,

$$|\eta_l * f(x)|$$
$$\le c\sum_{j=l}^{\infty} 2^{(l-j)b}2^{jn}\Big(\int_{\Omega}|\eta_j * f(y)|^p\,dy\Big)^{1/p}\Big(\int\frac{dy}{(1+2^j|x-y|)^{bp'}}\Big)^{1/p'}$$
$$\le cB\sum_{j=l}^{\infty} 2^{(l-j)b}2^{jn}2^{-js}2^{-jn/p'} = cB2^{l(n/p-s)}\sum_{j=l}^{\infty}2^{(l-j)(b-n/p+s)}$$
$$\le cB2^{l(n/p-s)}.$$

Hence $M_b(x,L) < \infty$ for all $x \in \Omega$ and $L \in \mathbb{N}_0$.

Furthermore, by the definition of $M_b(x,L)$, and (4.5) with b instead of a, we have, for every $z, x \in \Omega$ and $l \ge L$,

$$|\eta_l * f(z)| \le c\sum_{j=l}^{\infty} 2^{(l-j)b}2^{jn}\int_{\Omega}\frac{|\eta_j * f(y)|^r}{(1+2^L|z-y|)^b}$$
$$\times (M_b(x,L)(1+2^L|x-y|)^b 2^{(j-L)b})^{1-r}dy$$
$$\le cM_b(x,L)^{1-r}(1+2^L|x-z|)^b 2^{(l-L)b}$$
$$\times\sum_{j=l}^{\infty} 2^{(L-j)br}2^{jn}\int_{\Omega}\frac{|\eta_j * f(y)|^r}{(1+2^L|x-y|)^{br}}\,dy.$$

This implies that, for every $x \in \Omega$ and $L \in \mathbb{N}_0$,

$$M_b(x,L) \le cM_b(x,L)^{1-r}\sum_{j=L}^{\infty} 2^{(L-j)br}2^{jn}\int_{\Omega}\frac{|\eta_j * f(y)|^r}{(1+2^L|x-y|)^{br}}\,dy.$$

Thus, taking into account that $M_b(x,L) < \infty$, we get

$$(4.14)\qquad |\eta_L * f(x)|^r \le M_b(x,L)^r$$
$$\le c\sum_{j=L}^{\infty} 2^{(L-j)br}2^{jn}\int_{\Omega}\frac{|\eta_j * f(y)|^r}{(1+2^L|x-y|)^{br}}\,dy$$
$$\le c\sum_{j=L}^{\infty} 2^{(L-j)br/2}2^{jn}\int_{\Omega}\frac{|\eta_j * f(y)|^r}{(1+2^j|x-y|)^{br/2}}\,dy.$$

The constant c here does not depend on B, since we used (4.3) only to show the finiteness of $M_b(x,L)$. It remains to note that the right-hand side of (4.14) will only increase if we change b to $2a < b$. Lemma 4.1 is proved.

Remark 4.2. The $\mathbb{R}^n$ prototypes of estimates (4.5) and (4.13) were proved in the book by J.-O. Strömberg and A. Torchinsky [StT, Chapter V, Theorem 2(a)]. The scheme of our proof is essentially the same: first consider the case $r = 1$, and then derive the estimate for $r < 1$ by the "trick" with $M_a(x, L)$.

Proof of Theorem 2.2. Let $f \in \mathcal{D}'(\Omega)$ and $\|f|F^s_{pq}(\Omega)\|'' < \infty$. One can easily verify that for L and M from Theorem 2.2 it is possible to choose

$$r \in (0, \min(p, q)), \quad r = 1 \text{ if } p, q > 1,$$

and

$$a > \max(2n/r, -2s)$$

such that

$$M > 2\max(2a, n/p - s) - n - 1, \quad M, L \geq [|s| + 2a].$$

By Lemma 4.1(b) (note that (4.3) is satisfied) we have, for all $x \in \Omega$ and $j \in \mathbb{N}_0$,

$$\eta^\Omega_{l,a} f(x)^r \leq c \sum_{j=l}^{\infty} 2^{(l-j)ar/2} \mathcal{M}(|f_j|^r)(x), \tag{4.15}$$

where we put $f_j = (\eta_j * f) \cdot \chi_\Omega$. Consider the quasi-norm $\|f|F^s_{pq}(\Omega)\|'$ from Theorem 2.1 with the same η_0 and the chosen a. It follows from (4.15) that

$$\|f|F^s_{pq}(\Omega)\|' \leq c \Big\| \Big(\sum_{l=0}^{\infty} 2^{lsq} \Big(\sum_{j=l}^{\infty} 2^{(l-j)ar/2} \mathcal{M}(|f_j|^r)(\cdot) \Big)^{q/r} \Big)^{1/q} \Big| L_p(\Omega) \Big\|$$
$$= c \Big\| \Big(\sum_{l=0}^{\infty} 2^{l(s+a/2)q} \Big(\sum_{j=l}^{\infty} 2^{-jar/2} \mathcal{M}(|f_j|^r)(\cdot) \Big)^{q/r} \Big)^{1/q} \Big| L_p(\Omega) \Big\|.$$

By a discrete version of the Hardy inequality

$$\sum_{l=0}^{\infty} 2^{l\sigma} \Big(\sum_{j=l}^{\infty} |b_j| \Big)^{\tau} \leq c \sum_{l=0}^{\infty} 2^{l\sigma} |b_l|^{\tau},$$

which is valid for $\sigma > 0$, $\tau > 0$, the last expression is no greater than

$$c \Big\| \Big(\sum_{l=0}^{\infty} 2^{lsq} \mathcal{M}(|f_l|^r)(\cdot)^{q/r} \Big)^{1/q} \Big| L_p(\Omega) \Big\|.$$

We can now use the vector-valued maximal inequality by C. Fefferman and E. M. Stein [FeS] to conclude that

$$\|f|F^s_{pq}(\Omega)\|' \leq c \Big\| \Big(\sum_{l=0}^{\infty} 2^{lsq} |f_l(\cdot)|^q \Big)^{1/q} \Big| L_p \Big\| = c \|f|F^s_{pq}(\Omega)\|''.$$

The inverse inequality $\|f|F^s_{pq}(\Omega)\|'' \leq \|f|F^s_{pq}(\Omega)\|'$ is obvious. Thus, the quasi-norms $\|\cdot|F^s_{pq}(\Omega)\|''$ and $\|\cdot|F^s_{pq}(\Omega)\|'$ are equivalent. Now Theorem 2.2 follows from Theorem 2.1.

Remark 4.3. Note that in the case of B^s_{pq} spaces with $1 < p \leq \infty$, $0 < q \leq \infty$ and F^s_{pq} spaces with $1 < p < \infty$, $1 < q \leq \infty$ we can prove Theorem 2.2 with the help of Lemma 4.1(a) instead of Lemma 4.1(b) (the other arguments remain the same). This leads to somewhat milder conditions on the number of vanishing moments M. Say, $M = [5|s| + 4n]$ is sufficient in this situation.

5. Concluding remarks. (a) We now discuss the difficulties in the proof of Theorem 2.2 which prevented us from including the case $0 < p < 1$. The problem lies in the derivation of (4.13) from (4.5). In that derivation the assumption $p \geq 1$ was used to prove that $M_a(x, L) < \infty$ for some a for which (4.5) holds. In fact, we always have $M_a(x, L) < \infty$ for sufficiently large $a > a(f)$. Therefore all problems would disappear if we had (4.5) for all a. Why have we been unable to prove (4.5) for all a? Because in the "reproducing formula" (see (4.6), (4.7))

$$f = \sum_{i>l} \eta_i * \eta_i * f + 2 \sum_{i>j>l} \eta_i * \eta_j * f + \Psi_l * f, \tag{5.1}$$

used in the proof of (4.5), the first ("reproducing") kernel η_i has a limited number of vanishing moments. In Calderón's reproducing formula $f = \sum_i \psi_i * \phi_i * f$, used for similar purposes in [StT], [BPT2], the Fourier transform $\widehat{\psi}$ of the reproducing kernel ψ is compactly supported away from the origin, and thus all moments of ψ vanish. However, working in domains, we need reproducing formulas with compactly supported reproducing kernels, and Calderón's elegant formula is not applicable in this setting.

Some more reproducing formulas with compactly supported reproducing kernels can be found in D.-C. Chang, S. G. Krantz, and E. M. Stein [CKS], and in T. Schott [Sch], but they have the same shortcoming as (5.1), and cannot improve the situation.

(b) We now explain why we regard the atomic characterizations of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ from [TrW] (mentioned in the Introduction) as being nonconstructive.

Let us briefly recall what is meant by atomic characterizations. It was proved in [FrJ] that a distribution f on $\mathbb{R}^n$ belongs to F^s_{pq} if and only if it can be expanded in a series $f = \sum_Q c_Q \psi_Q$, where Q runs through all dyadic cubes in $\mathbb{R}^n$ with side length ≤ 1, compactly supported functions ψ_Q (called atoms) satisfy certain smoothness, moment and localization (around Q) conditions, and the sequence of coefficients c_Q lies in some discrete space $\mathbf{f}^s_{pq}$, with norm depending only on the magnitudes $|c_Q|$. There exist explicit, though complicated, formulas to calculate the coefficients c_Q from f.

In [TrW] a similar decomposition $f = \sum_Q c_Q \psi_Q$ for $f \in F^s_{pq}(\Omega) \subset \mathcal{D}'(\Omega)$ is established, where Q runs through dyadic cubes having nonempty intersection with Ω. But no way to determine the coefficients c_Q from f directly is pointed out. (Following [TrW], one should take first an extension $\widetilde{f} \in F^s_{pq}(\mathbb{R}^n)$, which is possible since $F^s_{pq}(\Omega)$ is defined as the restriction of F^s_{pq}, and then apply Frazier and Jawerth's decomposition to $\widetilde{f}$.)

(c) As we have already mentioned in Section 2, for Definition 2.2 of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ to work in the most perfect way, the boundary of Ω should be sufficiently smooth. However, there also exist certain results for nonsmooth domains, and we now list some of them (see also references in [TrW]).

To begin with, for $s > 0$ and $1 < p, q < \infty$ intrinsic characterizations of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ were obtained in [Bes3], [Kal1], [Kal2] at one stroke for the class of domains having Lipschitz boundary, and the same is true for what is done in [Mur].

For $1 < p, q < \infty$ and $s > 0$, $s - 1/p \notin \mathbb{N}_0$ there exist intrinsic characterizations even for certain domains with cusps (zero angles) due to G. A. Kalyabin [Kal3].

Turning to the case $0 < p \le 1$, we would like to mention investigations of the Hardy spaces $H_p(\Omega)$ carried out by A. Miyachi [Miy]. In that paper the spaces $H_p(\Omega)$ are defined in a constructive way (by means of certain maximal functions) for an arbitrary domain Ω. It is shown that for a wide class of domains (including domains with Lipschitz boundary) they coincide with the restrictions of the usual Hardy spaces $H_p(\mathbb{R}^n)$. Since $H_p(\mathbb{R}^n)$ coincides with the homogeneous space $\dot{F}^0_{p,2}(\mathbb{R}^n)$, Miyachi's result can be interpreted as an intrinsic characterization of $\dot{F}^0_{p,2}(\Omega)$.

(d) We also note that for $s > 0$ and $1 < p, q < \infty$ there exists another approach to the theory of B^s_{pq} and F^s_{pq} spaces on domains, not via restrictions. It consists in introducing explicit norms in $L_p(\Omega)$ and defining $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ as the sets of $L_p(\Omega)$ functions for which these norms are finite. This allows one to study $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ spaces on a very general class of domains having nonsmooth, even fractal boundary (the domains satisfying the so-called flexible cone condition). This theory is to a great extent due to O. V. Besov. We mention only his paper [Bes4] and the book by O. V. Besov, V. P. Il'in, and S. M. Nikol'skiĭ [BIN], where further details and references can be found. Probably, this approach can also be extended to $s \le 0$ and $0 < p < 1$. For Lipschitz domains and $s > 0$, $1 < p, q < \infty$, both approaches are equivalent.

Addendum (September 1997). We have recently found reproducing formulas with compactly supported kernels having arbitrarily many vanishing moments (cf. Section 5(a), where we discussed the necessity of such formulas). With the aid of these formulas we can improve Theorems 2.1 and 2.2. Namely, we can prove similar results for domains with Lipschitz boundary and under the natural assumption $M \ge [s]$. We can also dispense with the restriction $p \ge 1$ in Theorem 2.2. The details ([Ry]) will appear elsewhere.

References

[Bes1] O. V. Besov, *On a family of function spaces. Embedding theorems and extensions*, Dokl. Akad. Nauk SSSR 126 (1959), 1163–1165 (in Russian).

[Bes2] —, *On a family of function spaces in connection with embeddings and extensions*, Trudy Mat. Inst. Steklov. 60 (1961), 42–81 (in Russian).

[Bes3] —, *Extension of functions beyond a domain with preservation of difference-differential properties in L_p*, Mat. Sb. 66 (1965), 80–96 (in Russian).

[Bes4] —, *Embeddings of Sobolev–Liouville and Lizorkin–Triebel spaces on a domain*, Dokl. Akad. Nauk 331 (1993), 538–540 (in Russian).

[BIN] O. V. Besov, V. P. Il'in and S. M. Nikol'skiĭ, *Integral Representations of Functions and Embedding Theorems*, 2nd ed., Nauka–Fizmatgiz, Moscow, 1996 (in Russian).

[BPT1] H.-Q. Bui, M. Paluszyński and M. H. Taibleson, *A maximal function characterization of weighted Besov–Lipschitz and Triebel–Lizorkin spaces*, Studia Math. 119 (1996), 219–246.

[BPT2] —, —, —, *Characterization of the Besov–Lipschitz and Triebel–Lizorkin spaces. The case $q < 1$*, in: Proc. Conf. on Harmonic Analysis in Honor of M. de Guzmán (El Escorial, 1996), to appear.

[CKS] D.-C. Chang, S. G. Krantz and E. M. Stein, *H^p theory on a smooth domain in $\mathbb{R}^n$ and elliptic boundary value problems*, J. Funct. Anal. 114 (1993), 286–347.

[FeS] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), 107–115.

[FrR] J. Franke and T. Runst, *Regular elliptic boundary value problems in Besov–Triebel–Lizorkin spaces*, Math. Nachr. 174 (1995), 113–149.

[FrJ] M. Frazier and B. Jawerth, *A discrete transform and decomposition of distribution spaces*, J. Funct. Anal. 93 (1990), 34–170.

[Hei] N. J. H. Heideman, *Duality and fractional integration in Lipschitz spaces*, Studia Math. 50 (1974), 65–85.

[Hes] M. Hestenes, *Extension of the range of a differentiable function*, Duke Math. J. 8 (1941), 183–192.

[Kal1] G. A. Kalyabin, *Function spaces of Lizorkin–Triebel type in domains with Lipschitz boundary*, Dokl. Akad. Nauk SSSR 271 (1983), 795–798 (in Russian).

[Kal2] —, *Theorems on extensions, multipliers and diffeomorphisms for generalized Sobolev–Liouville classes in domains with Lipschitz boundary*, Trudy Mat. Inst. Steklov. 172 (1985), 173–186 (in Russian).

[Kal3] —, personal communication, 1994.

[Liz1] P. I. Lizorkin, *Operators connected with fractional derivatives and classes of differentiable functions*, Trudy Mat. Inst. Steklov. 117 (1972), 212–243 (in Russian).

[Liz2] —, *Properties of functions of the spaces $\Lambda^r_{p\theta}$*, ibid. 131 (1974), 158–181 (in Russian).

[Miy] A. Miyachi, *H^p spaces over open subsets of $\mathbb{R}^n$*, Studia Math. 95 (1990), 205–228.

[Mur] T. Muramatu, *On Besov spaces and Sobolev spaces of generalized functions defined on a general region*, Publ. Res. Inst. Math. Sci. Kyoto Univ. 9 (1974), 325–396.

[P1] J. Peetre, *Remarques sur les espaces de Besov. Le cas $0 < p < 1$*, C. R. Acad. Sci. Paris Sér. A–B 277 (1973), 947–950.

[P2] —, *On spaces of Triebel–Lizorkin type*, Ark. Mat. 13 (1975), 123–130.

[Rud] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.

[Ry] V. S. Rychkov, *On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domains*, preprint.

[Sch] T. Schott, *Function spaces with exponential weights I*, Math. Nachr., to appear.

[See] A. Seeger, *A note on Triebel–Lizorkin spaces*, in: Approximation and Function Spaces, Banach Center Publ. 22, PWN–Polish Sci. Publ., Warszawa, 1989, 391–400.

[StT] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer, Berlin, 1989.

[Tai] M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space I. Principal properties*, J. Math. Mech. 13 (1964), 407–479.

[Tri1] H. Triebel, *Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation*, Ark. Mat. 11 (1973), 13–64.

[Tri2] —, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.

[Tri3] —, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.

[Tri4] — , *Local approximation spaces*, Z. Anal. Anwendungen 8 (1989), 261–288.

[TrW] H. Triebel and H. Winkelvoß, *Intrinsic atomic characterizations of function spaces on domains*, Math. Z. 221 (1996), 647–673.

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